Asymptotically Optimal Cooperative Wireless Networks with Reduced Signaling Complexity

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Abstract—This paper considers an orthogonal amplify-and-forward (OAF) protocol for cooperative relay communication over Rayleigh-fading channels in which the intermediate relays are permitted to linearly transform the received signal and where the source and relays transmit for equal time durations. The diversity-multiplexing gain (D-MG) tradeoff of the equivalent space-time channel associated to this protocol is determined and a cyclic-division-algebra-based D-MG optimal code constructed. The transmission or signaling alphabet of this code is the union of the QAM constellation and a rotated version of QAM. The size of this signaling alphabet is small in comparison with prior D-MG optimal constructions in the literature and is independent of the number of participating nodes in the network.

Index Terms—cooperative diversity, distributed space-time code, orthogonal amplify and forward, diversity-multiplexing gain tradeoff, space-time codes, cyclic division algebra codes.

I. INTRODUCTION

In wireless communications networks with fading, cooperative diversity protocols seek to provide MIMO-diversity benefits without requiring multiple transmit or receive antennas at any of the nodes in the network.

A. Existing Cooperative Diversity Schemes

Several cooperative diversity protocols were recently presented (for example in [4]-[9]). These can be separated into two main categories: the amplify-and-forward category, where the assisting nodes perform a linear operation on the signal vector they receive from the information source, before forwarding it, and the decode-and-forward category, where the assisting nodes try to decode the received signal, and eventually re-encode it before sending it again. While the error performance analysis in [7] focused on the diversity of the protocols, and the analysis in [9] focused on the capacity of the network, the works in [4], [5], [6] applied the diversity-multiplexing gain (D-MG) tradeoff as a means of evaluating the fundamental limitations in the error performance of the different cooperation protocols, in a similar manner that the same tradeoff was used by Zheng and Tse for the point to point MIMO case [13]. D-MG analysis also led to the introduction of the orthogonal selection-decode-and-forward (O-SDAF, [5]) protocol, which asks the users to adapt to the channel’s outage behavior. D-MG analysis also highlighted the benefits of the non-orthogonal amplify-and-forward (NAF) scheme proposed in [8] and generalized in [6], where the protocol’s D-MG performance was established to be better than that of the O-SDAF and of the orthogonal amplify-and-forward (OAF) protocols, mainly due to the fact that it had the source transmitting continuously. As a consequence, the NAF protocol allowed for a non-zero diversity gain, even for the highest multiplexing gain regions.

The optimal D-MG performance of the protocols was established in the early papers through random coding arguments and the D-MG optimal implementation of the above O-SDAF and NAF protocols was thought to require infinite time duration and infinite decoding complexity.

Recently however, in ([10], [11]), D-MG optimal implementations of the OAF and NAF cooperation protocols were explicitly constructed to meet the corresponding protocol’s high-SNR outage region, for any network size, in finite time duration with finite sphere-decoding complexity.

In the distributed space-time codes designed for cooperative relay communication, the signals transmitted by the source and intermediate relays are typically drawn from a common alphabet $\mathcal{S}$ apart from scale factors that adjust for transmission power. We will refer to $\mathcal{S}$ as the signaling alphabet of the distributed space-time code (DSTC).

Definition 1: A DSTC operating at $R$ bits per channel use (bpcu) is said to have signaling set complexity $C_s$, if the cardinality of its signaling set $\mathcal{S}$ is:

$$|\mathcal{S}| = \kappa_s(2^R)^{C_s},$$

where $\kappa_s$ is a constant independent of the rate $R$.

In this work we present a reduced signaling complexity, D-MG optimal implementation of an OAF protocol (see [7], [10]) in which both source and relays transmit for equal durations of time. This implementation maintains a signaling complexity $C_s = 2$ that is independent of the number of users in the network, thereby avoiding the exponential increase in signaling alphabet incurred by other D-MG optimal constructions found in the literature, for example those found in [11] and [10]. This is discussed in greater detail in Section II-B.

Our code construction, like other prior constructions [11] and [10], is based cyclic division algebras (CDA). A key difference in our construction is that we make use of the matrix representation of a subset $\mathcal{F}$ of elements in the CDA which commute under multiplication. It turns out that the corresponding matrix representations can be simultaneously diagonalized.

Section II presents the network model we consider as well as the equivalent channel seen by the DSTC. An upper
bound on the D-MG tradeoff of the cooperative relay network operating under the OAF protocol described here is then presented. Section III shows that the upper bound on D-MG tradeoff derived in II-C is tight by presenting the construction of a CDA-based DSTC whose probability of error performance achieves the upper bound. Most proofs are found in the Appendix.

II. NETWORK AND CHANNEL MODELS

A. Network Model

As in [4], the network consists of a set
\[ \mathcal{R} = \{ R_1, R_2, \ldots, R_n, R_{n+1} \} \]
of \(n + 1\) different cooperating terminals/relays (see Figure 1), each with the ability to communicate over \(n + 1\) different orthogonal frequencies \( \mathcal{F} = \{ \nu_1, \nu_2, \ldots, \nu_n, \nu_{n+1} \} \). A certain relay \( R_i \), wishing to communicate with relay \( d(R_i) \), broadcasts its information over frequency \( \nu_i \). Depending on the availability of each intermediate relay, the set
\[ D(R_i) \subset \{ \mathcal{R} \setminus \{ R_i \cup d(R_i) \} \} \tag{2} \]
is then the set of all intermediate relays that cooperate with \( R_i \). Consequently, each relay \( R_j \in D(R_i) \) transmits a possibly modified version of the received signal over frequency \( \nu_j \). By the end of the transmission, \( d(R_i) \) has received the information from \( R_i \) over frequency \( \nu_i \), essentially in a form of a superposition of faded versions of signals originating from \( R_i \) and from \( D(R_i) \).

We consider the case where communication takes place in the presence of additive receiver noise, and in the presence of spatially independent quasi-static fading. Furthermore, we will assume complete knowledge of the fading channel at the receiver of the final destination, and no knowledge of the fading at the receivers of the assisting relays.

a) Assumptions: The overall rate of information transmission is \( R \) bpcu, which specifically means that for some time duration of \( T \) channel uses, from the beginning to the end of the communication between the source and the destination, the total number of information bits received by the destination (correctly or incorrectly), is \( RT \).

It is assumed that each node has a single receive-transmit antenna operating under the half-duplex constraint, being able to either transmit or receive but not both simultaneously. This is due to practical considerations such as the large ratio between the transmission and reception powers at the relay antennas ([4], [5], [6]). Furthermore, no feedback is permitted to any of the transmitters.

All channels are assumed to be Rayleigh fading and all fade coefficients are assumed to be i.i.d., circularly-symmetric, complex Gaussian \( \mathcal{CN}(0,1) \) random variables, i.e., they are i.i.d. with common density function
\[ p(u) = \frac{1}{\pi} e^{-|u|^2}. \]
It is also assumed that all fading coefficients remain fixed for the duration of communication, i.e., that encoding over multiple channel realizations is not permitted.

The noise vector at the receivers is assumed to be comprised of i.i.d., circularly-symmetric, complex Gaussian \( \mathcal{CN}(0,1) \) random variables as well.

b) Notation: The notation \( \leq \) and \( \geq \) corresponds to the exponential equality and inequalities describing the equivalence of \( y = \rho \) to \( \lim_{\rho \to \infty} \frac{\log(y)}{\log(\rho)} = x \). Matrices, vectors, and scalars are respectively denoted by capital letters, underlined small letters, and small letters. \( x^\dagger \) represents the complex conjugate of a scalar \( x \), and \( X^\dagger \) represents the conjugate transpose of some matrix \( X \). \( |x| \) represents the Frobenius norm of \( X \), \( |x|^2 \) the square of the magnitude of some vector \( x \), and \( |x|^2 \) denotes the square of the magnitude of some scalar \( x \). For \( Y \) a set, \( |Y| \) is its cardinality. Furthermore, if \( Y \) is a set of scalars, vectors or matrices with entries from the complex numbers, \( \Delta Y \) denotes the set of all differences of such elements, where the difference is taken in a componentwise manner. The symbol \( \mathbb{Z} \) represents the sets of integers, \( \mathbb{Q} \) represents the rationals, and \( t := \sqrt{-1} \).

c) Performance Measure: Performance of the DSTC constructed here will be given in the form of the diversity-multiplexing gain (D-MG) tradeoff. In a network where each user operates at rate \( R \) bpcu and at SNR \( \rho \), performance is described in terms of the diversity gain
\[ d(r) := -\lim_{\rho \to \infty} \frac{\log(\Pr(e))}{\log(\rho)} \]
which is a function of the multiplexing gain
\[ r := R/\log(\rho). \]

B. Channel Model Under the OAF protocol

Under this OAF protocol, communication takes place in two phases. In the broadcast phase, Phase I, lasting for \( n \) channel uses, the source broadcasts to the relays and destination. In the cooperation phase, Phase II, the relays broadcast to the destination, again for a duration of \( n \) channel uses. The source is assumed to remain silent in Phase II.

Let \( g_i \) denote the fading coefficient from source \( S \) to destination \( D \), \( g_i, i = 2, 3, \ldots, n \) denote the fading coefficient between \( S \) and \( i^{th} \) intermediate relay \( R_i \) and \( h_j \) the fading coefficient from \( R_j \) to \( D \). As mentioned above, the \( h_j \) are assumed to be independent and have a \( \mathcal{CN}(0,1) \) distribution. These fading coefficients remain constant throughout the transmission, and change in an i.i.d. manner for every new message vector.

The random vectors \( w_i \) and \( v_i, i = 2, 3, \ldots, n \), represent the zero-mean, additive white Gaussian noise seen at the destination and the relays \( R_i \) in Phase I respectively. The noise vector at the destination in Phase II is denoted \( w_2 \). The components \( w_{1j}, v_{ij}, w_{2j}, j = 1, 2, \ldots, n \) are also assumed
satisfying the Frobenius-norm constraint:

$$\|X\|_F^2 = \frac{1}{n^2}$$

which contains the information to be communicated to the destination. The $f_i$ are drawn from some alphabet $\mathcal{A}$ and the scalar $\theta$ normalizes for energy.

The destination receives the $n$-length vector

$$z^T = \theta f^T + w^T,$$

where the $A_i$, $2 \leq i \leq n$ are chosen to be $n \times n$ matrices satisfying the Frobenius-norm constraint:

$$\|A_j\|_F \leq \alpha^2 = \rho^0,$$

for some $\alpha > 0$.

Consequently, the receiver up to time $t = 2n$ has received

$$y = \theta z^T X' + w^T T$$

where

$$z^T = \left[ g_1 g_2 h_2 \cdots g_n h_n \right]_{1 \times n},$$

$$w^T = \left[ w_1^T \sum_{j=2}^{n} h_j A_j + w_2^T \right]_{1 \times 2n},$$

$$X' = \left[ \begin{array}{c|c} \theta f^T & 0 \cdots 0 \\ \hline 0 & \cdots & 0 & f^T A_2 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & f^T A_n \end{array} \right]_{n \times 2n}$$

Taking the transpose of both sides of equation (8) leads to

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \left[ \begin{array}{c} \sum_{j=2}^{n} g_j h_j A_j \\ \sum_{j=2}^{n} h_j A_j w_1 + w_2 \end{array} \right].$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}^T = \left[ \begin{array}{c} \sum_{j=2}^{n} g_j h_j A_j \end{array} \right] + \left[ \begin{array}{c} \sum_{j=2}^{n} h_j A_j w_1 + w_2 \end{array} \right].$$

Consequently, the receiver up to time $t = 2n$ has received

$$y = \theta z^T X' + w^T T$$

where

$$z^T = \left[ g_1 g_2 h_2 \cdots g_n h_n \right]_{1 \times n},$$

$$w^T = \left[ w_1^T \sum_{j=2}^{n} h_j A_j + w_2^T \right]_{1 \times 2n},$$

$$X' = \left[ \begin{array}{c|c} \theta f^T & 0 \cdots 0 \\ \hline 0 & \cdots & 0 & f^T A_2 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & f^T A_n \end{array} \right]_{n \times 2n}$$

Taking the transpose of both sides of equation (8) leads to

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \left[ \begin{array}{c} \sum_{j=2}^{n} g_j h_j A_j \\ \sum_{j=2}^{n} h_j A_j w_1 + w_2 \end{array} \right].$$

### C. Upper Bound on the D-MG Tradeoff of the OAF Protocol

**Lemma 2.1:** The D-MG tradeoff of the above OAF protocol satisfies the upper bound:

$$d(r) \leq n(1 - 2r).$$

**Proof:** See Appendix I.

In the next section, we will identify a DSTC whose codeword error probability when employed over the OAF channel described by (8), satisfies

$$P_e(r) \leq \rho^{-n(1-2r)}$$

thereby proving that the right hand side in equation (13) is indeed the D-MG tradeoff of the described OAF protocol.

### III. A D-MG Optimal Code for the OAF Protocol

Our code construction is based on cyclic division algebras. Some background on these algebraic objects can be found in Appendix II and we will assume in the present section, that the reader is familiar with the terminology and notation introduced there.

### A. Code Construction

For $M$ even, let $\mathcal{A}_{QAM}$ denote the $M^2$-QAM constellation given by

$$\mathcal{A}_{QAM} = \{a + ib \mid |a|, |b| \leq M - 1, \ a, b \ \text{odd} \}.$$  

Let $X$ be the collection of matrices

$$X = \{X(l) \mid l_i \in \mathcal{A}_{QAM}\},$$

where

$$X(l) = \left[ \begin{array}{cccc} l_0 & \gamma l_{n-1} & \cdots & \gamma l_1 \\ l_1 & l_0 & \cdots & \gamma l_2 \\ \vdots & l_1 & \cdots & \vdots \\ l_{n-1} & l_{n-2} & \cdots & l_0 \end{array} \right]$$

where $X(l)$ is the matrix representation of an element $l_0 + zl_1 + \cdots + z^{n-1}l_{n-1}$ belonging to a CDA and where $\gamma$ is a non-norm element of unit magnitude $|\gamma| = 1$ belonging to $\mathbb{Q}(i)$, both as described in Appendix II. Often for the sake of convenience, we will abbreviate and write $X$ in place of $X(l)$. 

\[ \]
The matrix $X(l)$ can be expressed in the form

$$X(l) = \begin{bmatrix} l^T B_1 & \vdots & l^T B_n \end{bmatrix}$$

where

$$l = [l_0, l_1, \ldots, l_{n-1}]^T,$$

and where each matrix $B_i$ has just one nonzero entry, either 1 or $\gamma$, in each row and column. Since $|\gamma| = 1$, it follows that each matrix $B_i$ is unitary. Next, set

$$f^T = l^T B_1$$

$$A_j = B_i^T B_j, \quad 2 \leq j \leq n.$$

Then each matrix $A_j$ is also unitary and we can write

$$X(l) = \begin{bmatrix} f^T A_2 & \vdots & f^T A_n \end{bmatrix}.$$  \hfill (19)

Since the elements $l_i$ belong to the subset $A_{QM}$ of $\mathbb{Z}[i]$, the matrices in $\mathcal{X}$ are the regular representations of elements in the center $\mathcal{F} = \mathbb{Q}(i)$ of the CDA and hence commute under multiplication, thus meeting a condition needed for simultaneous diagonalizability. As shown in Appendix IV, $\mathcal{X}$ is indeed a collection of normal matrices that can be simultaneously diagonalized by a unitary transformation $S$, i.e., every matrix $X \in \mathcal{X}$ can be expressed in the form

$$X = S^d X_d S$$

$$\Leftrightarrow X_d = SXS^d$$

for a suitable diagonal matrix $X_d$.

e) Usage of the code $\mathcal{X}$ in the OAF protocol: The DSTC to be employed in the OAF protocol is built upon the matrices in $\mathcal{X}$. Let $l \in A_{QM}$, represent the message vector. As we shall see below, in effect, the signals transmitted by the various relays correspond to different rows of the corresponding code matrix

$$X(l) = \begin{bmatrix} l_0 & \gamma l_{n-1} & \cdots & \gamma l_1 & l_1 & l_0 \cdots & \gamma l_2 & \vdots & l_1 & \cdots & \cdots & \vdots & l_{n-2} & \cdots & \gamma l_{n-1} & l_{n-1} & l_{n-2} & \cdots & l_0 \end{bmatrix}.$$  \hfill (20)

The source maps $l$ onto $f^T = l^T B_1$ and transmits $f^T$. The $j$th relay, $j = 2, 3, \ldots, n$ then applies the linear transformation represented by the unitary matrix $A_j$ to the received vector $y_j$.

The received signal at the destination is then given by:

$$[\xi^T \; \eta^T] = \theta X^T$$

$$= \begin{bmatrix} f^T I_1 & \vdots & f^T A_3 & \vdots & f^T A_n \end{bmatrix}$$

$$\quad + [w_1^T \; w_2^T + \sum_{j=2}^n h_j w_j^T A_j ]$$  \hfill (20)

where $\theta^T$ is as defined in (9).

B. Proof of D-MG Optimality

**Theorem 3.1 (Main Theorem):** The D-MG tradeoff of the OAF protocol described here is given by

$$d(r) = n(1 - 2r)$$

and the CDA-based DSTC described in (15) and used as suggested by equations (19) and (20) is D-MG optimal with respect to this tradeoff.

**Proof:** Let $P_e(r)$ denote the probability of error of the CDA-based code when used over the OAF channel in accordance with (19) and (20). Our aim is to show that

$$P_e(r) \leq \rho^{-n(1-2r)}.$$  \hfill (22)

By Lemma 2.1, this will not only establish D-MG optimality of the code, it will also establish that the D-MG tradeoff of this OAF channel is precisely given by the right hand side of (13), thereby proving the theorem.

We begin by outlining a decoding procedure to be followed at the destination and showing that this decoding procedure has probability of codeword error upper bounded by (22).

Given received vector $[\xi^T \; \eta^T]$ as in (20), we instruct the receiver to add the two halves of the vector leading to

$$[\xi^T + \xi_1^T] = \theta X^T$$

$$\quad + [w_1^T + w_2^T + \sum_{j=2}^n h_j w_j^T A_j ]$$ \hfill (23)

which with

$$X = \begin{bmatrix} f^T A_2 & \vdots & f^T A_n \end{bmatrix}$$ \hfill (24)

can be rewritten in the form

$$[\xi^T + \xi_1^T] = \theta X$$

$$+ [w_1^T + w_2^T + \sum_{j=2}^n h_j w_j^T A_j ].$$ \hfill (25)

Let us denote the noise vector appearing above by

$$n' = w_1 + w_2 + \sum_{j=2}^n h_j A_j^T v_j.$$
The covariance of this noise vector is then given by
\[ E \{ \mathbf{u} \mathbf{u}^\dagger \} = 2 + \sum_{j=2}^{n} |h_j|^2 \mathbf{I}_n := \beta \mathbf{I}_n \text{ (say)}. \]

Let \( \mathbf{u}'' \) denote the normalized noise vector
\[ \mathbf{u}'' := \frac{1}{\sqrt{\beta}} \mathbf{u}. \]

Then \( \mathbf{u}'' \) is white, distributed according to \( \mathcal{CN}(0,1) \). Scaling both sides of (23) by \( \sqrt{\beta} \) we obtain:
\[ \frac{1}{\sqrt{\beta}} \left[ \mathbf{z}^T + \mathbf{u}_n^T \right] = \frac{\theta}{\sqrt{\beta}} \mathbf{z}^T \mathbf{X} + [\mathbf{u}'']^T. \tag{26} \]

Making use of the fact that the collection of code matrices \( \{ \mathbf{X} \in \mathcal{X} \} \) is a family of commuting normal matrices that can be simultaneously diagonalized as \( \mathbf{X} = \mathbf{S}^\dagger \mathbf{X}_d \mathbf{S} \), we have:
\[ \frac{1}{\sqrt{\beta}} \left[ \mathbf{z}^T + \mathbf{u}_n^T \right] = \frac{\theta}{\sqrt{\beta}} \mathbf{z}^T \mathbf{S}^\dagger \mathbf{X}_d \mathbf{S} + [\mathbf{u}'']^T. \]

Post-multiplying by \( \mathbf{S}^\dagger \) yields
\[ \frac{1}{\sqrt{\beta}} \left[ \mathbf{z}^T + \mathbf{u}_n^T \right] \mathbf{S}^\dagger = \frac{\theta}{\sqrt{\beta}} \mathbf{z}^T \mathbf{X}_d + [\mathbf{u}'']^T \mathbf{S}^\dagger. \]

Since this is a unitary transformation, the transformed noise vector \( [\mathbf{u}]^T = [\mathbf{u}'']^T \mathbf{S}^\dagger \) remains white:
\[ y^T = \frac{\theta}{\sqrt{\beta}} \mathbf{z}^T \mathbf{X}_d + \mathbf{u}^T. \tag{27} \]

where we have set
\[ \mathbf{u}^T := \frac{1}{\sqrt{\beta}} \left[ \mathbf{z}^T + \mathbf{u}_n^T \right] \mathbf{S}^\dagger, \quad \text{and} \]
\[ y^T := [q_1, \ldots, q_n] = \mathbf{z}^T \mathbf{S}^\dagger. \]

Equation (27) can now be expressed in the form of an equation for a parallel channel [14]:
\[ y = \frac{\theta}{\sqrt{\beta}} \mathbf{z}^T \mathbf{X}_d \]
\[ + \mathbf{u}. \tag{28} \]

where \( X_{d,i} \) are the diagonal elements of \( \mathbf{X}_d \).

The probability of outage of this parallel channel is given by
\[ \mathbb{P}_\text{out}(\rho) = \mathrm{Pr} \left\{ \prod_{j=1}^{n} \left( 1 + \rho |q_j|^2 \right) < \rho^{2n} \right\} \]
\[ = \mathrm{Pr} \left\{ \prod_{j=1}^{n} (1 + \rho |q_j|^2) < \rho^{2n} \left( 2 + \sum_{i=2}^{n} |h_i|^2 \right) \right\} \]

Since the probability that
\[ |h_i|^2 = \rho', \quad \epsilon > 0 \]

vanishes exponentially fast with \( \rho \), it can be shown that
\[ \mathbb{P}_\text{out}(\rho) \triangleq \mathrm{Pr} \left\{ \prod_{j=1}^{n} (1 + \rho |q_j|^2) < \rho^{2n} \right\}. \tag{29} \]

In Appendix III it is shown that
\[ \mathbb{P}_\text{out}(\rho) \leq \rho^{-n(1-2r)}. \tag{29} \]

A space-time code is said to be approximately universal [14] if it is D-MG optimal for every statistical characterization of the matrix of channel fading coefficients. We will make use of the following sufficient criterion for approximate universality over the parallel channel:

**Theorem 3.2**: [14] Consider a parallel channel given by
\[ \mathbf{y} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \mathbf{u}. \tag{30} \]

where \( q_i \) represent the fading coefficients and where the components of \( \mathbf{u} \) are i.i.d. and \( \mathcal{CN}(0,1) \) distributed. Then a space-time code \( \mathcal{X} \) with code matrices \( \mathbf{X} \)
\[ \mathbf{X} = \begin{bmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \vdots \\ \mathbf{X}_n^T \end{bmatrix} \]

for this parallel channel is approximately universal if the product of the row-norms of every code-difference matrix satisfies
\[ \prod_{j=1}^{n} ||\mathbf{X}_j^T||^2 \geq \rho^{n-r_p}, \]

where \( \rho \) is the SNR and \( r_p \log_2(\rho) \) is the rate of the code over the parallel channel in bpcu.

The columnar version of the diagonal code
\[ \mathcal{X}_d = \{ X_d = \mathbf{S} \mathbf{X}^\dagger \mid X \in \mathcal{X} \} \]

turns out to satisfy this criterion of approximate universality. We regard the matrix
\[ \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{\beta} \\ \vdots \\ 1 \end{bmatrix} \]

as representing the channel matrix of the parallel channel. Each code matrix consists of a single column \( \theta \mathbf{X}_d \) whose entries are the diagonal elements of the matrices \( \theta \mathbf{X}_d \). Let \( \Delta \mathbf{X}_d \), \( \Delta_d \) denote the difference matrix in the diagonal and columnar versions respectively. Then the product of the squares of the row norms of \( \theta \Delta \mathbf{X}_d \) is given by
\[ \prod_{i=1}^{n} ||\theta \Delta \mathbf{X}_d||_i^2 \geq \rho^{2n} \det |\Delta \mathbf{X}_d|^2. \tag{31} \]

The code \( \mathcal{X} \) must be of size \( \rho^{-2n} \) to deliver a rate of \( r \log_2(\rho) \) bpcu over \( 2n \) uses of the MISO channel given by (20). Since
each code matrix represents the information content of \( n \) QAM symbols \((l_1, \cdots, l_n)\), it follows that the size \( M^2 \) of the QAM constellation must satisfy:

\[
[M^2]^n = \rho^{2rn}
\]
i.e., \( M^2 = \rho^{2r} \). Indeed this is the reason why the signaling complexity of the code constructed here equals 2. To ensure that the average energy per symbol equals \( \rho \) we must have

\[
\theta^2 M^2 \leq \rho \Rightarrow \theta^2 \leq \rho^{1-2r}.
\]

Since each code matrix \( X \) is drawn from a CDA-based code, its determinant is bounded from (34) by

\[
|\det(X)| = |\det(X_d)| = \prod_{i=1}^{n} |[X_d]_{ii}| \geq \frac{1}{b} \leq \rho^0.
\]
The same lower bound also applies to the magnitude of the determinant of the difference of two code matrices. It follows that we can lower bound the product of the row norms by

\[
\prod_{i=1}^{n} |[0 \Delta X_d]_{ii}| = \theta^{2n} |\det \Delta X|^2 \geq \rho^{n(1-2r)} = \rho^{n-2rn}.
\]

From Theorem 3.2 with \( r_p = 2rn \), it follows that the diagonal code when transmitting at rate \( 2rn \) is approximately universal for the parallel channel and therefore has probability of error equal to the probability of outage which from (29) is given by

\[
P_e(r) \leq P_{ou}(r) \leq \rho^{-n(1-2r)}.
\]

However, the outage probability of the equivalent space-time channel formed by the cooperating relay network in conjunction with our DSTC is lower bounded from Lemma 2.1, by \( \rho^{-n(1-2r)} \). This not only proves optimality of the DSTC constructed here, it also establishes that the outage probability bound derived in Lemma 2.1 is in fact the true value of outage probability.

**APPENDIX I**

**PROOF OF UPPER BOUND ON D-MG TRADEOFF (LEMMA 2.1)**

Our starting point is (12). We write the covariance matrices appearing in (12) as:

\[
\Sigma_w = \mathbb{E}([w w^\dagger]),
\Sigma_s = \mathbb{E}([s s^\dagger]) \quad \text{with} \quad \text{Tr}(\Sigma_s) = n\rho
\]

and we note that the mutual information

\[
I(y'; \tilde{x}) = \log |I_{2n} + \rho H H^\dagger \Sigma_w^{-1}|
\]
is bounded as [13]

\[
\log |I_{2n} + \rho H H^\dagger \Sigma_w^{-1}| \leq \sup_{\text{Tr}(\Sigma_s) \leq n\rho} I(y'; \tilde{x}) \leq \log |I_{2n} + n\rho H H^\dagger \Sigma_w^{-1}|.
\]

In the high-SNR scale of interest, the two bounds are essentially the same and hence we assume below that

\[
I(y'; \tilde{x}) = \log |I_{2n} + \rho H H^\dagger \Sigma_w^{-1}|
\]

where

\[
HH^\dagger = \begin{bmatrix} g_1 I_n \cr g_1 I_n \end{bmatrix} \begin{bmatrix} g_1 I_n & g_1 B^\dagger \\
B & B B^\dagger \end{bmatrix} = \begin{bmatrix} |g_1|^2 I_n & g_1 B^\dagger \\
g_1^* B & BB^\dagger \end{bmatrix}
\]

and where

\[
H H^\dagger \Sigma_w^{-1} = \begin{bmatrix} |g_1|^2 I_n & g_1 B^\dagger \\
g_1^* B & BB^\dagger \end{bmatrix} \begin{bmatrix} I_n & 0 \\
0 & C^{-1} \end{bmatrix}
\]

where \( C = I_n + \sum_{j=2}^{\infty} |h_j|^2 A_j A_j^\dagger \). Furthermore

\[
I_{2n} + \rho H H^\dagger \Sigma_w^{-1} = \begin{bmatrix} I_n (1 + \rho \gamma_1) & \rho g_1 B^\dagger C^{-1} \\
\rho g_1^* B & I_n + \rho B B^\dagger C^{-1} \end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} I_n (1 + \rho \gamma_1) & \rho g_1 B^\dagger C^{-1} \\
\rho g_1^* B & I_n + \rho B B^\dagger C^{-1} \end{bmatrix}
\]

upon row reduction (by multiplying the first row by \( \rho g_1 B \) and subtracting from the second row). Here we have set \( |g_1|^2 = \gamma_1, \gamma_2 = |g_2 h_2|^2, \ldots, \gamma_j = |g_j h_j|^2, \ldots \).

Let

\[
I = |I_{2n} + \rho H H^\dagger \Sigma_w^{-1}|.
\]

Then

\[
I = |I_n (1 + \rho \gamma_1)| \cdot |I_n + \frac{\rho}{1 + \rho \gamma_1} B B^\dagger C^{-1}|
\]

\[
= (1 + \rho \gamma_1)^n |C| + \frac{\rho}{1 + \rho \gamma_1} B B^\dagger |C|
\]

\[
\leq (1 + \rho \gamma_1)^n \cdot |I_n + \sum_{j=2}^{\infty} |h_j|^2 A_j A_j^\dagger|
\]

\[
\cdot |I_n + \sum_{j=2}^{\infty} |h_j|^2 A_j A_j^\dagger|
\]

where we have used the fact that by the Cauchy-Schwarz inequality,

\[
\sum_{j,k} c_j^* c_k A_j A_k^\dagger \leq \sum_{j} |c_j|^2 A_j A_j^\dagger
\]
as non-negative definite (n.n.d.) matrices and

\[
B \geq A \Rightarrow \log |I + B| \geq \log |I + A|
\]

when \( A, B \) are n.n.d. matrices.

By the Frobenius-norm constraint, have

\[
\sum_{j} |c_j|^2 A_j A_j^\dagger \leq (\sum_{j} |c_j|^2) \
\]

Putting these inequalities together gives us

\[
I \leq (1 + \rho \gamma_1 + \rho \sum_{j=2}^{n} \gamma_j)^n.
\]

We here note that the outage region is defined by the set of all channel realizations under which

\[
\log I < 2rn \log(\rho).
\]

We define \( \{u_j, v_j\} \) by

\[
\gamma_1 = \rho^{-p_1}, \quad \gamma_j = \rho^{-(u_j + v_j)}, \quad 2 \leq j \leq n, \quad p_j = u_j + v_j.
\]
We thus have the equivalent formulation of the outage region

\[ O = \{ \{p_j\} : n \{ \max_{1 \leq j \leq n} (1 - p_j)^t \} < 2rn \}. \]

This means that for \( d_{ou}(r) := \lim_{\rho \to \infty} \log(Pr(\log \mathcal{I})) / \log(\rho) \) then

\[ d_{ou}(r) = \inf \sum_{j=1}^{n} p_j \]

where the infimum is taken over the outage region, i.e., in the region over which

\[ n \min_{j=1}^{n} (1 - p_j)^t > 2rn. \]

Consequently, \( d_{ou}(r) = n(1 - \frac{\sqrt{n}}{\pi}) \) and since we have an upper bound on the mutual information, it is the case that we have an upper bound on the optimal diversity gain. This bound is given by

\[ d_{ou}(r) \leq n(1 - 2r). \]

**APPENDIX II**

**BACKGROUND ON CYCLIC DIVISION ALGEBRAS**

**A. The General CDA**

Division algebras are rings with identity in which every nonzero element has a multiplicative inverse. The center \( F \) of any division algebra \( D \), i.e., the subset comprising of all elements in \( D \) that commute with every element of \( D \), is a field. The division algebra is a vector space over the center \( F \) of dimension \( n^2 \) for some integer \( n \). A field \( L \) such that \( F \subseteq L \subseteq D \) and such that no subfield of \( D \) contains \( L \) is called a maximal subfield of \( D \). Every division algebra is also a vector space over a maximal subfield and the dimension of this vector space is the same for all maximal subfields and equal to \( n \). This common dimension \( n \) is known as the index of the division algebra.

Division algebras in which the center \( F \) and a maximum subfield \( L \) are such that \( L/F \) is a (finite) cyclic (Galois) extension are called Cyclic Division Algebras (CDA). CDAs have a simple characterization that aids in their construction, see [20], Proposition 11 of [21], or Theorem 1 of [19].

Let \( F, L \) be number fields, with \( L \) a finite, cyclic Galois extension of \( F \) of degree \( n \). Let \( \sigma \) denote the generator of the Galois group \( \text{Gal}(L/F) \). Let \( z \) be an indeterminate satisfying

\[ \ell z = z\sigma(\ell) \quad \forall \ell \in L \quad \text{and} \quad z^n = \gamma, \]

for some non-zero element \( \gamma \in F^* \), by which we mean some element \( \gamma \) having the property that the smallest positive integer \( t \) for which \( \gamma^t \) is the relative norm \( N(z, \ell)(u) \) of some element \( u \in L^* \), is \( n \). Then a CDA \( D(L/F, \sigma, \gamma) \) with index \( n \), center \( F \) and maximal subfield \( L \) is the set of all elements of the form

\[ \sum_{i=0}^{n-1} z^i \ell_i, \quad \ell_i \in L. \]  

Moreover it is known that every CDA has this structure. It can be verified that \( D \) is a right vector space (i.e., scalars multiply vectors from the right) over the maximal subfield \( L \).

The matrix corresponding to an element \( d \in D \) corresponds to the left multiplication by the element \( d \) in the division algebra. Let \( \lambda_d \) denote this operation, \( \lambda_d : D \to D \), defined by

\[ \lambda_d(e) = de, \quad \forall e \in D. \]

It can be verified that \( \lambda_d \) is a \( \mathbb{L} \)-linear transformation of \( D \). From (32), a natural choice of basis for the right-vector space \( D \) over \( \mathbb{L} \) is \( \{ 1, z, z^2, \ldots, z^{n-1} \} \). A typical element in the division algebra \( D \) is \( d = \ell_0 + z\ell_1 + \cdots + z^{n-1}\ell_{n-1} \), where the \( \ell_i \in L \). By considering the effect of multiplying \( d \times 1, d \times z, \ldots, d \times z^{n-1} \), one can show that the \( \mathbb{L} \)-linear transformation \( \lambda_d \) under this basis has the matrix representation

\[
\begin{bmatrix}
\ell_0 & \gamma\sigma(\ell_{n-1}) & \gamma\sigma^2(\ell_{n-2}) & \cdots & \gamma\sigma^{n-1}(\ell_1) \\
\ell_1 & \sigma(\ell_0) & \gamma\sigma^2(\ell_{n-2}) & \cdots & \gamma\sigma^{n-1}(\ell_2) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\ell_{n-1} & \sigma(\ell_{n-2}) & \sigma^2(\ell_{n-3}) & \cdots & \sigma^{n-1}(\ell_0)
\end{bmatrix},
\]

known as the left regular representation of \( d \). It is known that despite the fact that the entries of the left-regular representation belong to \( L \), the determinant of every such matrix lies in the subfield \( F \).

**B. CDA Specific to the DSTC Construction**

In the CDA relevant to the construction of the DSTC described in Section III, we choose

\[ F = \mathbb{Q}(i). \]

The field \( \mathbb{L} \) can be taken to be any number field that is a cyclic Galois extension of \( \mathbb{Q}(i) \) of degree \( n \). A general construction, valid for any integer \( n \), for such cyclic extensions can be found in [15]. The elements \( \ell_i \) are chosen to belong to \( A_{QAM} \subseteq \mathbb{Z}[i] \). A non-norm element \( \gamma \in \mathbb{Q}(i) \) having unit magnitude, i.e., \( |\gamma| = 1 \), is chosen. Such a \( \gamma \) can always be found, see [17]. Under these conditions, it can be shown that the determinant \( D(l(z)) \) of the left-regular representation of an element

\[ l(z) = \sum_{i=0}^{n-1} z^i \ell_i, \quad l_i \in A_{QAM} \]

is of the form

\[ D(l(z)) = \frac{a(l(z))}{b}, \quad a(l(z)), b \in \mathbb{Z}[i] \]

where \( b \) is fixed and independent of the specific choice of elements \( \{ l_i \} \). This determinant is moreover, nonzero if any of the \( l_i \) is nonzero. As a result, we obtain the lower bound

\[ |D(l(z))| \geq \frac{1}{|b|}, \]

on the magnitude of the determinant, that applies to all \( l(z) \) provided at least one \( l_i \neq 0 \).

We observe that the above remarks on the determinant also apply to the matrix that is the difference of the left-regular-representation of two distinct elements of the CDA, since the difference matrix is the left-regular representation of the difference element.
APPENDIX III
OUTAGE PROBABILITY OF THE PRODUCT-FADING, PARALLEL CHANNEL

Our goal is to prove that the probability of outage of the parallel channel given by (28) satisfies

\[ P_{z,\text{out}}(r) = \Pr \left\{ \prod_{j=1}^{n} \left( 1 + \rho |q_j|^2 \right) \cdot 2 + \sum_{i=1}^{n} |h_i|^2 \right\} < \rho^{-2n} \]

\[ \leq \rho^{-n(1-2r)}. \]  

(35)

We begin with some preliminaries:

a) Density function of the product of two complex-Gaussian random variables: Let \( z_1 = h_1 \) and \( z_j = g_j h_j, j = 2, \cdots, n \) where \( g_j, h_j \) are the fading coefficients introduced in Section II-B.

We will focus on the joint distribution of the real and imaginary parts of \( z_j \), and for short, will write \( z, h, g \) in place of \( z_j, h_j, g_j \). We have \( h = h_R + ih_I, g = g_R + ig_I, z = z_R + iz_I \), with \( z_R = h_R g_R - h_I g_I, z_I = h_R g_I + h_I g_R \).

The characteristic function \( \phi_z(u) \) is given by:

\[ \phi_{z_R,z_I}(u_R,u_I) = \mathbb{E}_q \{ e^{i(u_R z_R + u_I z_I)} \} = \mathbb{E}_q \{ e^{i(h_R u_R g_R + u_I g_I) + i(h_I u_R g_I + u_I h_I g_R)} \} = \mathbb{E}_q \{ e^{i(z_R u_R g_R + z_I u_I g_I) + i(z_R u_I g_I + z_I u_I h_I g_R)} \} = \mathbb{E}_q \{ e^{i(z_R u_R g_R + z_I u_I g_I) + i(z_R u_I g_I + z_I u_I h_I g_R)} \} = \mathbb{E}_q \{ e^{i(z_R u_R g_R + z_I u_I g_I) + i(z_R u_I g_I + z_I u_I h_I g_R)} \} \]

since

\[ \int e^{-\nu^2/(2\sigma^2)} e^{-\alpha \nu^2} \cdot (\sqrt{2\pi\sigma^2})^{-1} d\nu = \sqrt{2\pi/[(2(\alpha + 1/2\sigma^2)]/(\sqrt{2\pi\sigma^2})} = 1/\sqrt{1 + 2\alpha \sigma^2}. \]

The characteristic function is directly related to the Fourier transform:

\[ \phi_{z_R,z_I}(u_R,u_I) = \mathbb{E}_z \{ e^{i(u_R z_R + u_I z_I)} \} = \int \phi_{z_R,z_I}(z_R,z_I) e^{i(u_R z_R + u_I z_I)} d\mathbb{R} d\mathbb{I} \Rightarrow \phi_{z_R,z_I}(2\pi u_R,2\pi u_I) = \mathcal{F}_{(u_R,u_I)} \{ \phi_{z_R,z_I}(z_R,z_I) \}. \]

The density function can thus be obtained by taking the Fourier inverse:

\[ p_{z_R,z_I}(z_R,z_I) = \int \phi_{z_R,z_I}(2\pi u_R,2\pi u_I) e^{-i(2\pi u_R z_R + 2\pi u_I z_I)} d\mathbb{R} d\mathbb{I} = \int [1 + (2\pi)^2 |u|^2/4]^{-1} e^{-i(2\pi u_R z_R + 2\pi u_I z_I)} d\mathbb{R} d\mathbb{I} = 2\pi \int \frac{1}{1 + (2\pi)^2 |u|^2/4} J_0(2\pi |z||u|) |u||d|u| = 2\pi \int \frac{4}{1 + (2\pi)^2 |u|^2/4} J_0(2\pi |z||u|) |u||d|u| = \frac{4}{(2\pi)^2} 2\pi \left\{ 1 \cdot \frac{1}{s^2} \right\} J_0(2\pi |z||u|) |u||d|u| = \frac{4}{(2\pi)^2} 2\pi K_0(2\pi |z|) \]

where \( J_0(\cdot) \) is the Bessel function of the first kind and \( K_0(\cdot) \) is the modified Bessel function of the second kind, both of order 0 [26]. We have used information about the Hankel transform taken from [24]. By independence of fading coefficients, we have that

\[ p_{z_R,z_I}(z_R,z_I) = \frac{1}{\pi} e^{-\pi |z|^2} \prod_{j=2}^{n} \frac{4}{2\pi} K_0(2|z_j|). \]

b) Change of Variables: Given a complex-valued vector \( \mu \), we define

\[ \hat{\mu} = \left[ \frac{\mu_R}{\mu_I} \right]. \]

Similarly given a complex-valued matrix \( S \), we define

\[ \hat{S} = \left[ \begin{array}{cc} S_R & -S_I \\ S_I & S_R \end{array} \right]. \]

Since \( q^T = \hat{S}^T \hat{S} \Rightarrow z = S^T \hat{q} \) and \( S \) is unitary, it follows that

\[ \hat{z} = \hat{S}^T \hat{q} \]

with \( \hat{S} \) orthogonal and thus

\[ p_{q}(q_R,q_I) = p_Z([S^T q_R, [S^T q_I]] = \frac{1}{\pi} e^{-|q|^2} \prod_{j=2}^{n} \frac{4}{2\pi} K_0(2|S_q^T q_j|), \]

where \( s_i \) denotes the \( i \)th column vector of \( S \).

Next, switching to polar coordinates, we get:

\[ (q_R,q_I) = (r_j \cos(\theta_j), r_j \sin(\theta_j)), \quad 1 \leq j \leq n \quad q_j := r_j e^{i\theta_j}, \quad p_{r,J}(r,\theta) = r_1 \frac{1}{\pi} e^{-r^2/2} \prod_{j=2}^{n} \frac{4}{2\pi} r_j K_0(2|S_q^T q_j|). \]

A final change of variables \((r_j, \theta_j) \rightarrow (\rho^{-a_i/2}, \theta_j)\) (i.e.,\(|q_j|^2 = \rho^{-a_i}\), gives us

\[ p_{\rho,\omega}(\rho,\omega) = (\log \rho)^n \rho^{-a_i} \frac{1}{2\pi} e^{-|\omega|^2} \prod_{j=2}^{n} \frac{2}{2\pi} \rho^{-a_j} K_0(2|S_q^T q_j|). \]

(36)

c) Outage probability: The parallel channel characterized by

\[ \mathbb{Y} = \left[ \begin{array}{cccc} q_1 & q_2 & \cdots & q_n \\ x_1 & x_2 & \cdots & x_n \end{array} \right] + \mathbb{W} \]

has outage region given by

\[ \mathcal{O} = \{(\alpha, \theta) : \prod_{j=1}^{n} (1 + \rho^{1-a_j}) < \rho^{r_p}\}, \]

(37)

where \( r_p \log(\rho) \) is the desired rate in bits per channel use over the parallel channel and a corresponding probability of outage.
given by
\[ P_{\text{out},2}(r_p) = \int_{\mathcal{O}} p_{\omega,\phi}(\omega, \phi) \, d\omega \, d\phi = \rho^{-d_{\text{out},2}(r_p)} \]  
(38)
where \( p_{\omega,\phi}(\omega, \phi) \) is as in (36).

We first show that in the above integral, it is sufficient to consider the region where \( \alpha_i \geq 0, \forall i \). Recall that \( |z_i|^2 = \rho^{-p_0} \) and \( p_0 = \alpha_0 \). Let \( \min \{ p_i \} = p_0, \min \{ \alpha_i \} = \alpha_0 \) and assume \( \alpha_0 < 0 \). Since
\[ \sum_{i=1}^{n} |\eta_i|^2 = \sum_{i=1}^{n} |z_i|^2 \]

it follows that for large \( \rho \), \( p_0 = \alpha_0 \). Consider the term \( K_0(2|z_0^T q_0|) = K_0(2|z_{i0}|) \) where \( z_{i0} = \rho^{-p_0}, \ p_0 > 0 \). For large values of the argument \( |z_{i0}| = x \), we have the bound \( |\sqrt{x}e^x K_0(x) - P(1/x)| < \epsilon \) where \( \epsilon \) is a constant much smaller than 1, and \( P(1) \) is a polynomial of degree greater than 2 (see [25, Section 6.6]). This implies that \( |K_0(x) - x^{-1/2} e^{-x} P(1/x)| < e^{-x/2} e^{-x} \). It follows from the presence of the exponential term \( e^{-x} \) and an application of Varadhan’s Lemma, that for the purposes of determining \( d_{\text{out},2}(r) \) it suffices to restrict attention to the region \( \alpha_i \geq 0 \).

Accordingly, we define the effective outage region
\[ \mathcal{O}' = \mathcal{O} \cap \{ (\omega, \phi) : \alpha_i \geq 0, \forall i \} \]  
(39)
For any \( (\omega, \phi) \), we have \( e^{-|\omega|^2 \rho^2} \leq 1 \approx \rho^0 \). We will now proceed to establish that
\[ K_0(2|z_0^T q_0|) \leq \rho^0 \quad \text{all } (\omega, \phi) \in \mathcal{O}' \]  
(40)
We list below two properties of \( K_0(\cdot) \) that will prove useful:

1) \( \lim_{x \to 0} K_0(x) = 1 \).
2) \( K_0(x) \) is monotonically decreasing with increasing \( x, x \geq 0 \).

Let \( \epsilon > 0 \) and define
\[ \mathcal{O}'' = \mathcal{O}' \cap \{ (\omega, \phi) : 2|\omega^T q_0| \leq S_{\text{max}} \rho^{-\epsilon} \} \]
where \( S_{\text{max}} = \max \{|S_{ij}| : 1 \leq i, j \leq n\} \). Then for \( (\omega, \phi) \in \mathcal{O}'' \), we can make the approximation
\[ K_0(2|z_0^T q_0|) \approx \ln\left( \frac{2}{2|\omega^T q_0|} \right) = \rho^0 \].

For \( (\omega, \phi) \in \mathcal{O}' \setminus \mathcal{O}'' \), we have that
\[ 2|\omega^T q_0| > S_{\text{max}} \rho^{-\epsilon} \]
and it follows from the monotonically decreasing property of \( K_0(x) \) that \( K_0(2|z_0^T q_0|) \leq \ln\left( \frac{2}{2|\omega^T q_0|} \right) = \rho^0 \). Substituting (36), (37) and (40) into (38), we conclude that for the parallel channel defined by diagonal matrix \( \text{diag}(q_1, q_2, \ldots, q_n) \),
\[ P_{\text{out},2}(r_p) \leq \rho^{-d_{\text{out},2}(r_p)} \]
so that
\[ P_{\text{out},2}(r_p) \leq \rho^{-(n-r_p)} \].
Equation (35) now follows by inspecting (35) and (37) and consequently setting \( r_p = 2rn \).

**APPENDIX IV**

**PROOF OF SIMULTANEOUS DIAGONALIZABILITY**

Our goal here is to show that the matrices \( X \) belonging to the CDA-based code \( X \) (see (15)) are simultaneously diagonalizable, i.e., can be diagonalized by a single unitary matrix \( S \). We will do so by explicitly constructing this matrix.

Let \( \zeta_n \) be the complex, primitive \( n \)th root of unity given by \( \zeta_n = e^{2\pi i/n} \). Let the complex number \( \phi \) be given by \( \phi = \zeta_n \gamma^{1/n} \). Then \( \phi \) is a \( n \)th root of \( \gamma \), that is \( \phi^n = \gamma \). In the field of complex numbers, the equation \( x^n = \gamma \) has the \( n \) solutions \( \zeta_n \gamma^{1/n}, i = 1, \ldots, n \). For \( i = 1, 2, \ldots, n \), let \( \sigma_i \) denote the mapping given by:
\[ \sigma_i(\phi) = \zeta_n^{i} \gamma^{1/n} \]
\[ \sigma_i(\phi^k) = \zeta_n^{ki} \gamma^{k/n} \]
Note that \( \sigma_i(\phi^k) = [\sigma_i(\phi)]^k \) and recall that a matrix \( X \) in the code \( X \) is of the form
\[ X = \begin{bmatrix} l_0 & \gamma_{l_{n-1}} & \cdots & \gamma_{l_1} \\ l_1 & l_0 & \gamma_{l_2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n-1} & l_{n-2} & \cdots & l_0 \end{bmatrix} , \ l_i \in \mathcal{A}_\text{OM} \]

**Proposition 1:** The \( n \) vectors
\[ [1, \sigma_i(\phi), \sigma_i(\phi^2), \ldots, \sigma_i(\phi^{n-1})], \ i = 1, \ldots, n \]
are eigenvectors of each matrix \( X \in X \).

**Proof:** Let \( x = l_0 + l_1 \phi + l_2 \phi^2 + \ldots + l_{n-1} \phi^{n-1} \). First notice that since \( \phi^n = \gamma \),
\[ [1, \phi, \phi^2, \ldots, \phi^{n-1}] X = x[1, \phi, \phi^2, \ldots, \phi^{n-1}] \]
so that \( [1, \phi, \phi^2, \ldots, \phi^{n-1}] \) is an eigenvector of \( F \) associated to the eigenvalue \( x \). Similarly, since \( [\sigma_i(\phi)]^n = \gamma \), we have
\[ [1, \sigma_i(\phi), \sigma_i(\phi^2), \ldots, \sigma_i(\phi^{n-1})] X = \sigma_i(x)[1, \sigma_i(\phi), \sigma_i(\phi^2), \ldots, \sigma_i(\phi^{n-1})] \]
where \( \sigma_i(x) = l_0 + l_1 \sigma_i(\phi) + l_2 \sigma_i(\phi^2) + \ldots + l_{n-1} \sigma_i(\phi^{n-1}) \), \( i = 1, \ldots, n \). 

Let \( S \) be the matrix whose rows are scaled versions of these \( n \) eigenvectors as shown below:
\[ S = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & \sigma_1(\phi) & \sigma_1(\phi^2) & \cdots & \sigma_1(\phi^{n-1}) \\ 1 & \sigma_2(\phi) & \sigma_2(\phi^2) & \cdots & \sigma_2(\phi^{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_n(\phi) & \sigma_n(\phi^2) & \cdots & \sigma_n(\phi^{n-1}) \end{bmatrix} \]
It follows that \( SX = X \sigma_d S \) where \( \sigma_d \) is the diagonal matrix \( X_d = \text{diag}(x[\sigma_i(\phi), \ldots, \sigma_i(n-1)](x)) \). We now show that \( S \) is a unitary matrix, from which it follows that we can write
\[ X = S^T X_d S \]  
(41)
and $S$ is then the common unitary diagonalizing matrix for the matrices in $X$ we are looking for.

**Lemma 4.1:** The matrix $S$ is unitary, that is $S^\dagger S = SS^\dagger = I_n$.

**Proof:** The element in the $i$th row and $j$ column of $S$, $1 \leq i, j \leq n$, is given by

$$
\sigma_i(\phi^{-1}) = \sigma_i(\zeta_n^{j-1} \gamma^{(j-1)/n}) = \zeta_n^{(j-1) \gamma^{(j-1)/n}}.
$$

As a result it follows that we can decompose $S$ according to

$$
\begin{pmatrix}
1 & \zeta_n & \cdots & \zeta_n^{n-1} \\
1 & \zeta_n^2 & \cdots & (\zeta_n^n)^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \zeta_n^{n-1} & \cdots & (\zeta_n^n)^{-(n-1)} \\
1 & 1 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
\gamma^{1/n} \\
\vdots \\
\gamma^{(n-1)/n}
\end{pmatrix}
$$

The first matrix on the right is a row-permuted version of the familiar unitary Fourier-transform matrix and the second is also unitary since $|\gamma| = 1$. It follows that their product $S$ is unitary as well.

**References**


