## IEEE ISAC 2026 Tutorial

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# Next-Generation Chirp-based Waveforms for ISAC in 6G and Beyond:

*Fundamentals, Models, and Algorithms (with Live Demonstrations!)*




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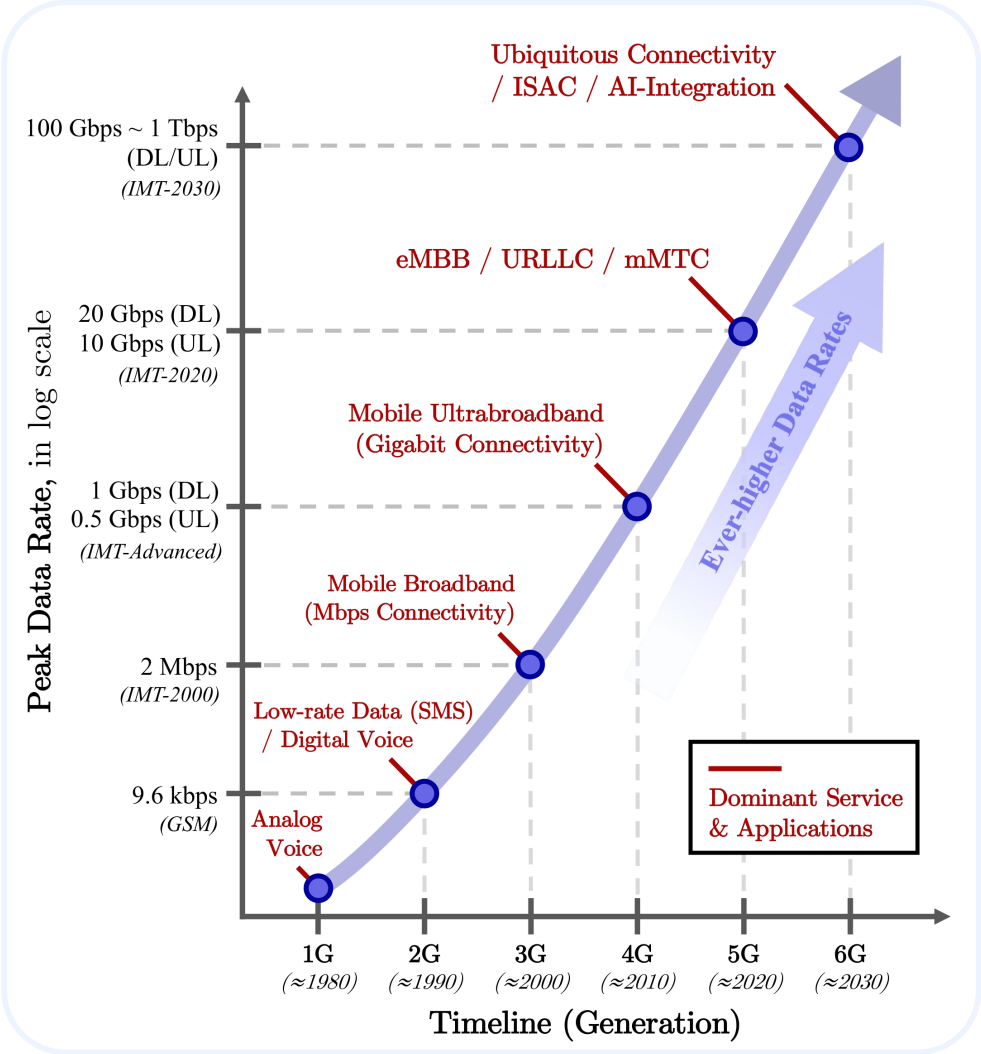
**Prof. Giuseppe T. F. de Abreu**
Constructor University, Germany



**Prof. Marios Kountouris**
University of Granada, Spain / EURECOM, France

# Why Consider New Waveforms for 6G and Beyond?

Wireless evolution is *not* just about exponentially increasing **data rates**.




- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.

# Why Consider New Waveforms for 6G and Beyond?

Wireless evolution is *not* just about exponentially increasing **data rates**.

- There also exist the underlying robustness axis:

**2G: *Robustness* against transmission error**

**3G: *Robustness* against interference**

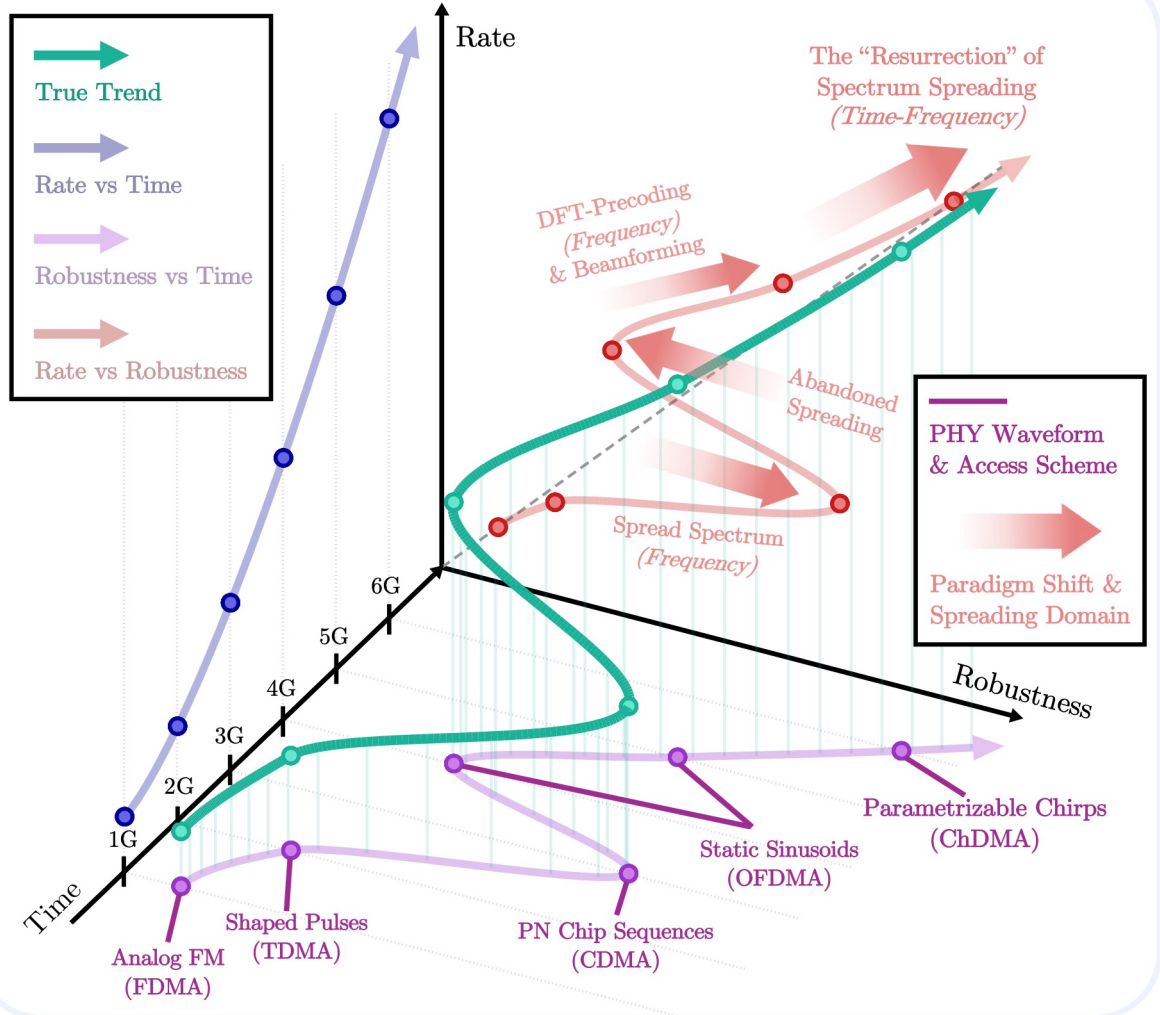
**4G: *Extreme rate* via multiplexing**

**5G: Regain *robustness* via MIMO/precoding**

**6G and Beyond: ???**

**Need for Doppler robustness!**

- 6G **high mobility** use cases: V2X, NTN, etc.
- **ISAC** pillar requirements
- Higher frequency bands (FR3+, mmWave, subTHz)




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# Analyzing Signals in Time-Frequency

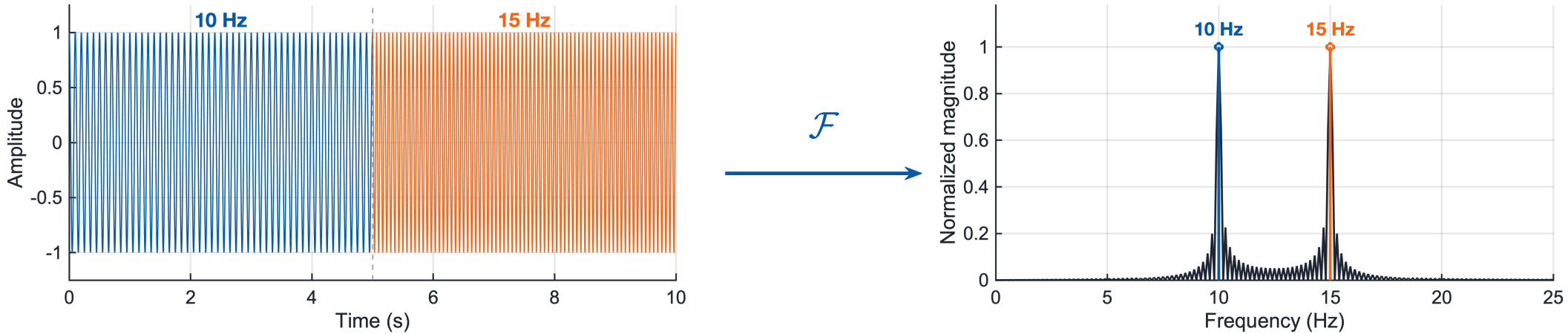


To understand the limitations of sinusoids and OFDM, we first dive into the **Fourier Transform** to see what it reveals and what it hides.

A signal can be represented in either the time domain or the frequency domain

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

The Fourier Transform reveals which frequencies are present, but not directly when they occur.




What if, instead of Fourier-transforming the entire signal, we analyzed only a **small portion** at a time?

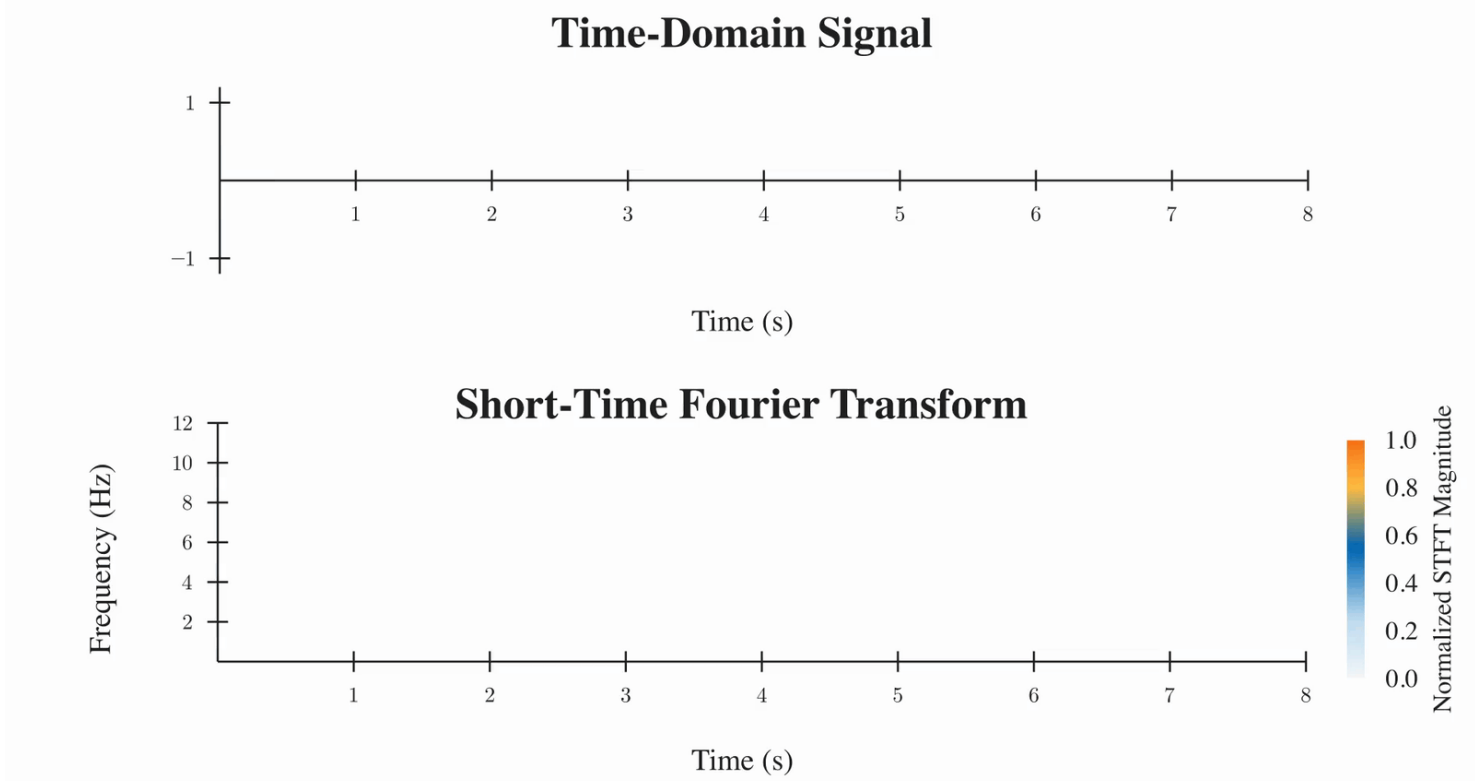
# Short-Time Fourier Transform (STFT)



The STFT analyzes a signal locally in time by applying a sliding window before computing its Fourier Transform.

$$X_{\mathrm{STFT}}(t_0, f) = \int_{-\infty}^{\infty} x(\tau) w^*(\tau - t_0) e^{-j2\pi f \tau}\, d\tau$$

$$x(t) = \begin{cases} \cos(2\pi \cdot 3t), & 0 \le t < 4 \text{ s}, \\ \cos(2\pi \cdot 7t), & 4 \le t \le 8 \text{ s}. \end{cases}$$



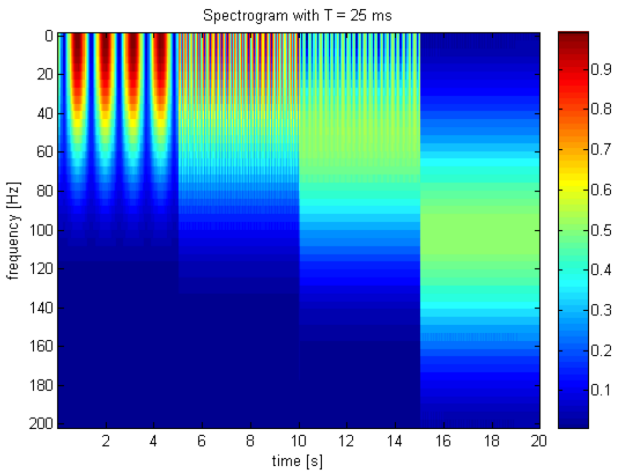
# The Time-Frequency Resolution Trade-off

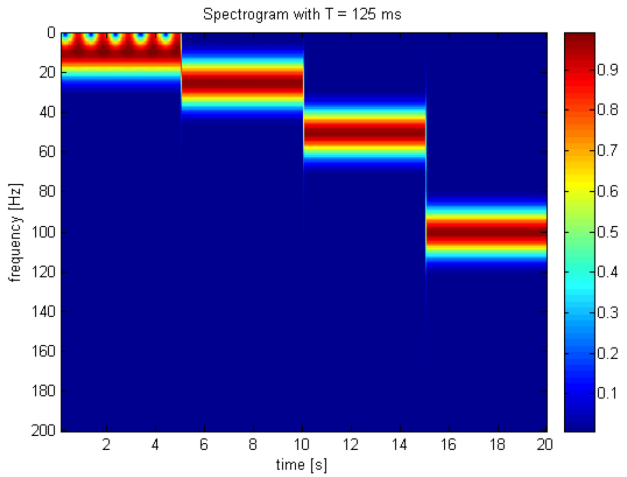
A signal cannot be localized arbitrarily precisely in both time and frequency. The **Fourier uncertainty principle** establishes a fundamental lower bound:

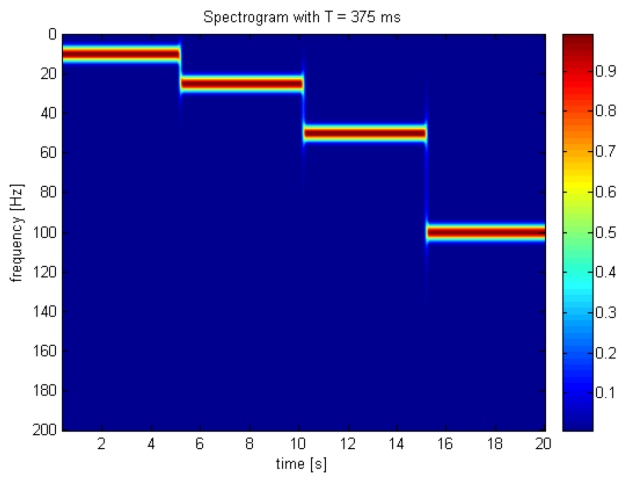
$$x(t) = \begin{cases} \cos(2\pi 10t), & 0s \leq t < 5s, \\ \cos(2\pi 25t), & 5s \leq t < 10s, \\ \cos(2\pi 55t), & 10s \leq t < 15s, \\ \cos(2\pi 100t), & 15s \leq t \leq 20s \end{cases}$$

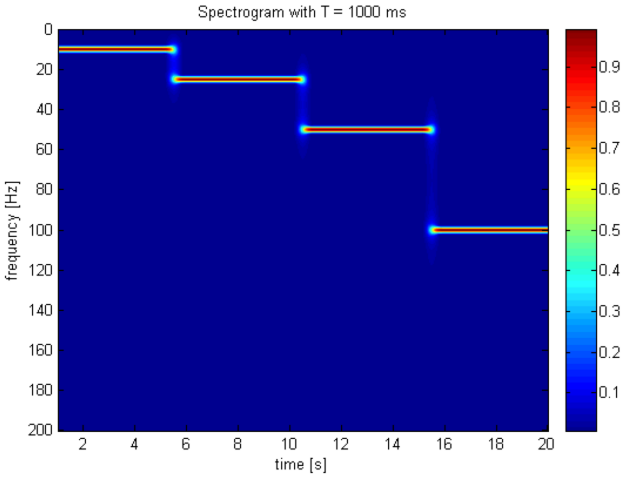
$$\boxed{\sigma_t \sigma_f \geq \frac{1}{4\pi}}$$

Where $\sigma_t$ and $\sigma_f$ quantify temporal and spectral spread respectively



# The Wigner-Ville Distribution (WVD)

Now a natural question is whether we can represent a signal's time–frequency structure without introducing an analysis window?

$$W_x(t_0, f) = \int_{-\infty}^{\infty} x\left(t_0 + \frac{\tau}{2}\right) x^*\left(t_0 - \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau$$

Why is it called a **distribution**?

$$\int_{-\infty}^{\infty} W_x(t, f)\, df = |x(t)|^2$$

$$\int_{-\infty}^{\infty} W_x(t, f)\, dt = |X(f)|^2$$

# The Wigner-Ville distribution of a Linear Chirp

Consider a linear chirp whose instantaneous frequency varies linearly with time. How does its energy appear in the time–frequency plane?

$$x(t) = e^{j\pi k t^2}$$

$\mathrm{Hz/s}$

$$W_x(t,f) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau$$

$$f_{\text{inst}}(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\pi k t^2\right)$$

$$= \boxed{kt}$$

# The Wigner-Ville distribution of a Linear Chirp

Consider a linear chirp whose instantaneous frequency varies linearly with time. How does its energy appear in the time–frequency plane?

$$x(t) = e^{j\pi k t^2}$$

$$W_x(t,f) = \int_{-\infty}^{\infty} e^{j\pi k(t+\tau/2)^2} e^{-j\pi k(t-\tau/2)^2} e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{j2\pi(kt-f)\tau}\, d\tau$$

# The Wigner-Ville distribution of a Linear Chirp

Consider a linear chirp whose instantaneous frequency varies linearly with time. How does its energy appear in the time–frequency plane?

$$x(t) = e^{j\pi k t^2}$$

$$\boxed{W_x(t, f) = \delta(f - kt)}$$

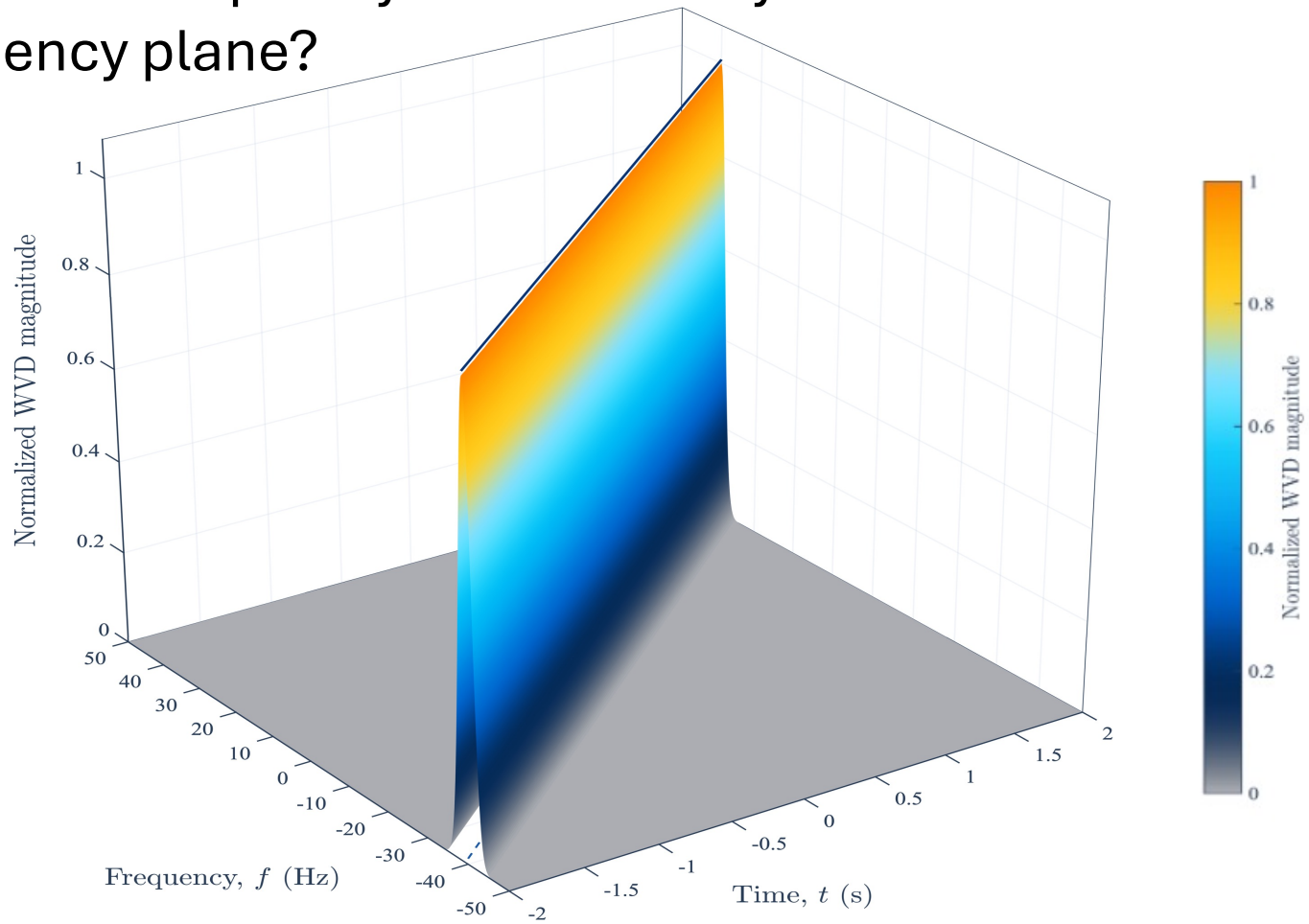
# The Wigner-Ville distribution of a Linear Chirp

Consider a linear chirp whose instantaneous frequency varies linearly with time. How does its energy appear in the time–frequency plane?

$$x(t) = e^{j\pi k t^2}$$

$$f_{\text{inst}}(t) = kt$$

$$\boxed{W_x(t, f) = \delta(f - kt)}$$




The Wigner–Ville distribution of an ideal linear chirp is perfectly concentrated along its instantaneous-frequency trajectory, f=kt. The finite ridge width shown here is introduced solely for visualization.

# From Chirps to Rotations in the Time-Frequency plane

The Wigner–Ville distribution reveals that a linear chirp occupies a tilted line in the time–frequency plane. Can we **rotate** our **coordinate system** so that this energy becomes concentrated along a single frequency component?

Rotation matrix

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\text{Output basis}} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \underbrace{\begin{bmatrix} t \\ f \end{bmatrix}}_{\text{Initial t,f basis}}$$

$$f = kt$$

# From Chirps to Rotations in the Time-Frequency plane

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$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix}$$

$$v = -t\sin\alpha + f\cos\alpha$$

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$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix}$$

$$v = t(-\sin\alpha + k\cos\alpha)$$

What happens if we choose $\tan\alpha = k$?

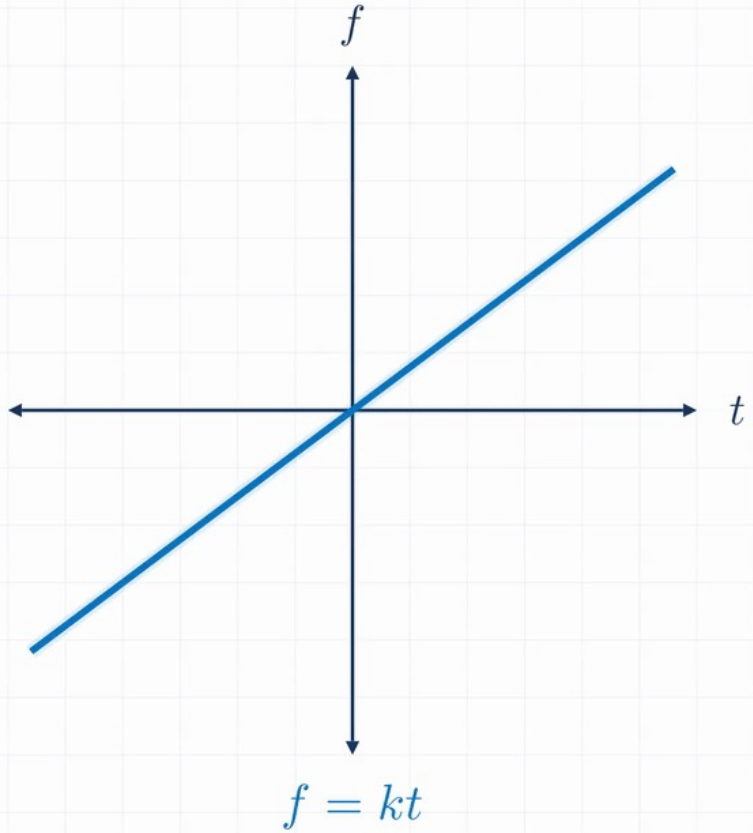
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$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix}$$

$$-\sin\alpha + k\cos\alpha = 0,$$

What happens if we choose $\tan\alpha = k$?



# The Fractional Fourier Transform: Beyond a 90°Rotation

The Fourier transform connects the time and frequency domains. The Fractional Fourier Transform **generalizes** this idea to **arbitrary rotation angles** in the time–frequency plane.

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(u,t)\, dt$$

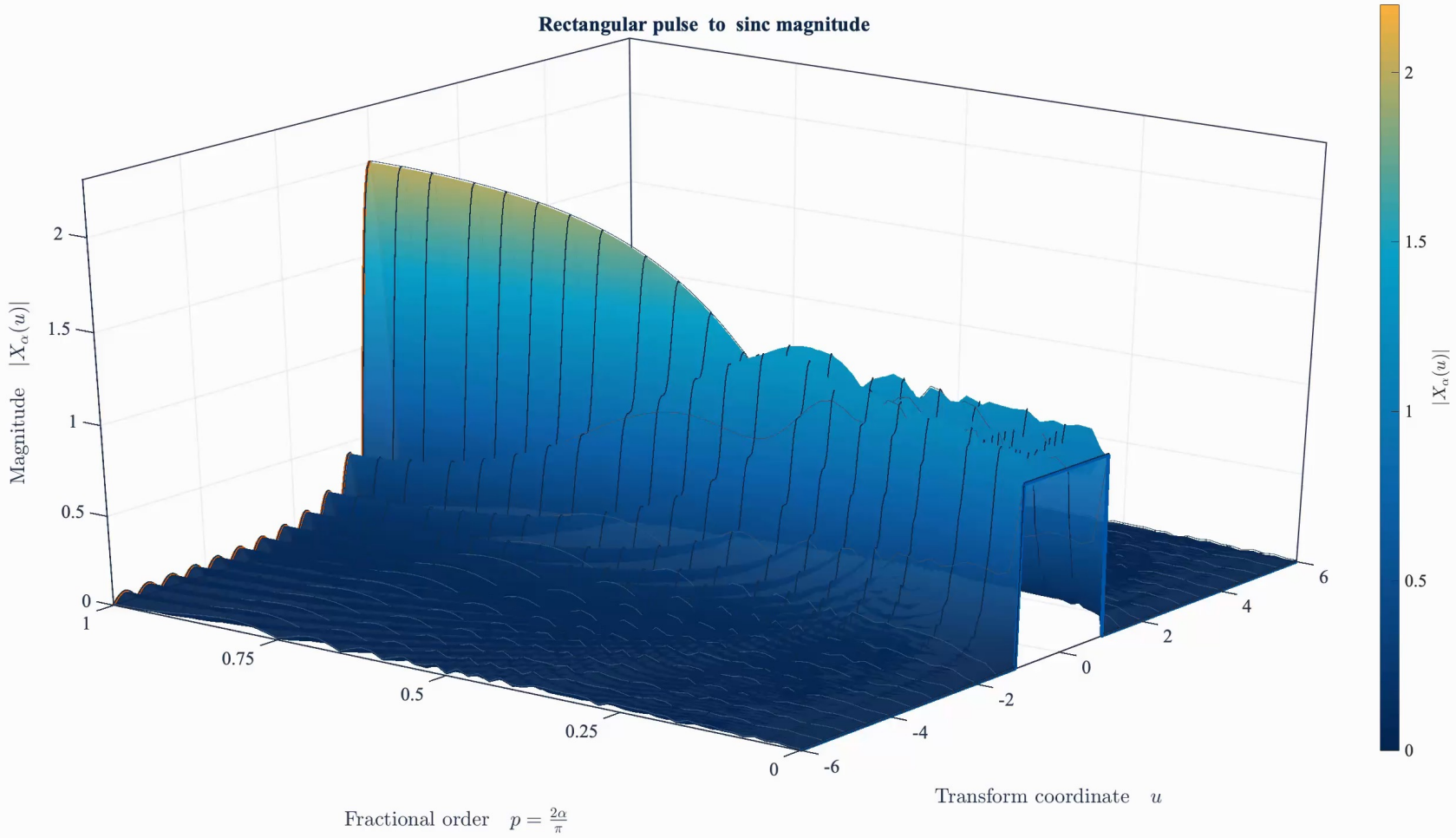
$$K_\alpha(u,t) = A_\alpha \exp\left[j\pi\left((u^2+t^2)\cot\alpha - 2ut\csc\alpha\right)\right]$$

$$A_\alpha = \sqrt{1 - j\cot\alpha}$$

Order

$$\alpha = \frac{a\pi}{2}$$

# Visualizing the FrFT: From Rectangular to Sinc



# Decomposing the FrFT Rotation

$$K_\alpha(u,t) = A_\alpha \exp\left[j\pi\left((u^2+t^2)\cot\alpha - 2ut\csc\alpha\right)\right]$$

$$e^{j\pi(a+b+c)} = e^{j\pi a}e^{j\pi b}e^{j\pi c}.$$

# Decomposing the FrFT Rotation

$$K_\alpha(u,t) = A_\alpha \underbrace{e^{j\pi u^2 \cot\alpha}}_{\text{Output chirp}} \underbrace{e^{-j2\pi ut \csc\alpha}}_{\text{Scaled Fourier kernel}} \underbrace{e^{j\pi t^2 \cot\alpha}}_{\text{Input chirp}}$$

$$e^{j\pi(a+b+c)} = e^{j\pi a} e^{j\pi b} e^{j\pi c}.$$

Remembering what multiplying with a chirp did to the time frequency plane in our WVD example, multiplying with a chirp corresponds to a **shear**, and a sheering matrix along the y axis is given by:

$$S_f(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \qquad x(t)e^{j\pi kt^2} \iff f' = f + kt$$

Here $k = \cot\alpha$

# Decomposing the FrFT Rotation

$$K_\alpha(u,t) = A_\alpha \underbrace{e^{j\pi u^2 \cot\alpha}}_{\text{Output chirp}} \underbrace{e^{-j2\pi ut \csc\alpha}}_{\text{Scaled Fourier kernel}} \underbrace{e^{j\pi t^2 \cot\alpha}}_{\text{Input chirp}}$$

$$e^{j\pi(a+b+c)} = e^{j\pi a} e^{j\pi b} e^{j\pi c}.$$

The Fourier Transform corresponds to a 90° rotation of the time-frequency plane, which corresponds to the following transformation matrix:

$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \mathcal{F}\{x(t)\}(u) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ut}\, dt$$

# Decomposing the FrFT Rotation

$$K_\alpha(u,t) = A_\alpha \underbrace{e^{j\pi u^2 \cot\alpha}}_{\text{Output chirp}} \underbrace{e^{-j2\pi ut \csc\alpha}}_{\text{Scaled Fourier kernel}} \underbrace{e^{j\pi t^2 \cot\alpha}}_{\text{Input chirp}}$$

$$e^{j\pi(a+b+c)} = e^{j\pi a} e^{j\pi b} e^{j\pi c}.$$

A canonical scaling stretches one time–frequency axis while compressing the other by the reciprocal factor, preserving phase-space area:

$$D(s) = \begin{bmatrix} \overset{s = \sin\alpha}{s} & 0 \\ 0 & 1/s \end{bmatrix} \qquad \det D(s) = s\frac{1}{s} = 1$$

# Decomposing the FrFT Rotation

$$K_\alpha(u,t) = A_\alpha \underbrace{e^{j\pi u^2 \cot\alpha}}_{\text{Output chirp}} \underbrace{e^{-j2\pi ut \csc\alpha}}_{\text{Scaled Fourier kernel}} \underbrace{e^{j\pi t^2 \cot\alpha}}_{\text{Input chirp}}$$

$$e^{j\pi(a+b+c)} = e^{j\pi a} e^{j\pi b} e^{j\pi c}.$$

So the whole FrFT transformation matrix boils down to a product of elementary transformations of the time frequency plane:

$$R_\alpha = S_f(\cot\alpha)\, D(\sin\alpha)\, F\, S_f(\cot\alpha)$$

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cot\alpha & 1 \end{bmatrix} \begin{bmatrix} \sin\alpha & 0 \\ 0 & \csc\alpha \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \cot\alpha & 1 \end{bmatrix}$$

*Valid for $\sin\alpha \neq 0$ ; singular cases are defined by continuity.*

# Beyond Rotations: The Linear Canonical Transform (LCT)

Recall the FrFT rotation matrix: $R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

The generalized Linear Canonical Transform matrix is give by the following transformation matrix

$$\boxed{M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} t \\ f \end{bmatrix}$$

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$$u = at + bf,$$

$$v = ct + df.$$

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Recall the FrFT rotation matrix: $R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

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Note that not every arbitrary matrix corresponds to an LCT, it also satisfy the **symplectic condition** in two dimentions

$$\boxed{\det M = ad - bc = 1}$$

# The Linear Canonical Transform Kernel

Our LCT is a 4 parameter transform with one constraint

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1$$

For $b \neq 0$,

$$\boxed{X_M(u) = \int_{-\infty}^{\infty} x(t) K_M(u,t)\, dt}$$

$$\boxed{K_M(u,t) = \frac{1}{\sqrt{jb}} \exp\left[ j\pi \left( \frac{a}{b} t^2 - \frac{2}{b} ut + \frac{d}{b} u^2 \right) \right]}$$

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$\cos\alpha$ $\cos\alpha$ $\sin\alpha$

# The Linear Canonical Transform Kernel

The LCT is characterized by four matrix entries subject to one constraint, leaving three independent parameters

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1$$

For $b \neq 0$,

$$\boxed{X_M(u) = \int_{-\infty}^{\infty} x(t) K_M(u,t)\, dt}$$

$$K_\alpha(u,t) = A_\alpha \exp\left[j\pi \left(t^2 \cot\alpha - 2ut \csc\alpha + u^2 \cot\alpha\right)\right]$$

$$\boxed{K_M(u,t)\big|_{M=R_\alpha} = K_\alpha(u,t)}$$

# The Linear Canonical Transform Kernel

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Shearing Shearing

$$K_M(u,t) = \frac{1}{\sqrt{jb}} \exp\left[ j\pi \left( \frac{a}{b} t^2 - \frac{2}{b} ut + \frac{d}{b} u^2 \right) \right]$$

Scaled FT

# Familiar Transforms as Special Cases of the LCT

Fourier Transform:
90° rotation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \int_{-\infty}^{\infty} x(t) e^{-j2\pi ut}\, dt$$

Fractional Fourier Transform:
Arbitary rotation of time frequency plane

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \qquad \frac{1}{\sqrt{j\sin\alpha}} \int_{-\infty}^{\infty} x(t) e^{j\pi\left[\cot\alpha(t^2+u^2)-2ut\csc\alpha\right]}\, dt$$

Chirp multiplication:
Frequency shear

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \qquad e^{j\pi k u^2} x(u)$$

Scaling:
Reciprocal Scaling

$$\begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix} \qquad \frac{1}{\sqrt{s}} x\left(\frac{u}{s}\right)$$

# Familiar Transforms as Special Cases of the LCT

Fresnel transform:
Time Shear

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad \frac{1}{\sqrt{jb}} \int_{-\infty}^{\infty} x(t) e^{j\pi(u-t)^2/b}\, dt$$

Identity:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad x(u)$$

All these transformations satisfy the canonical condition $\det M = 1$

# Fundamental Properties of the LCT

- Unitarity:

$$\int_{-\infty}^{\infty} |X_M(u)|^2 \, du = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

The total signal energy is preserved.

- Composition:

$$\mathcal{L}_{M_2} \mathcal{L}_{M_1} = \pm \mathcal{L}_{M_2 M_1}$$

Consecutive LCTs correspond to matrix multiplication

- Inverse:

$$M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Every LCT can be reversed by its inverse canonical matrix

# Fundamental Properties of the LCT

- Phase-Space action: $$W_{X_M}(u,v) = W_x(du - bv, -cu + av)$$

$$W_{X_M}\left(M \begin{bmatrix} t \\ f \end{bmatrix}\right) = W_x(t,f)$$

# From Continuous LCT to the DAFT

Starting with out continuous LCT: $$X_M(u) = \frac{1}{\sqrt{jb}} \int_{-\infty}^{\infty} x(t) \exp\left[ j\pi \left( \frac{a}{b} t^2 - \frac{2}{b} ut + \frac{d}{b} u^2 \right) \right] dt$$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1, \qquad b \neq 0$$

A digital communications system operates on N samples.

$$\boxed{t = n\Delta t, \qquad u = m\Delta u} \qquad n, m \in \{0, 1, \ldots, N-1\}$$

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# From Continuous LCT to the DAFT

$$\frac{a}{b}t^2 - \frac{2}{b}ut + \frac{d}{b}u^2 = \frac{a\Delta t^2}{b}n^2 - \frac{2\Delta t\Delta u}{b}nm + \frac{d\Delta u^2}{b}m^2$$

$$K_M(m,n) \propto \exp\left[j\pi\left(\frac{a\Delta t^2}{b}n^2 - \frac{2\Delta t\Delta u}{b}nm + \frac{d\Delta u^2}{b}m^2\right)\right]$$

# From Continuous LCT to the DAFT

$$\frac{a}{b}t^2 - \frac{2}{b}ut + \frac{d}{b}u^2 = \frac{a\Delta t^2}{b}n^2 - \frac{2\Delta t\Delta u}{b}nm + \frac{d\Delta u^2}{b}m^2$$

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Imposing $\Delta t\Delta u = \frac{b}{N}$

$$e^{\left(-j2\pi\frac{\Delta t\Delta u}{b}nm\right)}$$

Comparing with the DFT Kernel

$$e^{\left(-j2\pi\frac{nm}{N}\right)}$$

# From Continuous LCT to the DAFT

$$\frac{a}{b}t^2 - \frac{2}{b}ut + \frac{d}{b}u^2 = \frac{a\Delta t^2}{b}n^2 - \frac{2\Delta t\Delta u}{b}nm + \frac{d\Delta u^2}{b}m^2$$

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$$e^{(-j2\pi\frac{nm}{N})}$$

# From Continuous LCT to the DAFT

$$\frac{a}{b}t^2 - \frac{2}{b}ut + \frac{d}{b}u^2 = \frac{a\Delta t^2}{b}n^2 - \frac{2\Delta t\Delta u}{b}nm + \frac{d\Delta u^2}{b}m^2$$

$$K_M(m,n) \propto \exp\left[j\pi\frac{a\Delta t^2}{b}n^2\right]\exp\left[-j2\pi\frac{nm}{N}\right]\exp\left[j\pi\frac{d\Delta u^2}{b}m^2\right]$$

$$e^{(-j2\pi\frac{nm}{N})}$$

# From Continuous LCT to the DAFT

$$X_M(u) = \int_{-\infty}^{\infty} x(t) K_M(u,t)\, dt$$

In a discrete setting we work with N samples: $n, m \in \{0, 1, \ldots, N-1\}$.

We construct a unitary discrete transform using the sampled quadratic phase

$$X_M[m] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \exp\left[ j\pi \left( \frac{a\Delta t^2}{b} n^2 - \frac{2nm}{N} + \frac{d\Delta u^2}{b} m^2 \right) \right]$$

# Matrix Representation of the Discrete Transform

$$X_M[m] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \exp\left[ j\pi \left( \frac{a\Delta t^2}{b} n^2 - \frac{2nm}{N} + \frac{d\Delta u^2}{b} m^2 \right) \right]$$

$$\Lambda_{\text{in}} = \text{diag}\left( e^{j\pi \frac{a\Delta t^2}{b} n^2} \right)_{n=0}^{N-1}$$

$$\Lambda_{\text{out}} = \text{diag}\left( e^{j\pi \frac{d\Delta u^2}{b} m^2} \right)_{m=0}^{N-1}$$

$$F[m,n] = \frac{1}{\sqrt{N}} e^{-j2\pi mn/N}$$

$$A_M = \Lambda_{\text{out}} F \Lambda_{\text{in}}$$

$$\mathbf{X}_M = A_M \mathbf{x}$$

$$A_M^{\text{H}} A_M = I$$

$$\mathbf{x} = A_M^{\text{H}} \mathbf{X}_M$$

# Constructing a Waveform from the AFT



**After the coffee break:**

- How is AFT used to multiplex symbols unto a new waveform?
- What do the AFT-based subcarriers imply?
- How can we model the IO relation and analyze the signal?
- What properties does it have?
- How can sensing (ISAC) be achieved and how is the new process beneficial?
- What other applications can the new waveform be used for?

…

- So can you actually realize the waveform?

# The OFDM Legacy

For OFDM, *N* information symbols are multiplexed into the time-domain signal via:

- ***N*-point Inverse Discrete Fourier Transform (IDFT)**

*Time domain samples*

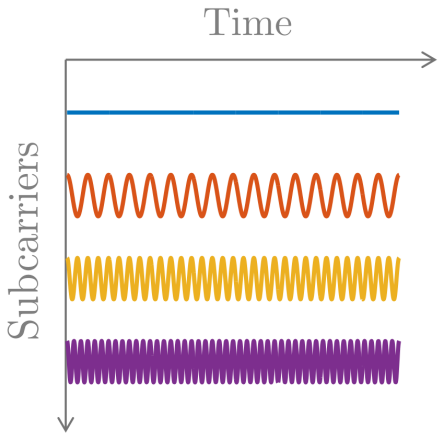
$$\mathbf{s} \triangleq [s_0, s_1, \ldots, s_{N-1}]^{\mathsf{T}}$$

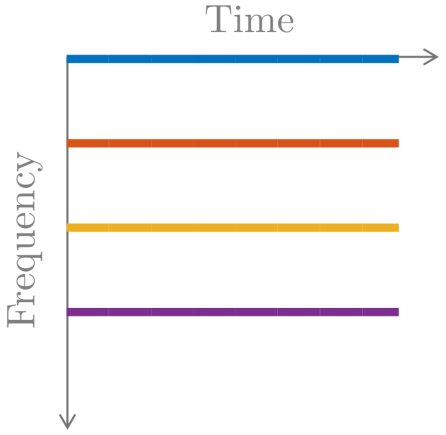
$$s_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{j2\pi\left(\frac{nm}{N}\right)} \cdot x_m$$

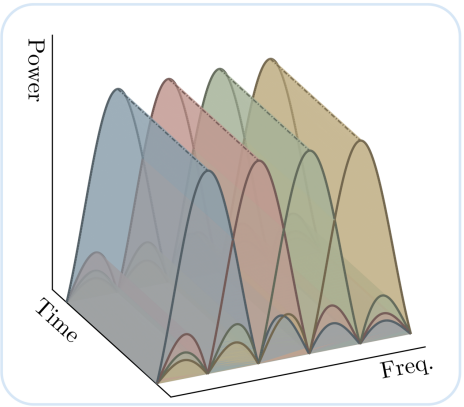
*Information symbols*

$$\mathbf{x} \triangleq [x_0, x_1, \ldots, x_{N-1}]^{\mathsf{T}}$$

- Each symbol is multiplexed unto unique sinusoids of different central frequencies:



# Why not OFDM in LTV?

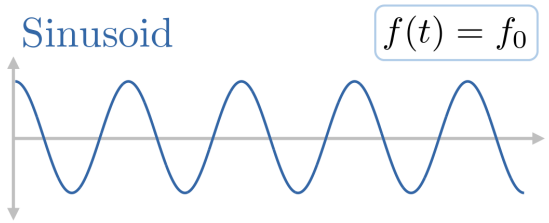
Why was **Orthogonal Frequency Division Multiplexing (OFDM)** so successful?

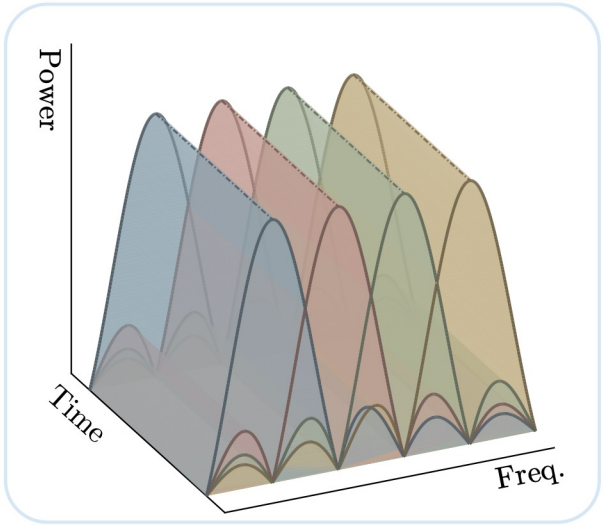
**OFDM sinusoidal subcarrier = eigenfunction of linear time-invariant (LTI) channels**

→ Efficient **FFT-based** implementation

→ Efficient **multiplexing** with **orthogonal** subcarriers

→ Elegant 1-tap equalisation, in **static multipath** (delay only) environments.




(c) OFDMA

*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

# Why not OFDM in LTV?

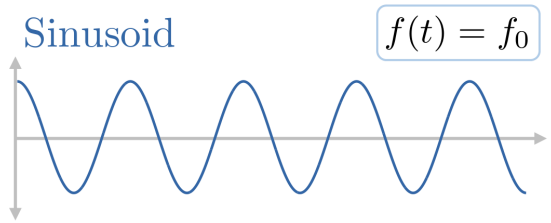
Why was **Orthogonal Frequency Division Multiplexing (OFDM)** so successful?

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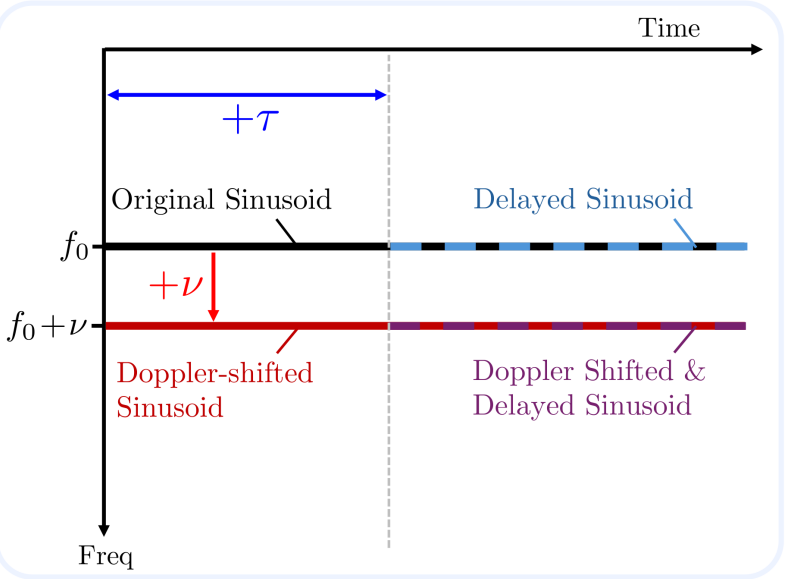



**However, in linear time-variant (LTV) doubly-dispersive channels:**

- Doppler shift destroys subcarrier orthogonality

  → uncontrolled **inter-carrier interference (ICI)**

Inherent deficit of the waveform (sinusoid), and mitigation techniques must be applied (long channel codes, Doppler compensation)




*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

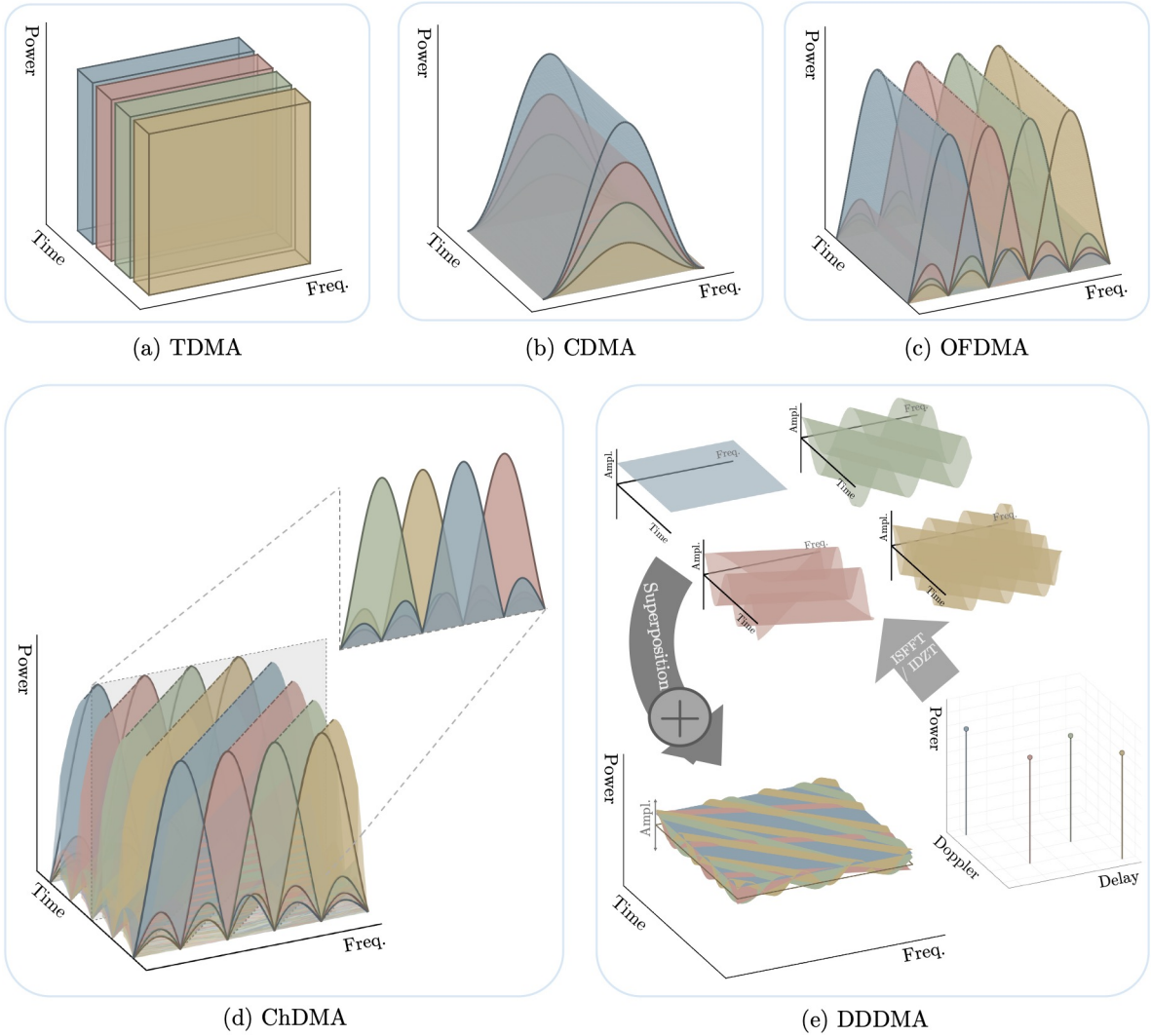
# Return of Spectrum Spreading

**Solution:**

- **Spread** the signal over **both** frequency and time!
- While keeping the **multiplexing gains** of OFDM.

**Two alternative approaches:**

- "2D **isotropic** spreading"
  - (Delay-Doppler approaches, i.e., **OTFS**, ODDM , etc.)
- "**<u>Sheared</u>** spreading"
  - (Chirp-based approaches, i.e., **AFDM**, OCDM, etc.)




*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

# How to Achieve Sheared Spreading and Multiplexing?

In 2021, Bemani et al. proposed **Affine Frequency Division Multiplexing (AFDM):**

In AFDM, *N* information symbols are multiplexed into the time-domain signal via:

- ***N*-point Inverse Discrete <u>Affine</u> Fourier Transform (IDAFT)**

*Time domain samples*

$$\mathbf{s} \triangleq [s_0, s_1, \ldots, s_{N-1}]^{\mathsf{T}}$$

$$s_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \underbrace{e^{j2\pi\left(c_2 m^2 + \frac{nm}{N} + c_1 n^2\right)}}_{\triangleq \phi_{n,m}(c_1,c_2)} \cdot x_m$$

($c_2 = 0$, $c_1 = 0$)

$$\mathbf{x} \triangleq [x_0, x_1, \ldots, x_{N-1}]^{\mathsf{T}}$$

We now know that when $c_1$ and $c_2$ (two tunable chirp parameters) are equal to zero:

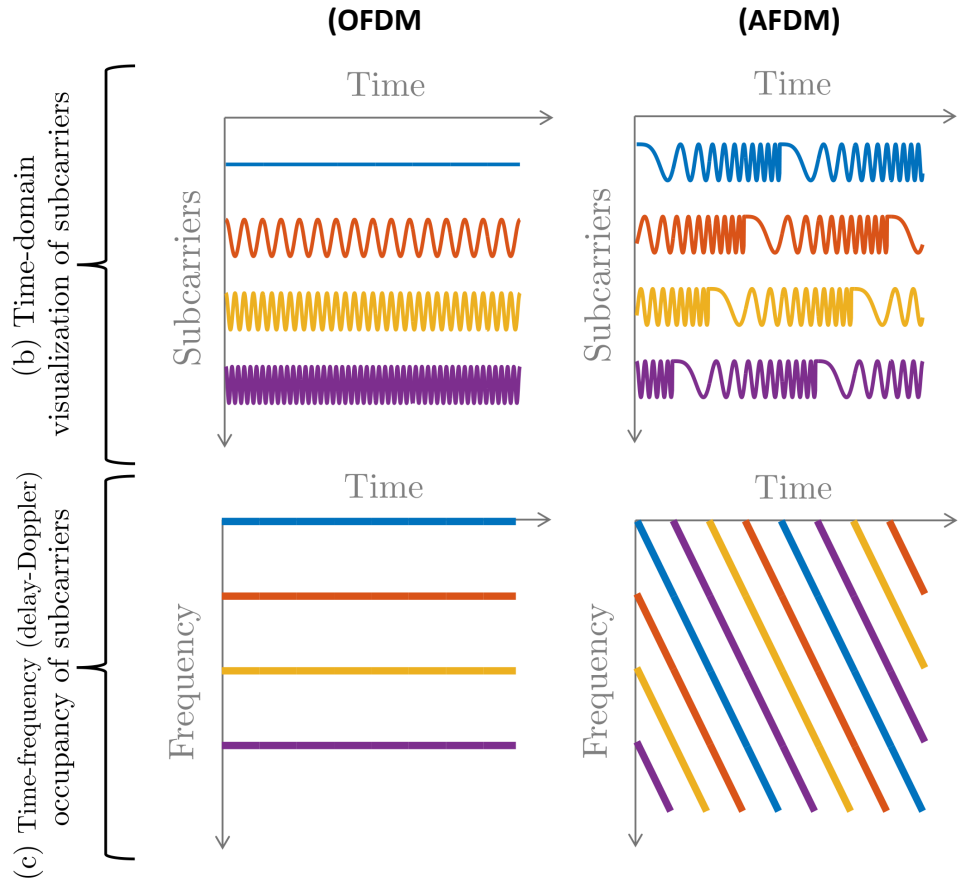
$$s_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{j2\pi\left(\frac{nm}{N}\right)} \cdot x_m$$

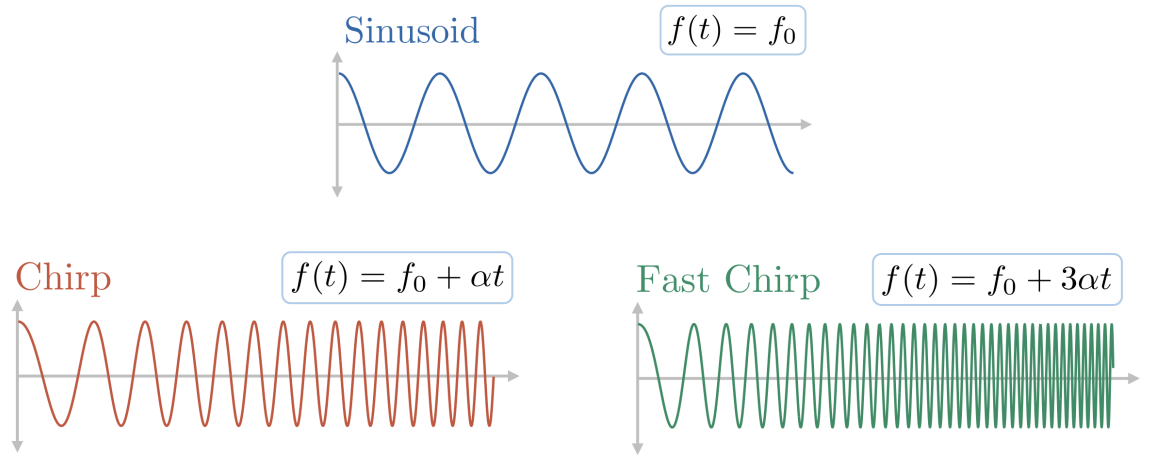
*The IDAFT reduces to the IDFT!*

- A. Bemani, N. Ksairi, and M. Kountouris, "Affine frequency division multiplexing for next generation wireless communications," IEEE Trans. Wireless Commun., vol. 22, no. 11, pp. 8214–8229, Nov. 2023

# Affine Frequency Division Multiplexing (AFDM)

**In other words:** AFDM evolves the fixed sinusoids of OFDM to **parametrizable** **linear chirps**.

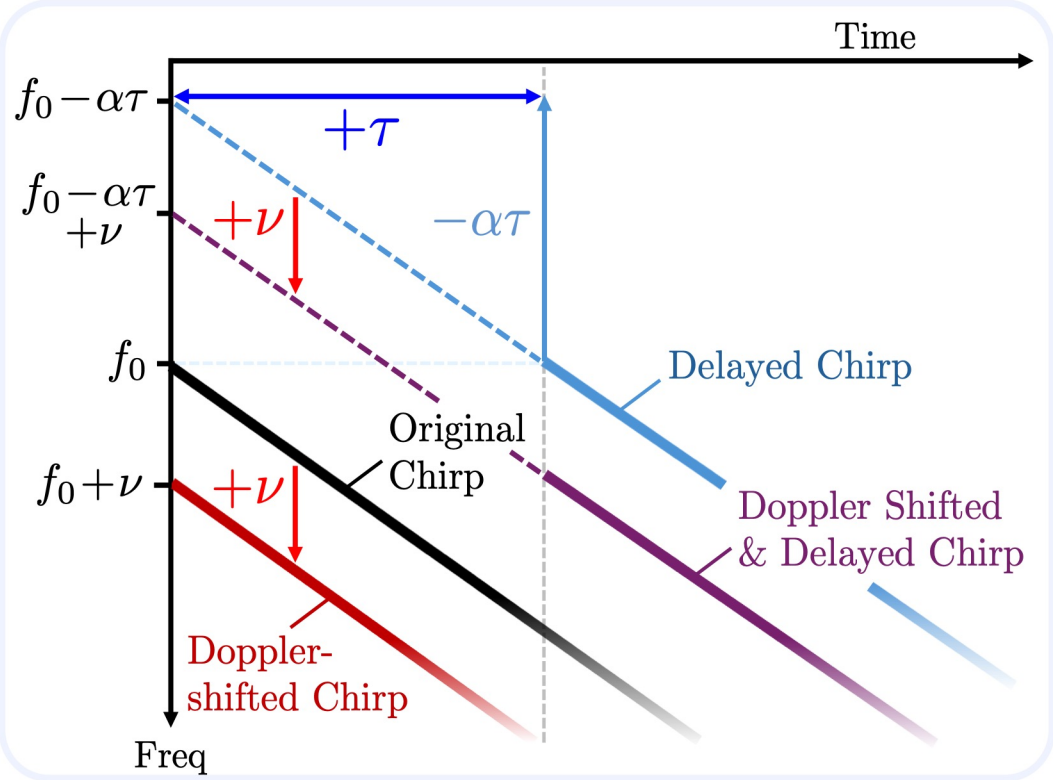


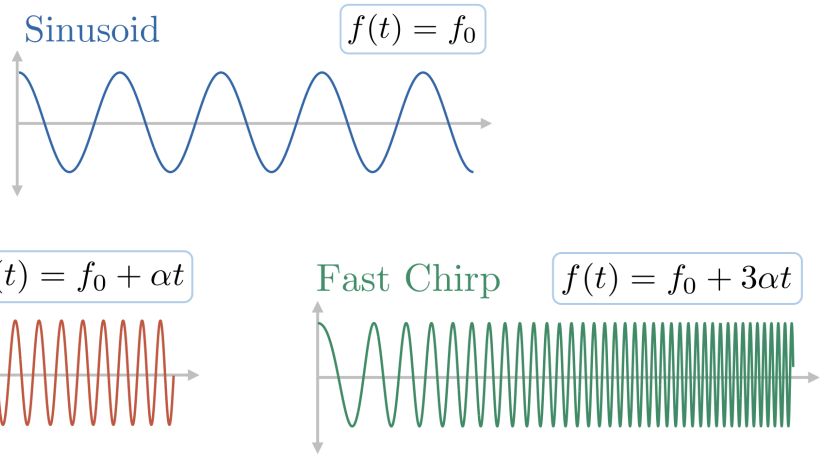




- H. S. Rou, K. R. R. Ranasinghe, V. Savaux, D. Gonzalez G., G. T. F. de Abreu, C. Masorous, "Affine frequency division multiplexing (AFDM) for 6G: Properties, features, and challenges," IEEE Commun. Standards Mag., 2025.

# Affine Frequency Division Multiplexing (AFDM)

**In other words:** AFDM evolves the fixed sinusoids of OFDM to **parametrizable linear chirps.**




**Chirp signals have well-known properties:**

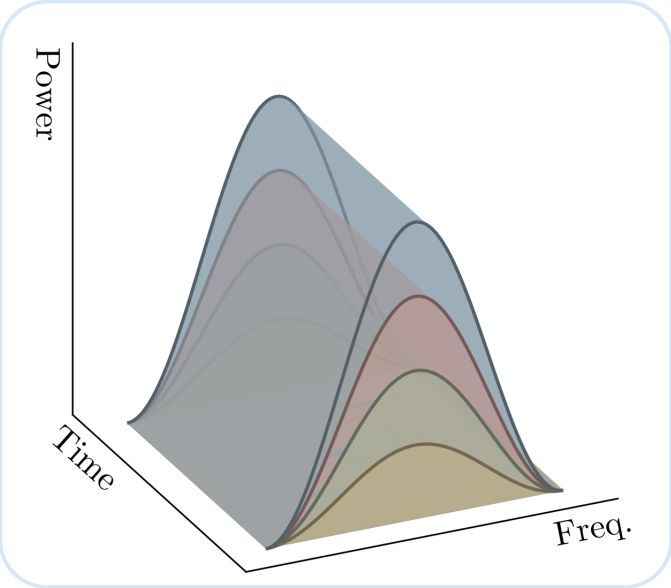
- Great delay-Doppler resolution (i.e., FMCW radar)
- **A Doppler shift translates to an equivalent delay shift!**

*- H. S. Rou, K. R. R. Ranasinghe, V. Savaux, D. Gonzalez G., G. T. F. de Abreu, C. Masorous, "Affine frequency division multiplexing (AFDM) for 6G: Properties, features, and challenges," IEEE Commun. Standards Mag., 2025.*

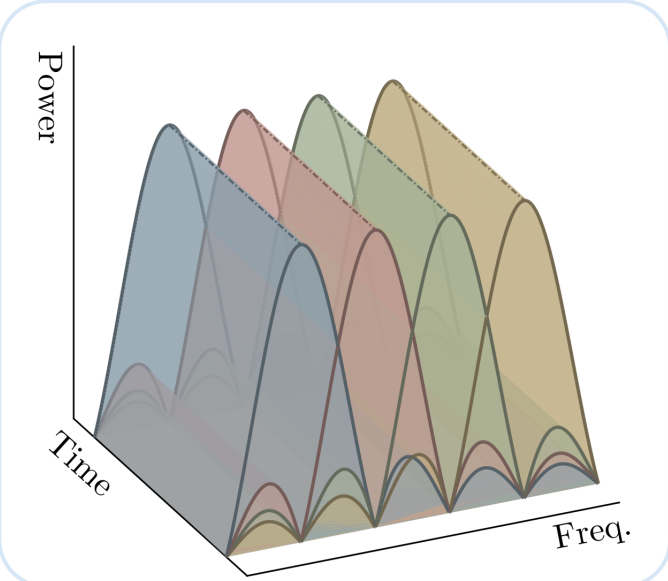
# Affine Frequency Division Multiplexing (AFDM)

The chirp subcarriers of AFDM naturally spread in **both frequency and time** domains:

- Achieve the **“spreading” gains** alike to CDMA.
- Achieve the **“multiplexing” gains** alike to OFDM.

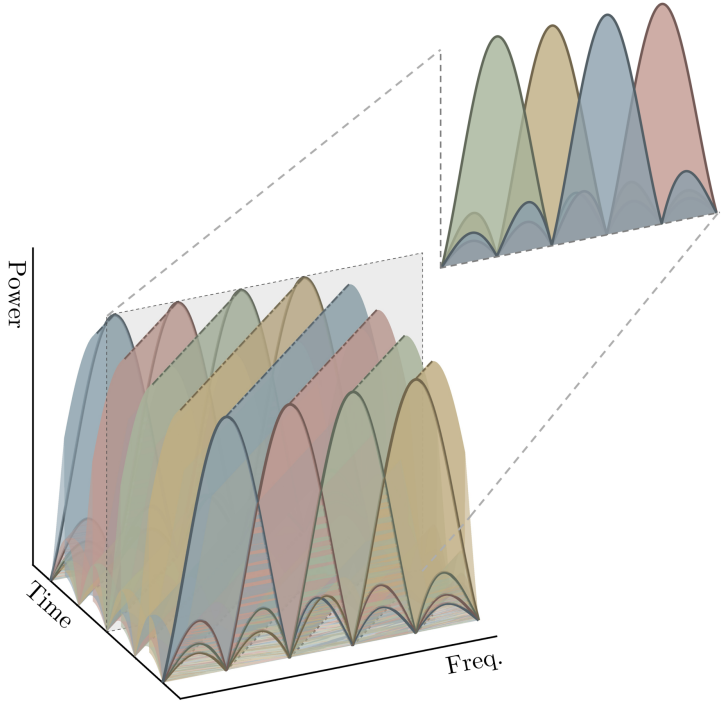



*CDMA*




*OFDMA*

*Notice the similarities to OFDM!!*




*ChDMA (Chirp-division MA)*

*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

# Affine Frequency Division Multiplexing (AFDM)

But the new transform (DAFT) needs the coupling of $c_1$ and $c_2$, different to the **FFT implementation** of OFDM..

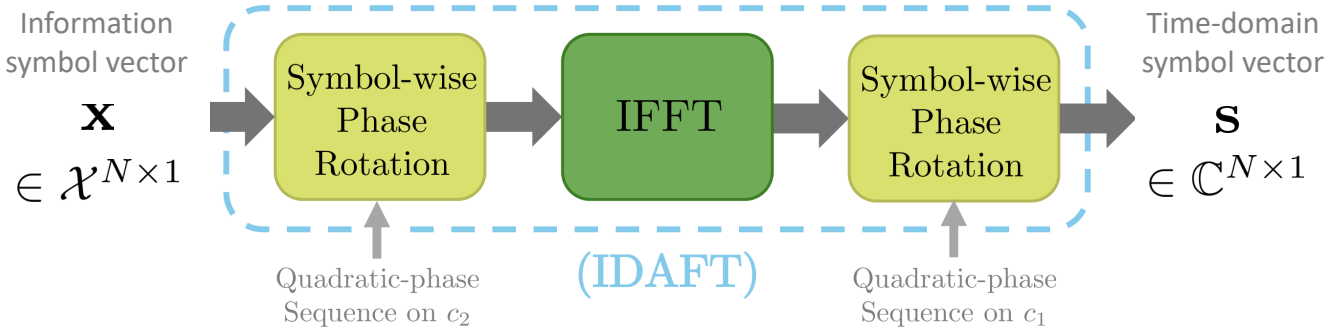
$$s_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \underbrace{e^{j2\pi\left(c_2 m^2 + \frac{nm}{N} + c_1 n^2\right)}}_{\triangleq \phi_{n,m}(c_1,c_2)} \cdot x_m$$

The **IDAFT matrix** can be decomposed as a sequence of **IDFT matrix** and **two element-wise products**!

$$\mathbf{A}^{\mathsf{H}} = \mathrm{diag}(\boldsymbol{\lambda}_{c_1}^*) \cdot \mathbf{F}_N^{\mathsf{H}} \cdot \mathrm{diag}(\boldsymbol{\lambda}_{c_2}^*) \in \mathbb{C}^{N \times N}$$

*Chirp sequence of AFDM*

$$\boldsymbol{\lambda}_{c_i} \triangleq \left[1, e^{-j2\pi c_i (1)^2}, \ldots, e^{-j2\pi c_i (N-1)^2}\right]$$




In other words, we can keep the core FFT/IFFT block and only apply two element-wise phase rotations.

$$5N \log_2 N \rightarrow 5N \log_2 N + 12N$$

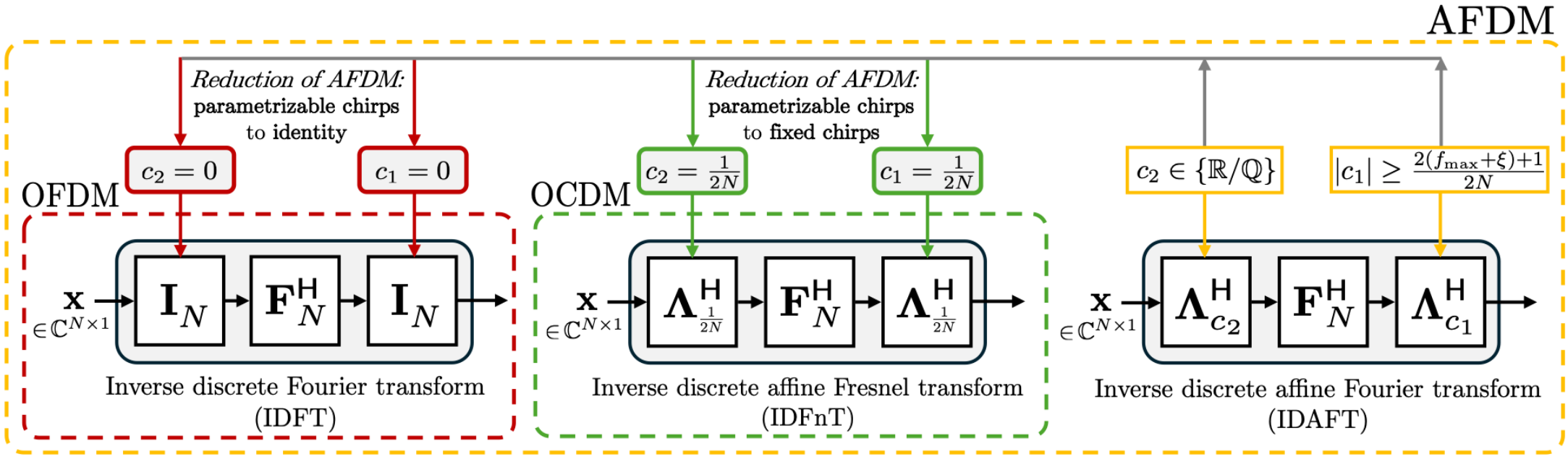
*OFDM complexity* *AFDM complexity*

***See more on implementation in Part IV!***

*- H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

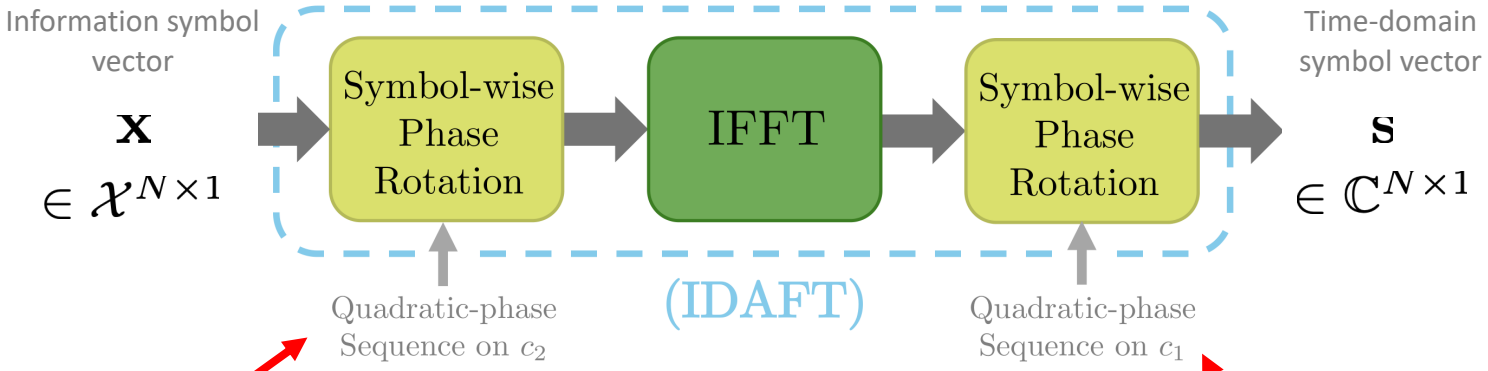
# Affine Frequency Division Multiplexing (AFDM)

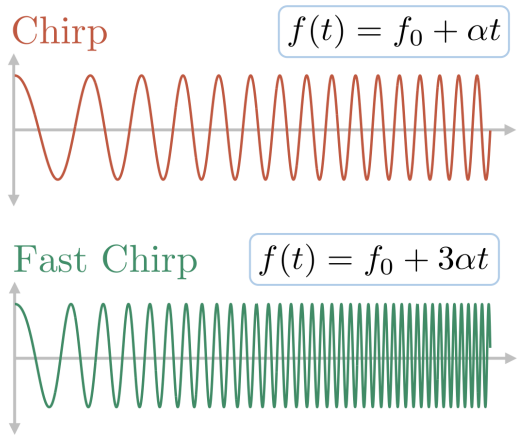
***Im all, we can see that AFDM is a general case of OFDM! (and OCDM)***



# Affine Frequency Division Multiplexing (AFDM)

What do the **two chirp parameters** actually do?




*$c_2$ is the "chirp-domain precoder"*

*$c_1$ controls the chirp rate (slope in TF grid)*

- **does not directly affect the "robustness"**

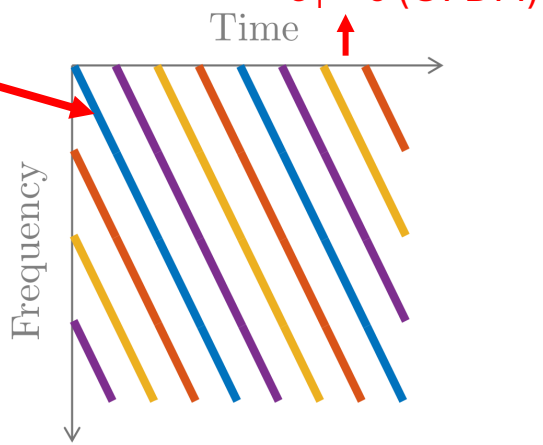
**Controls secondary signal behaviour such as:**

- **Waveform shaping** (ambiguity function)
- **Also opens door to secondary functionalities**

- **Can be optimally tuned to match the channel statistics for full diversity**

**Known closed-form expression for $c_1$**

- Depends on **Doppler spread** and **number of subcarriers**



# Affine Frequency Division Multiplexing (AFDM)

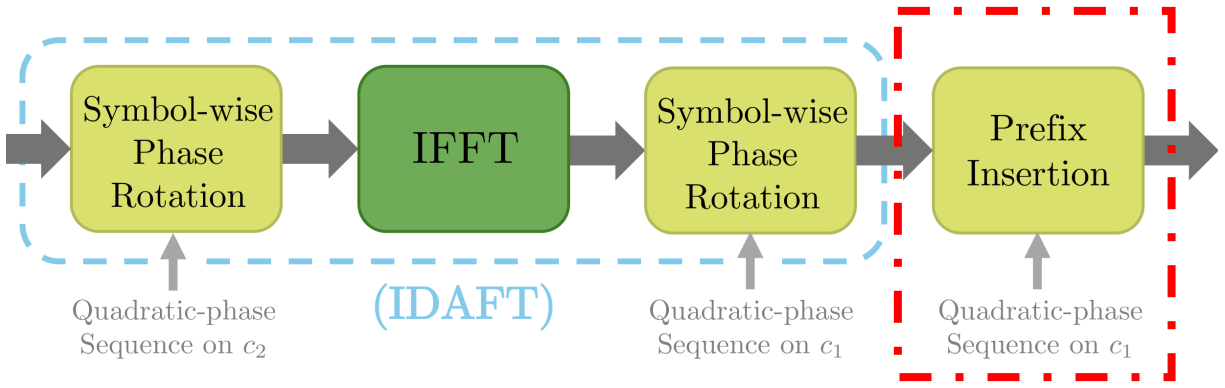
Due to the chirp-based subcarriers, AFDM requires a **modified prefix**:

- **Chirp periodic prefix (CPP):** conventional **OFDM cyclic prefix (CP) with a phase correction**

$$\mathbf{s} \in \mathbb{C}^N$$

$$[s_{-N_{\mathrm{cp}}}, \ldots, s_{n'}, \ldots s_{-1}] \quad [s_0, \ldots\ldots\ldots, s_n, \ldots\ldots\ldots, s_{N-1}]$$

$$s_{n'} \triangleq s_{N-n'} \cdot e^{j2\pi c_1 (N^2 - 2Nn')}$$

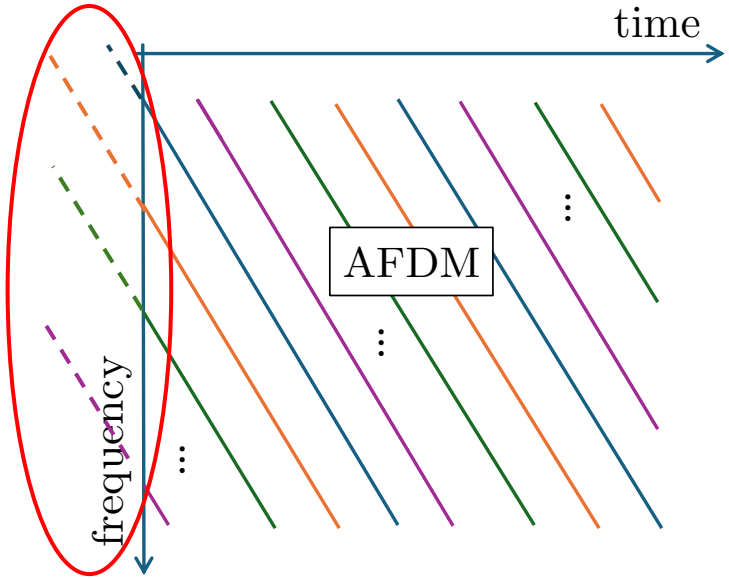



*The CPP reduces to the conventional CP if

$$c_1(N^2 - 2Nn') \in \mathbb{Z}, \forall n$$

satisfied when :

$$2Nc_1 \in \mathbb{Z} \;\; \& \;\; N \text{ is even}$$

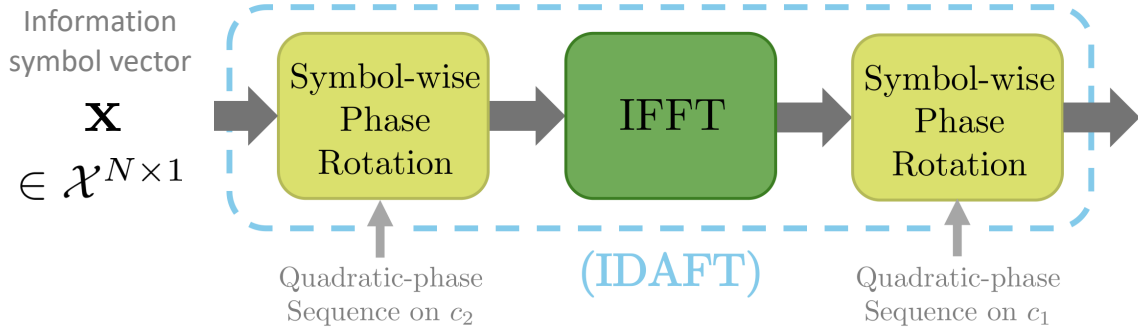


# AFDM in Doubly-Dispersive Channels

Based on the AFDM modulator structure (IDAFT):

$$s_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \underbrace{e^{j2\pi\left(c_2 m^2 + \frac{nm}{N} + c_1 n^2\right)}}_{\triangleq \phi_{n,m}(c_1,c_2)} \cdot x_m$$

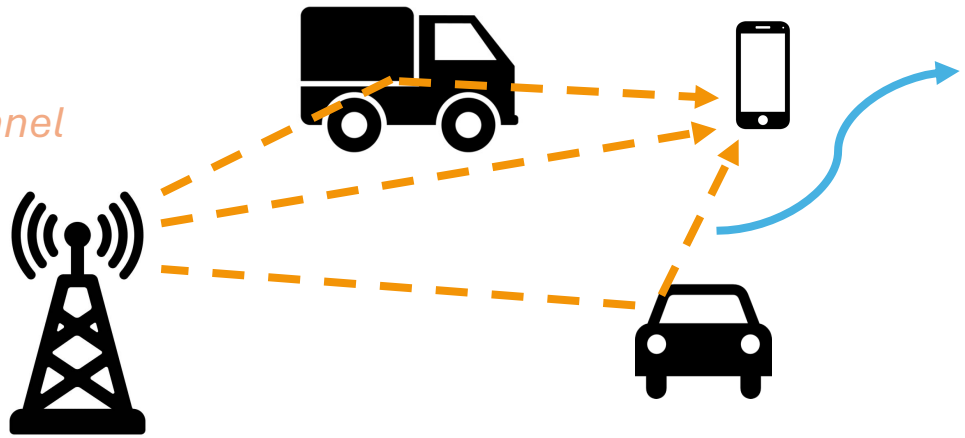
$$\mathbf{x} \triangleq [x_0, x_1, \ldots, x_{N-1}]^{\mathrm{T}}$$




$$\mathbf{s} \triangleq [s_0, s_1, \ldots, s_{N-1}]^{\mathrm{T}}$$

Let us analyse the full IO relationship between the Tx and Rx:

*Heterogeneous and mobile (time-variant) multipath channel*



*(DD channel parameters)*

Path gain $h_p \in \mathbb{C}$

Path delay $\tau_p \in [0, \tau^{\max}]$

Doppler shift $\nu_p \in [-\nu^{\max}, +\nu^{\max}]$

# Doubly-Dispersive Channel

The doubly-dispersive channel is mathematically described by:

Path gain $h_p \in \mathbb{C}$

Path delay $\tau_p \in [0, \tau^{\max}]$

Doppler shift $\nu_p \in [-\nu^{\max}, +\nu^{\max}]$

Time-variant impulse response function (TV-IRF)

$$h(t,\tau) \triangleq \sum_{p=1}^{P} h_p \cdot e^{j2\pi\nu_p t} \cdot \delta(\tau - \tau_p),$$

The received signal over the doubly-dispersive channel is given by a convolution:

*Received signal* *TV-IRF* *Transmit signal* *AWGN*

$$r(t) \triangleq h(t,\tau) * s(t) + w(t)$$

$$r(t) \triangleq \int_{-\infty}^{+\infty} s(t-\tau) \underbrace{\left( \sum_{p=1}^{P} h_p \cdot e^{j2\pi\nu_p t} \cdot \delta(t - \tau_p) \right)}_{\text{TV-IRF}} d\tau + w(t)$$

*- D. Bliss and S. Govindasamy, Dispersive and Doubly-Dispersive Channels. Cambridge University Press, 2013.*

# Signal Processing Over Doubly-Dispersive Channels

$$r(t) \triangleq \int_{-\infty}^{+\infty} s(t-\tau) \left( \sum_{p=1}^{P} h_p \cdot e^{j2\pi\nu_p t} \cdot \delta\left(t-\tau_p\right) \right) d\tau + w(t)$$

If one assumes the sampling interval to be: $\frac{\tau_p}{T_s} - \left\lfloor \frac{\tau_p}{T_s} \right\rceil \approx 0$ ***Integer normalised <u>delay</u> assumption!!***

- Normalised delay: $\ell_p \triangleq \frac{\tau_p}{T_s} \in \mathbb{N}_0$
- Normalised digital Doppler shift: $f_p \triangleq \frac{N\nu_p}{f_s} \in \mathbb{R}$

*Integer/Fractional Doppler shift*

The discrete convolution is rewritten in terms of the normalised delay-Doppler parameters:

$$r[n] \triangleq \sum_{\ell=0}^{\infty} s[n-\ell] \left( \sum_{p=1}^{P} h_p \cdot e^{j2\pi f_p \frac{n}{N}} \cdot \delta\left[\ell-\ell_p\right] \right) + w[n] \quad \Rightarrow \quad r[n] = \sum_{p=1}^{P} h_p \, e^{j2\pi f_p \frac{n}{N}} \, s[n-\ell_p] + w[n]$$


- *M. Mirabella et al., "Continuous-Time Analysis of AFDM: Fundamental Bounds, and Effects of Pulse-Shaping and Hardware Impairments," arXiv preprint arXiv:2602.20909, 2026.*
- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Signal Processing Over Doubly-Dispersive Channels

Under prefix insertion and discarding, rewrite as **circular convolutional matrix:**

Path-wise convolutional channel (**shifted** diagonal)

Received signal

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Bigg(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Bigg)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

Transmit signal

Forward cyclic-shift matrix: $\mathbf{\Pi}^0 \equiv \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{\Pi}^1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{\Pi}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Roots-of-unity diagonal matrix: $\mathbf{W} \triangleq \mathrm{diag}(\mathbf{w}) = \mathrm{diag}\left(\left[1, e^{-j2\pi/N}, \cdots, e^{-j2\pi(N-1)/N}\right]\right) \in \mathbb{C}^{N\times N}$

Prefix phase diagonal matrix: $\mathbf{\Phi}_p \triangleq \mathrm{diag}\Big(\big[\underbrace{e^{-j2\pi\cdot\phi_{\mathrm{cp}}(\ell_p)}, \cdots, e^{-j2\pi\cdot\phi_{\mathrm{cp}}(1)}}_{\ell_p \text{ terms}}, \underbrace{1, \cdots, 1}_{N-\ell_p \text{ ones}}\big]\Big) \in \mathbb{C}^{N\times N}$

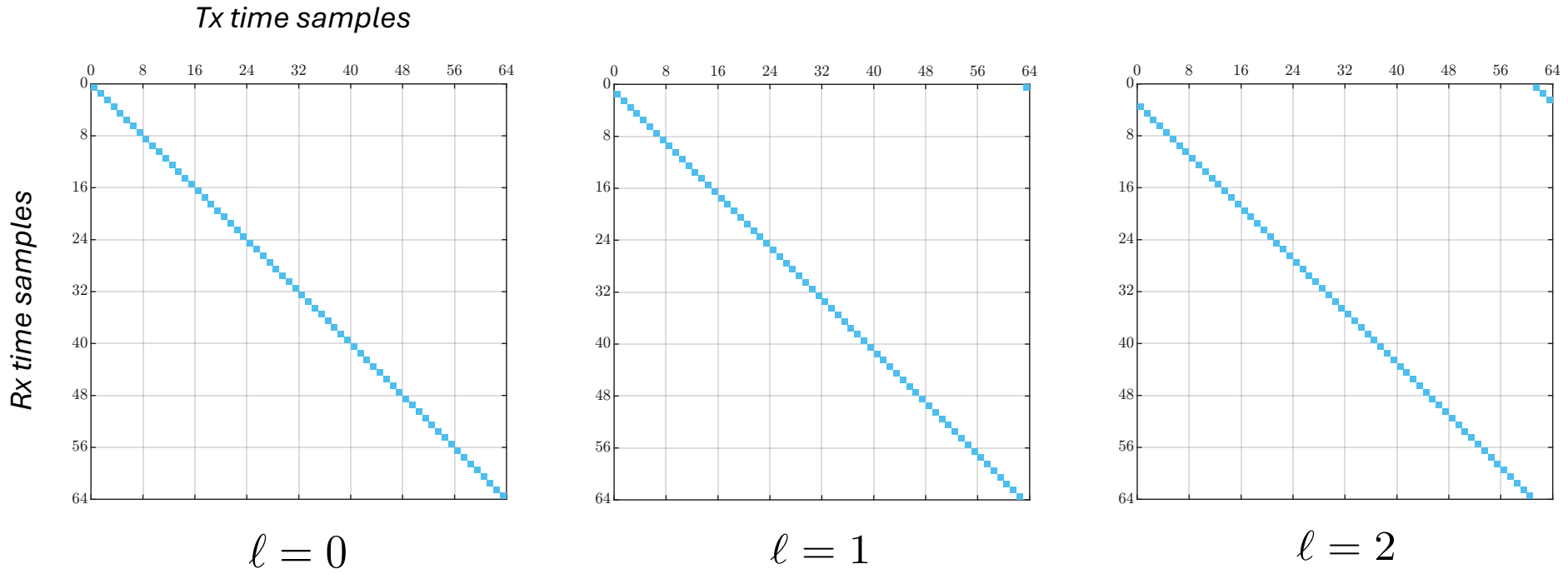
Complex fading coefficient: $h_p \in \mathbb{C}$

- *H. S. Rou et al., "From Orthogonal Time–Frequency Space to Affine Frequency-Division Multiplexing: A comparative study of next-generation waveforms for integrated sensing and communications in doubly dispersive channels," IEEE Signal Processing Magazine, vol. 41, no. 5, 2024.*

# Signal Processing Over Doubly-Dispersive Channels

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Bigg(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Bigg)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

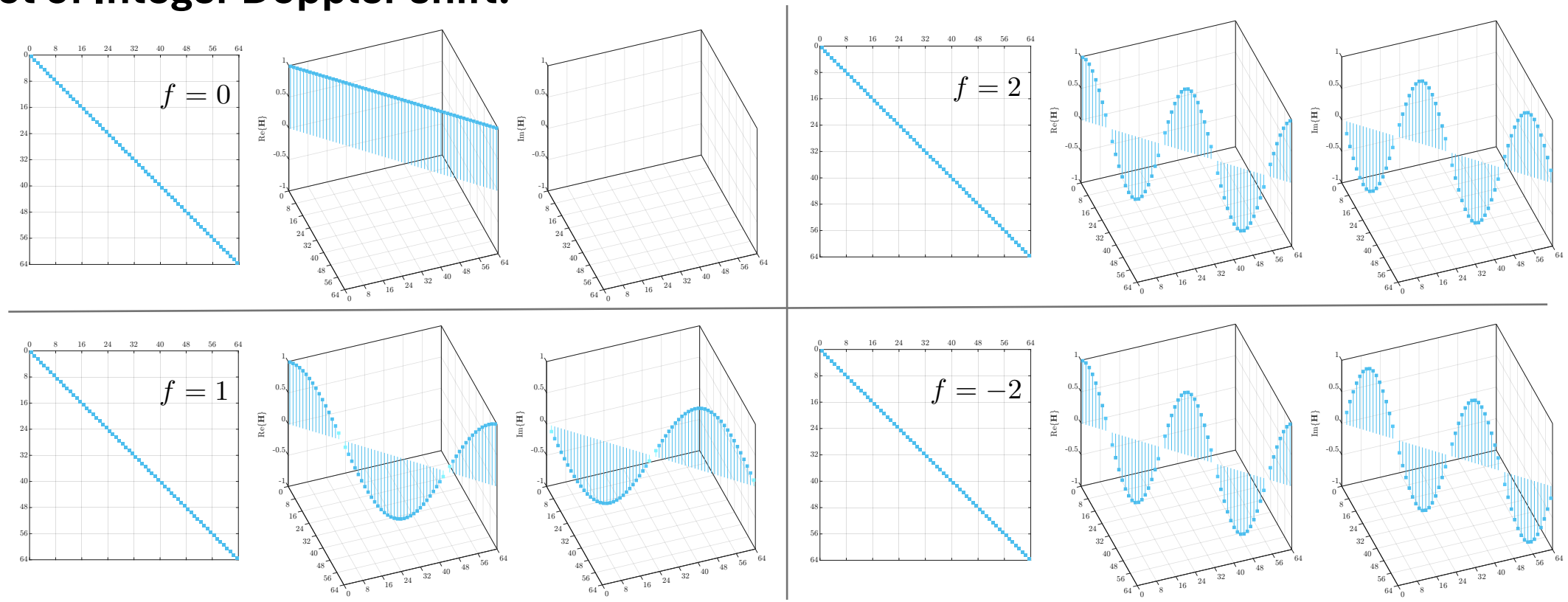
**Effect of Integer Delay:**



# Signal Processing Over Doubly-Dispersive Channels

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Bigg(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Bigg)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

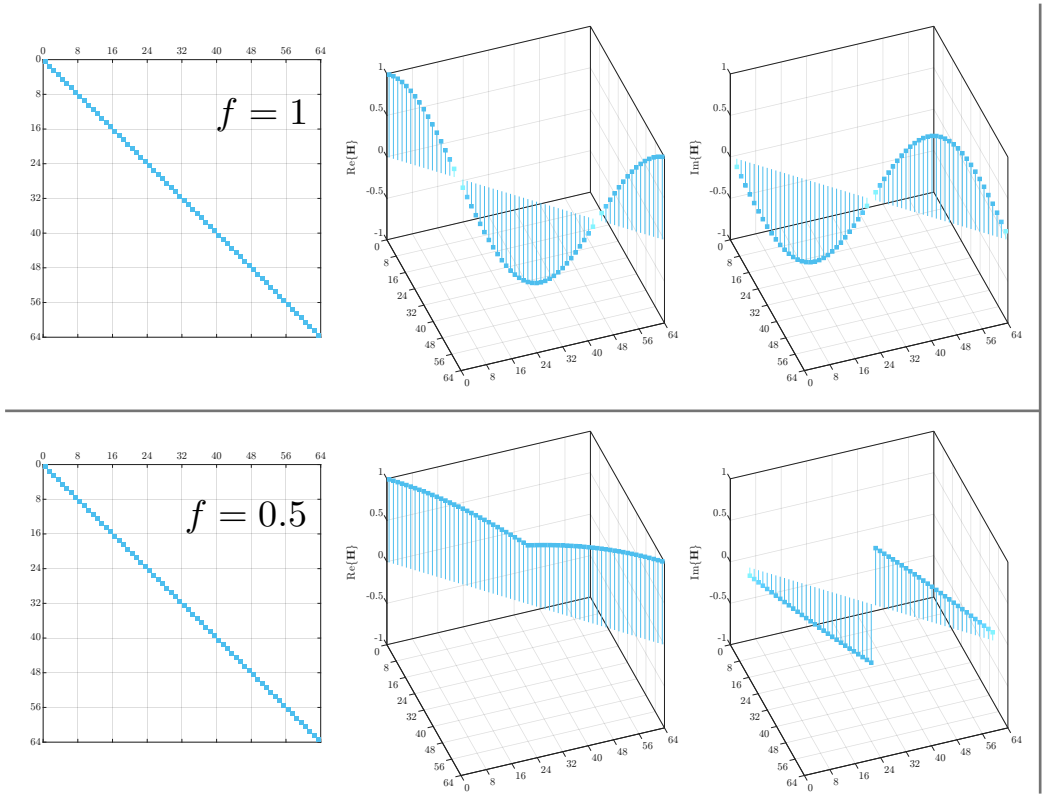
**Effect of Integer Doppler shift:**

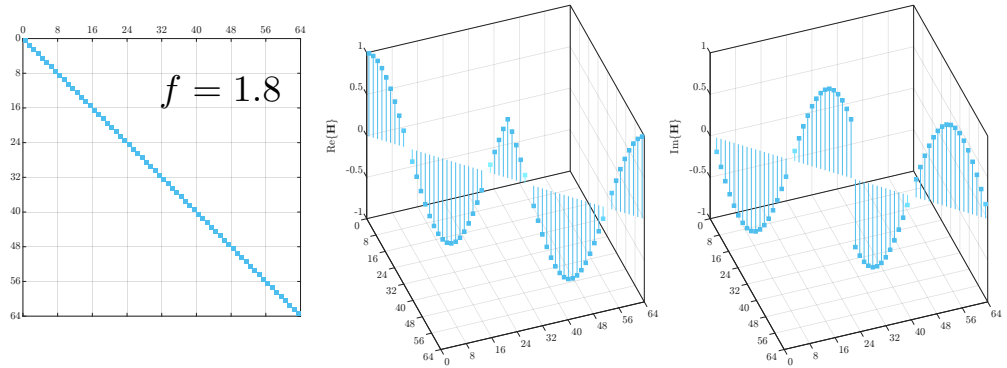


# Signal Processing Over Doubly-Dispersive Channels

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Bigg(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Bigg)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

**Effect of <u>Fractional</u> Doppler shift:**

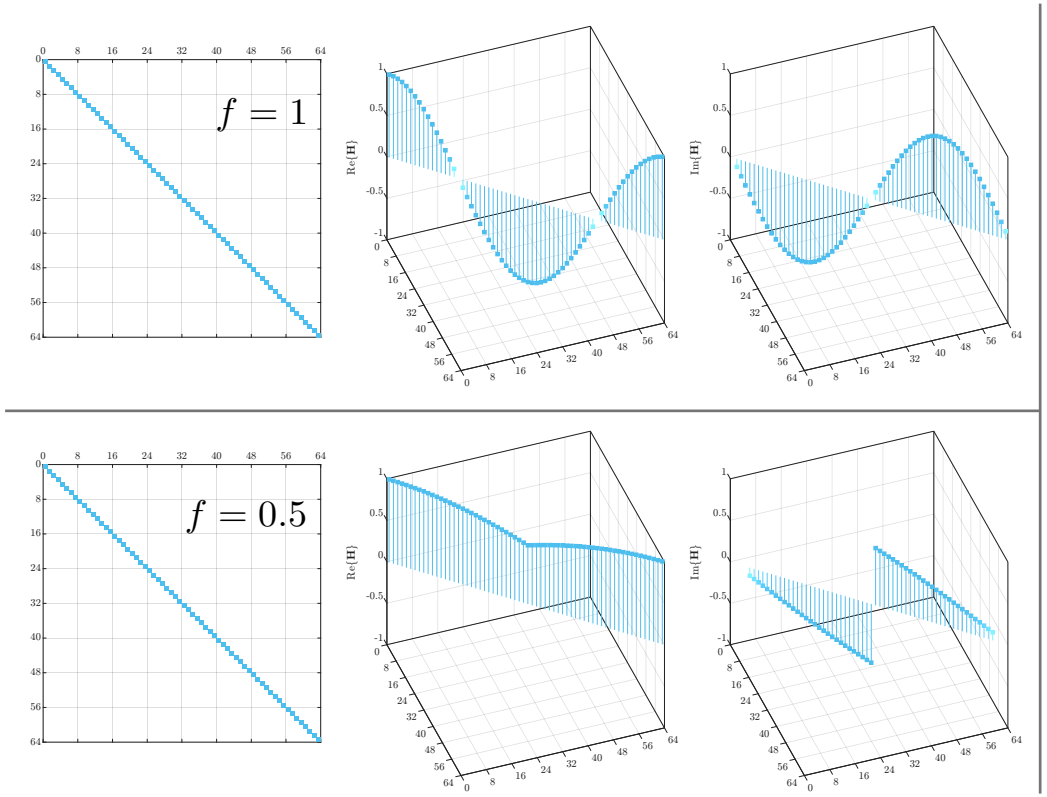


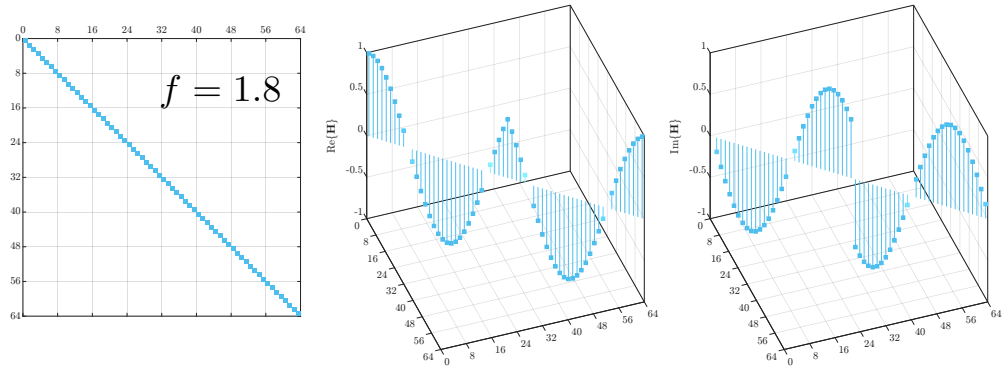



# Signal Processing Over Doubly-Dispersive Channels

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Bigg(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Bigg)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

**Effect of <u>Fractional</u> Doppler shift:**

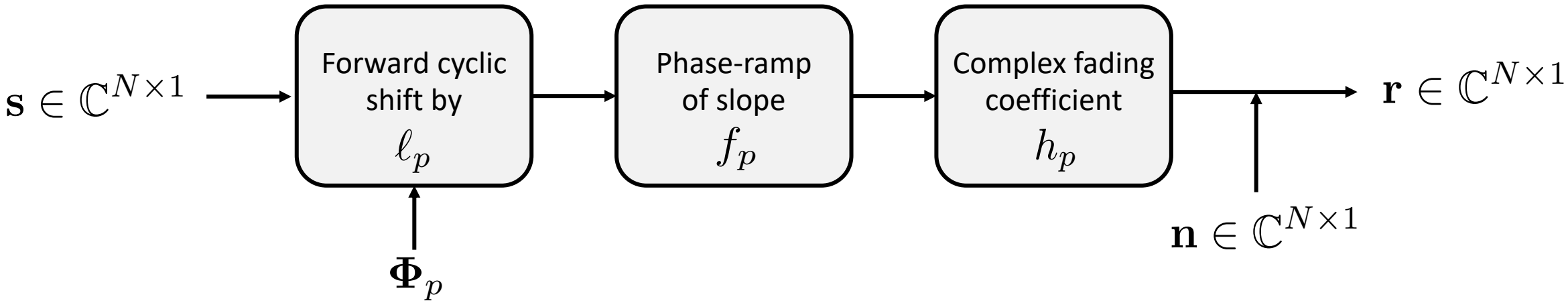


# Signal Processing Over Doubly-Dispersive Channels

$$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Big(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Big)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

*In all:* the total effect unto a transmit signal is a **circular shift**, then a **phase-ramp:**



# Signal Processing Over Doubly-Dispersive Channels

*Ideal integer-delay assumption:*

$$r[n] = \sum_{p=1}^{P} h_p \, e^{j2\pi f_p \frac{n}{N}} \, s[n - \ell_p] + w[n]$$

!!

However, if the sampling interval is assumed as: $\frac{\tau_p}{T_s} - \left\lfloor \frac{\tau_p}{T_s} \right\rceil \neq 0$

***Fractional* normalised delay assumption!!**

- The continuous-time signal must be reconstructed, described via the discrete-time interpolation

$$s[n - \ell_p] = \sum_{m=0}^{N-1} s[m] g\big((n-m) - \ell_p\big)$$

Dependent on the pulse shaping filter!!

***General*** *Fractional-delay -Fractional-Doppler (FDFD) Model:*

$$r[n] = \sum_{p=1}^{P} h_p \, e^{j2\pi f_p \frac{n}{N}} \sum_{m=0}^{N-1} s[m] g\big((n-m) - \ell_p\big) + w[n]$$


- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Signal Processing Over Doubly-Dispersive Channels

Similarly, we can rewrite as "**circular convolutional**"-like matrix form:

*Integer delay model:* $$\mathbf{r} \triangleq \mathbf{H}\cdot\mathbf{s} = \Big(\sum_{p=1}^{P} \overbrace{h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}}^{\triangleq \mathbf{H}_p \in \mathbb{C}^{N\times N}}\Big)\cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

*Under FDFD, not possible to decouple the CPP phase-offset and the delays*

*Delays are no longer clean cyclic shifts*

$$\mathbf{\Pi}^0 \equiv \mathbf{I}_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \mathbf{\Pi}^1 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \text{ and } \mathbf{\Pi}^2 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$s[n-\ell_p] = \sum_{m=0}^{N-1} s[m]g\big((n-m)-\ell_p\big)$$

*FDFD per-path channel:*

$$\mathbf{H}_p \triangleq \mathbf{V}^{f_p}\cdot\big(\mathbf{G}(\ell_p)+\mathbf{\Phi}(\ell_p)\big) \in \mathbb{C}^{N\times N}$$

$$[\mathbf{G}(\ell_p)]_{n,m} = g(n-m-\ell_p)$$

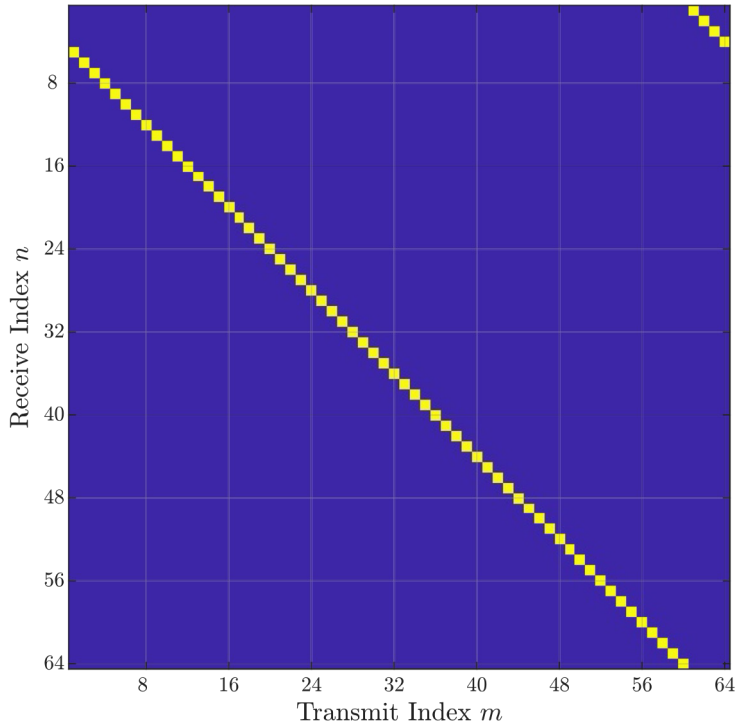
$$[\mathbf{\Phi}_p(\ell_p)]_{n,m} = g(n-m-\ell_p+N)\cdot e^{j2\pi\phi_{\text{cp}}(n-m-\ell_p)}$$

- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Signal Processing Over Doubly-Dispersive Channels

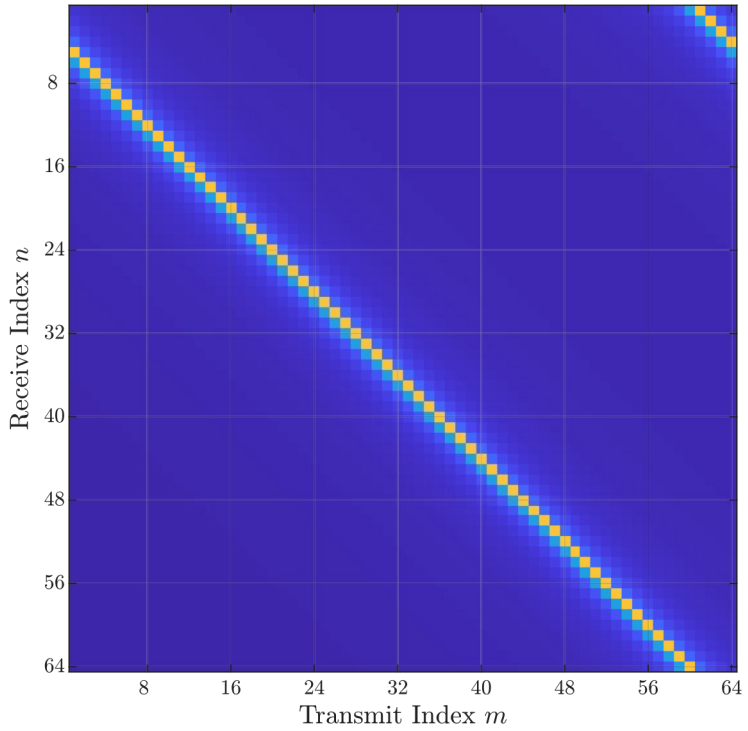
## IDID/IDFD Assumption

$$\mathbf{H}_p \triangleq h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{W}^{f_p} \cdot \mathbf{\Pi}^{\ell_p} \in \mathbb{C}^{N \times N}$$




## FDFD Assumption

$$\mathbf{H}_p \triangleq h_p \cdot \mathbf{V}^{f_p} \cdot \left(\mathbf{G}(\ell_p) + \mathbf{\Phi}(\ell_p)\right) \in \mathbb{C}^{N \times N}$$



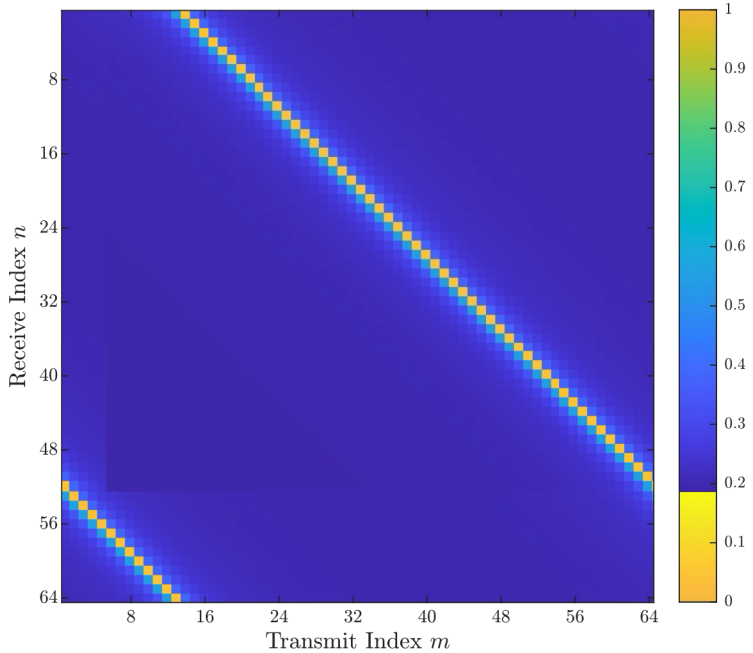

- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Signal Processing Over Doubly-Dispersive Channels

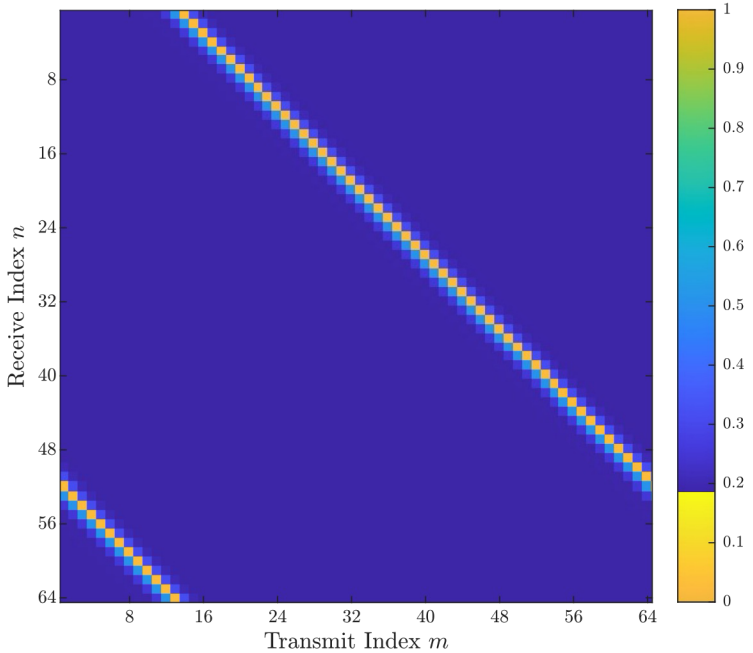
## FDFD model with different pulses

$$\mathbf{H}_p \triangleq h_p \cdot \mathbf{V}^{f_p} \cdot \left(\mathbf{G}(\ell_p) + \mathbf{\Phi}(\ell_p)\right) \in \mathbb{C}^{N \times N}$$

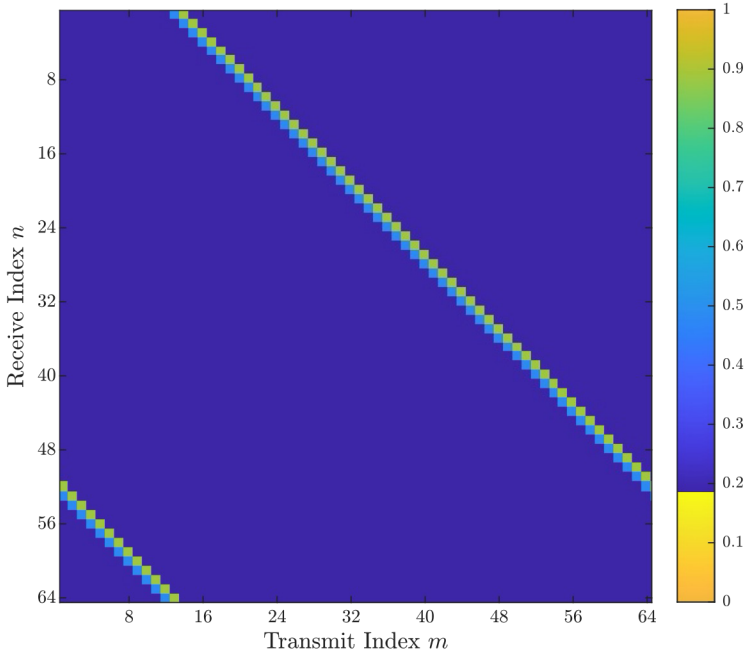
$$[\mathbf{G}(\ell_p)]_{n,m} = g(n - m - \ell_p)$$
$$[\mathbf{\Phi}_p(\ell_p)]_{n,m} = g(n - m - \ell_p + N) \cdot e^{j2\pi\phi_{\mathrm{cp}}(n-m-\ell_p)}$$




(a) Sinc (Ideal Band-limited)




(b) Raised Cosine ($\alpha = 0.5$)




(c) Rectangular (No Pulse Shaping)

- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Effective DD Channels of OFDM and AFDM

What happens for a given multicarrier waveform? $\mathbf{r} \triangleq \mathbf{H} \cdot \mathbf{s} = \left( \sum_{p=1}^{P} \mathbf{H}_p \right) \cdot \mathbf{s} + \mathbf{n} \in \mathbb{C}^{N \times 1},$

- Insert the modulation and demodulation operations

For **OFDM** (IDFT/DFT): $\mathbf{r}^{\mathrm{OFDM}} \triangleq \left( \sum_{p=1}^{P} \mathbf{H}_p \right) \cdot \underbrace{\mathbf{F}_N^{\mathsf{H}} \cdot \mathbf{x}^{\mathrm{OFDM}}}_{\mathbf{s}^{\mathrm{OFDM}}} + \mathbf{n} \in \mathbb{C}^{N \times 1},$ *Modulation*

$$\mathbf{y}^{\mathrm{OFDM}} \triangleq \underbrace{\mathbf{F}_N \cdot \left( \sum_{p=1}^{P} \mathbf{H}_p \right) \cdot \mathbf{F}_N^{\mathsf{H}} \cdot \mathbf{x}^{\mathrm{OFDM}} + (\mathbf{F}_N \cdot \mathbf{n})}_{\mathbf{G}^{\mathrm{OFDM}} \triangleq \sum_{p=1}^{P} \mathbf{F}_N \mathbf{H}_p \mathbf{F}_N^{\mathsf{H}}} \in \mathbb{C}^{N \times 1},$$

*Demodulation*

*OFDM effective channel*

# Effective DD Channels of OFDM and AFDM

For **AFDM** (IDAFT/DAFT):

$$\mathbf{A}_N \triangleq \mathbf{\Lambda}_{c_2}\mathbf{F}_N\mathbf{\Lambda}_{c_1}$$

$$\mathbf{A}_N^{\mathsf{H}} \triangleq \left(\mathbf{\Lambda}_{c_2}\mathbf{F}_N\mathbf{\Lambda}_{c_1}\right)^{\mathsf{H}} = \mathbf{\Lambda}_{c_1}^{\mathsf{H}}\mathbf{F}_N^{\mathsf{H}}\mathbf{\Lambda}_{c_2}^{\mathsf{H}}$$

$$\mathbf{\Lambda}_{c_i} \triangleq \mathrm{diag}[e^{-j2\pi c_i(0)^2}, \cdots, e^{-j2\pi c_i(N-1)^2}]$$

Diagonal matrices with AFDM chirp sequences

*Modulation*

$$\mathbf{r}^{\mathrm{AFDM}} \triangleq \left(\sum_{p=1}^{P}\mathbf{H}_p\right)\cdot \underbrace{\mathbf{A}_N^{\mathsf{H}}\cdot\mathbf{x}^{\mathrm{AFDM}}}_{\mathbf{s}^{\mathrm{AFDM}}} + \mathbf{n} \in \mathbb{C}^{N\times 1},$$

*Demodulation*

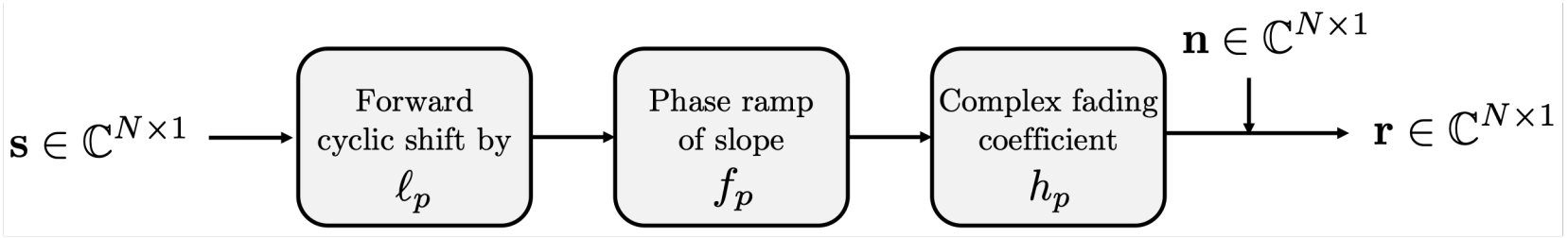
$$\mathbf{y}^{\mathrm{AFDM}} \triangleq \underbrace{\mathbf{A}_N\cdot\left(\sum_{p=1}^{P}\mathbf{H}_p\right)\cdot\mathbf{A}_N^{\mathsf{H}}}\cdot\mathbf{x}^{\mathrm{AFDM}} + (\mathbf{A}_N\cdot\mathbf{n}) \in \mathbb{C}^{N\times 1},$$

*AFDM effective channel*

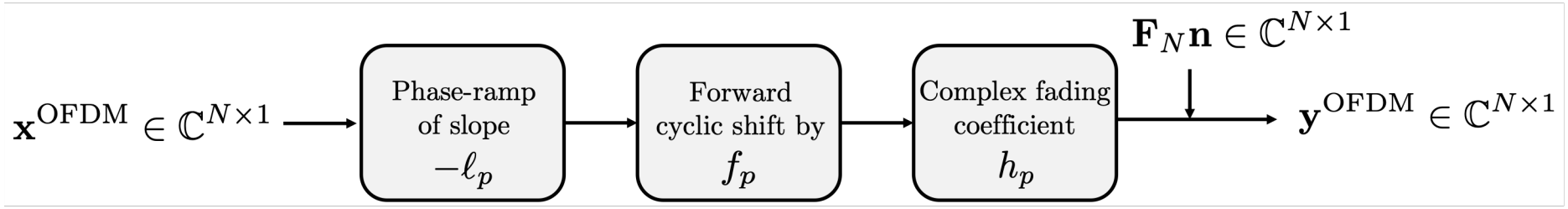
$$\mathbf{G}^{\mathrm{AFDM}} \triangleq \sum_{p=1}^{P}\left(\mathbf{\Lambda}_{c_2}\cdot\mathbf{F}_N\cdot\mathbf{\Lambda}_{c_1}\right)\mathbf{H}_p\left(\mathbf{\Lambda}_{c_1}^{\mathsf{H}}\cdot\mathbf{F}_N^{\mathsf{H}}\cdot\mathbf{\Lambda}_{c_2}^{\mathsf{H}}\right)$$

- *H. S. Rou, G. T. F. de Abreu, "AFDM as a Software Upgrade of OFDM: One Firmware Patch, a New Frontier," arXiv preprints, arXiv:2605.23062, 2026.*

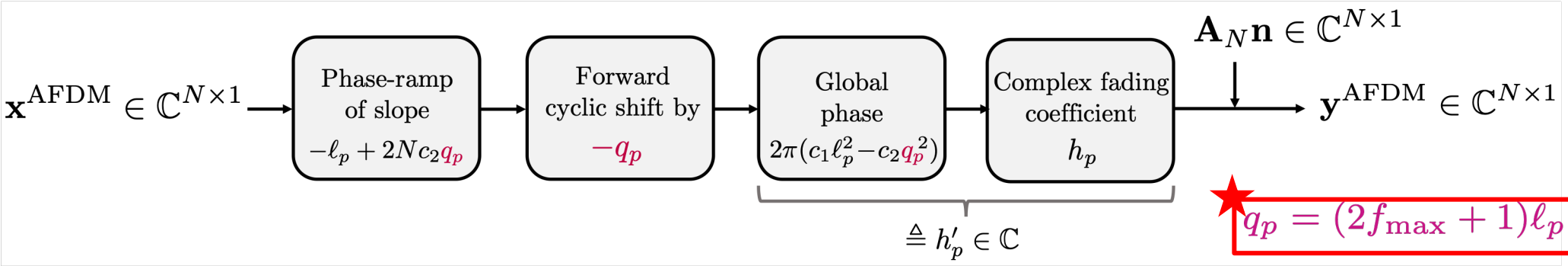
# Effective DD Channels of OFDM and AFDM




(a) Doubly-dispersive channel $\mathbf{H} = \sum_p h_p \mathbf{W}^{f_p} \mathbf{\Pi}^{\ell_p}$: paths with same delay collide on the same off-diagonal (*delay collision*).



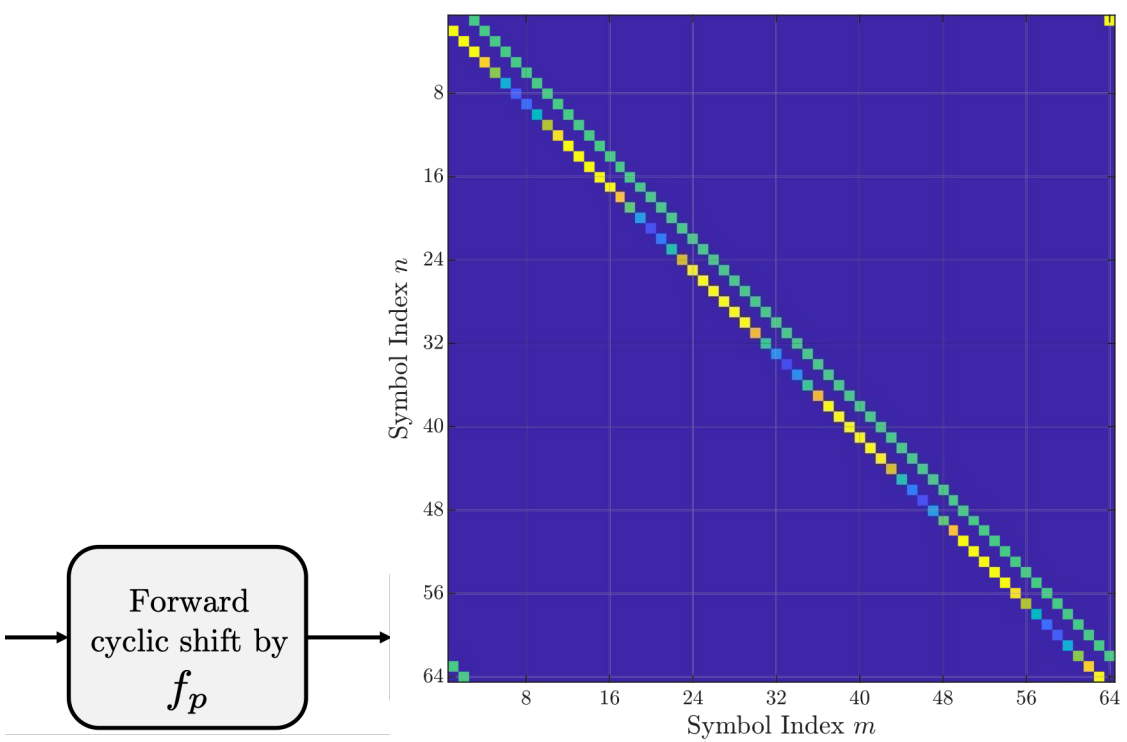

(b) OFDM effective channel $\mathbf{G}^{\mathrm{OFDM}} = \mathbf{F}_N \mathbf{H} \mathbf{F}_N^{\mathsf{H}}$: the IDFT/DFT pair swaps delay and Doppler roles; paths with same Doppler collide on the same off-diagonal (*Doppler collision*).



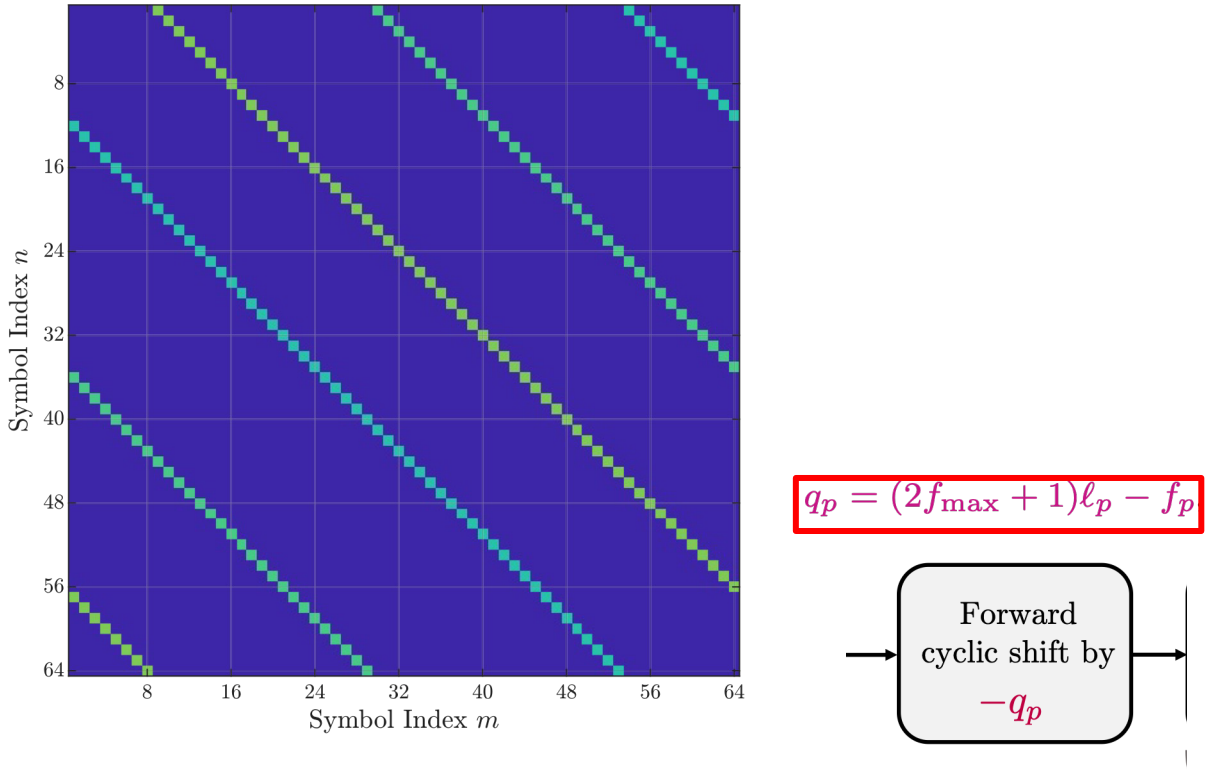

- *H. S. Rou, G. T. F. de Abreu, "AFDM as a Software Upgrade of OFDM: One Firmware Patch, a New Frontier," arXiv preprints, arXiv:2605.23062, 2026.*

# Effective DD Channels of OFDM and AFDM

*For a three-path IDID channel:*




(a) OFDM effective channel (IDID).




(d) AFDM effective channel (IDID).

- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

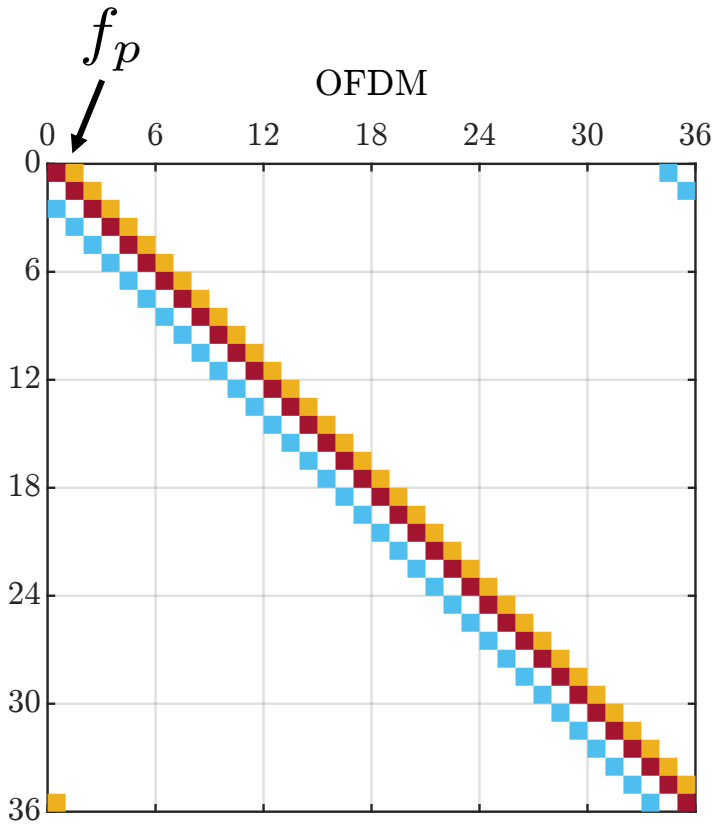
# Effective DD Channels of OFDM and AFDM

*Simplifying slightly for better understanding:*

Visualisation of effective channel structure with **integer delays and Dopplers:**

Three paths: $f^{\max} = 4$
$\{\ell_1 = 0,\ f_1 = 0\}$ (red),
$\{\ell_2 = 1,\ f_2 = 2\}$ (blue),
$\{\ell_3 = 3,\ f_3 = -1\}$ (yellow)

$$(2f_{\max}+1)\ell_p + f_p = q_p$$
$$9\ell_p + f_p = q_p$$

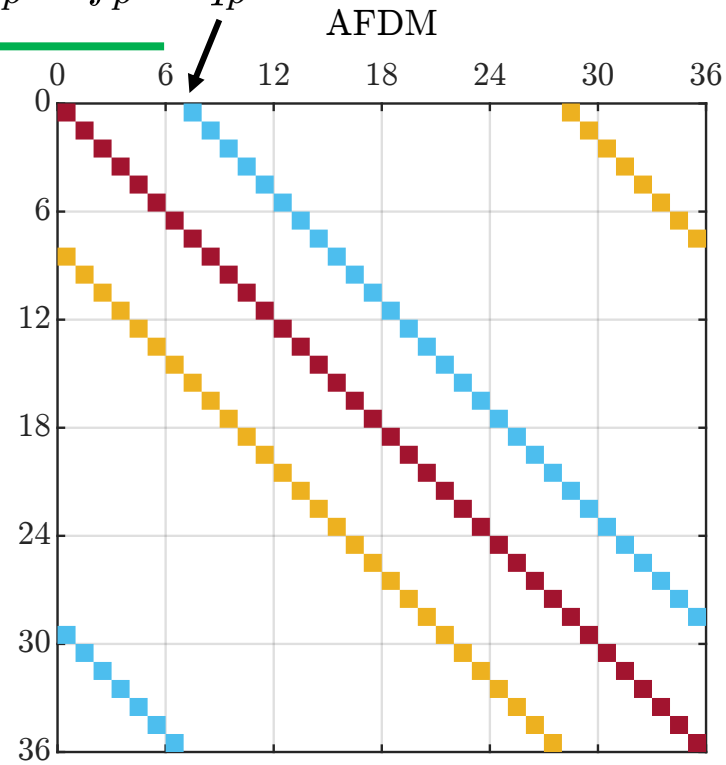



*If the effective channel is estimated, The integer delay-Doppler taps come for free! (ISAC)*

Forward cyclic shift by $f_p$

Forward cyclic shift by $-q_p$

$q_p \triangleq (2f_{\max}+1)\ell_p - f_p$




**OFDM:**
- Each integer delay results in **no shift**
- Each integer Doppler results in a **right-shift**

**AFDM:**
- Each integer delay results in a **left-shift** by $2f^{\max}+1$
- Each integer Doppler results in a **right-shift**

*- H. S. Rou et al., "From Orthogonal Time–Frequency Space to Affine Frequency-Division Multiplexing: A comparative study of next-generation waveforms for integrated sensing and communications in doubly dispersive channels," IEEE Signal Processing Magazine, vol. 41, no. 5, 2024.*

# Effective DD Channels of OFDM and AFDM

Paths with **fractional** Doppler shifts, with **Dirichlet kernel**-based **band leakage** along the main diagonal:

Target parameters:

$\{\ell_1 = 0,\ f_1 = +0.26\}$ (red),

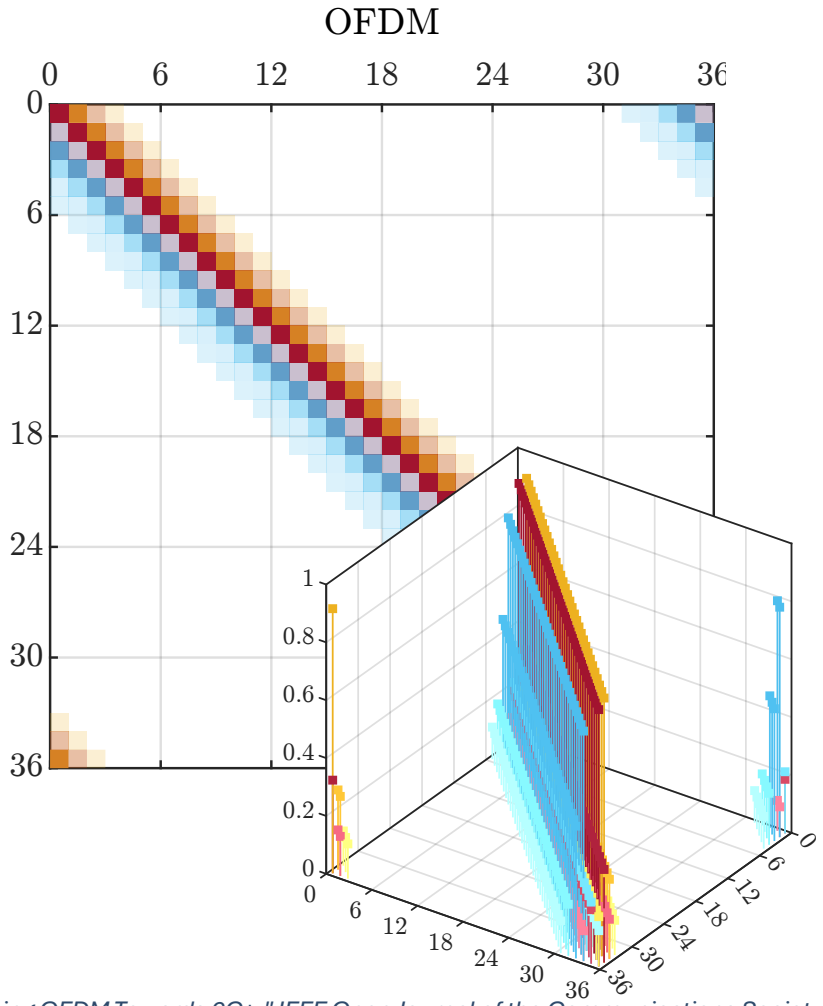
$\{\ell_2 = 1,\ f_2 = +2.35\}$ (blue),

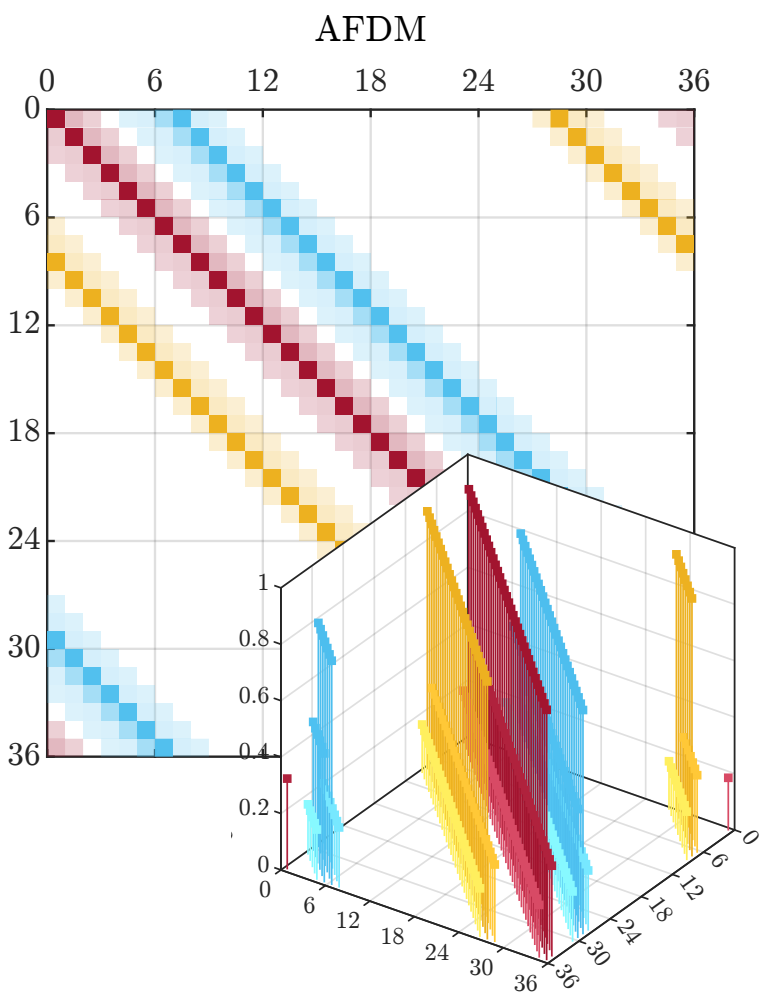
$\{\ell_3 = 3,\ f_3 = -1.23\}$ (yellow)

*The amount of leakage depends on the magnitude of the fractional component.*

$$-0.5 \longleftrightarrow 0 \longleftrightarrow +0.5$$

*Max leakage* *No leakage* *Max leakage*

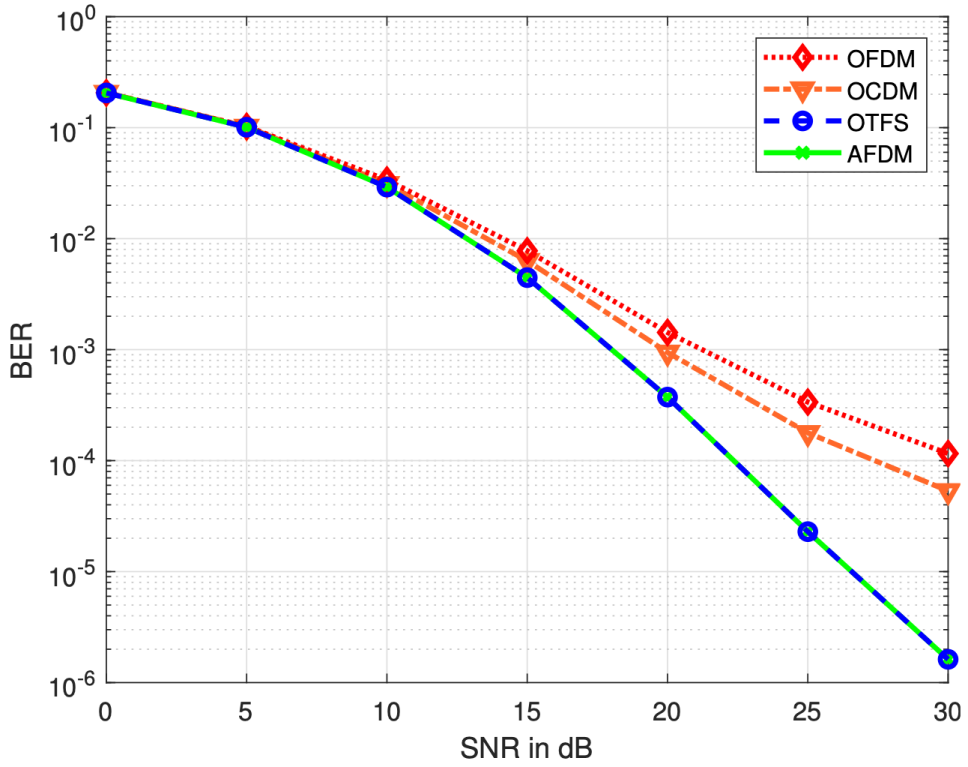


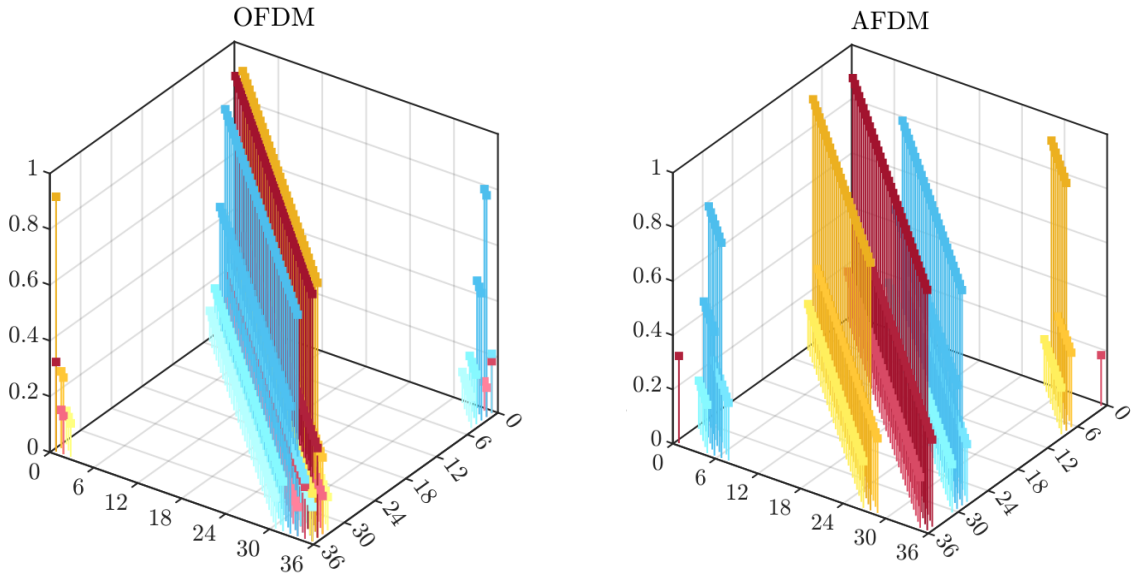

- *H. S. Rou, V. Savaux, Z. Sui, G. T. F. de Abreu, Z. Liu, "AFDM: Evolving OFDM Towards 6G+," IEEE Open Journal of the Communications Society, 2026.*

# Robustness over Doubly-Dispersive Channel

AFDM achieves **excellent delay-Doppler robustness** in doubly-dispersive channels **– like OTFS**




OFDM suffers from severe **inter-carrier interference (ICI)** and loss of performance, and also **loses its 1-tap equaliser.**





*- H. S. Rou, G. T. F. de Abreu, J. Choi, D. Gonzalez G., M. Kountouris, Y. L. Guan, O. Gonsa, "From Orthogonal Time Frequency Space to Affine Frequency Division Multiplexing: A comparative study of next-generation waveforms for ISAC in doubly dispersive channels," IEEE Signal Processing Magazine, vol. 41, no. 5, pp. 71–86, Sep. 2024.*

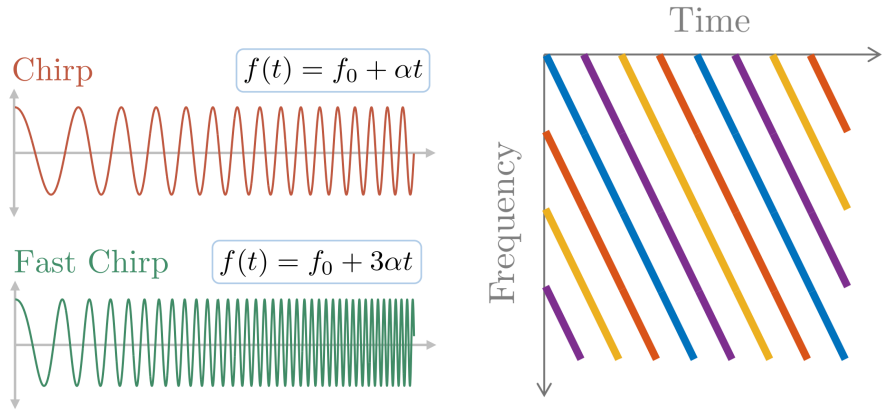
*- A. Bemani, N. Ksairi, and M. Kountouris, "Affine frequency division multiplexing for next generation wireless communications," IEEE Trans. Wireless Commun., vol. 22, no. 11, pp. 8214–8229, Nov. 2023.*

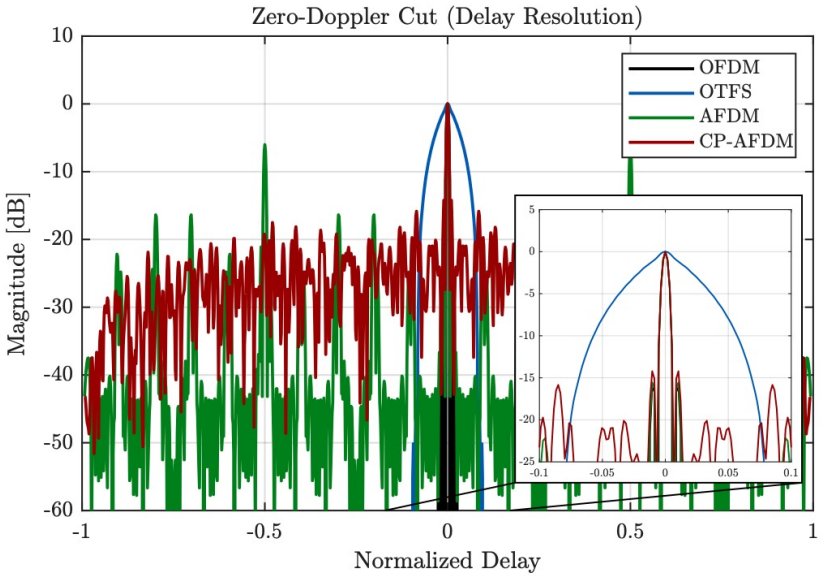
# Implications for ISAC in DD Channels

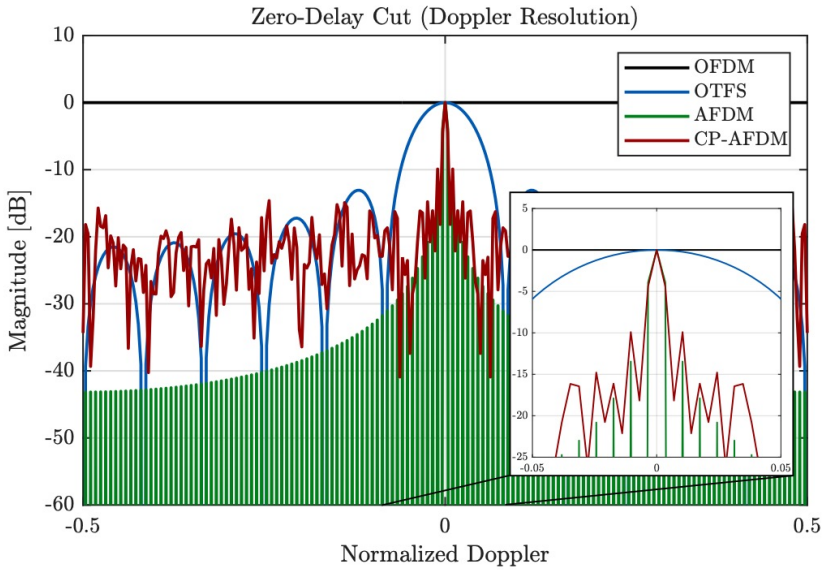
In addition to the channel-based delay-Doppler estimation,

**AFDM** is fundamentally a **chirp** multicarrier waveform:

- **excellent <u>fast-time</u> ambiguity properties** in **<u>both</u> delay** (range estimation) and **Doppler** (velocity estimation)..



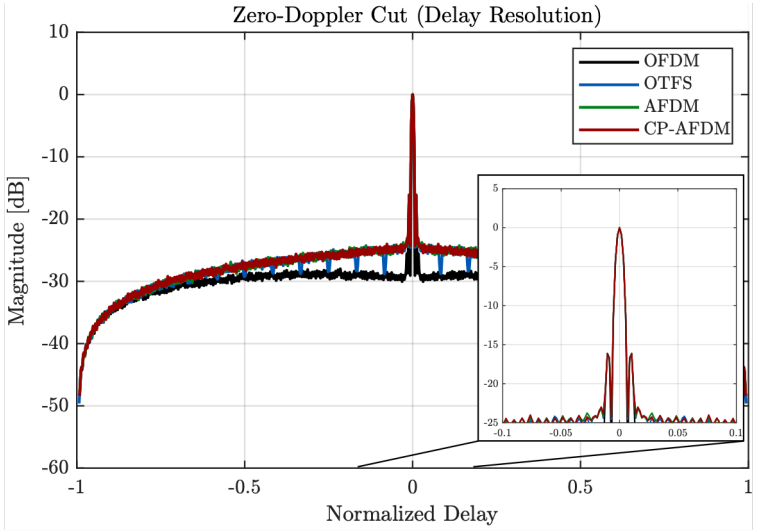



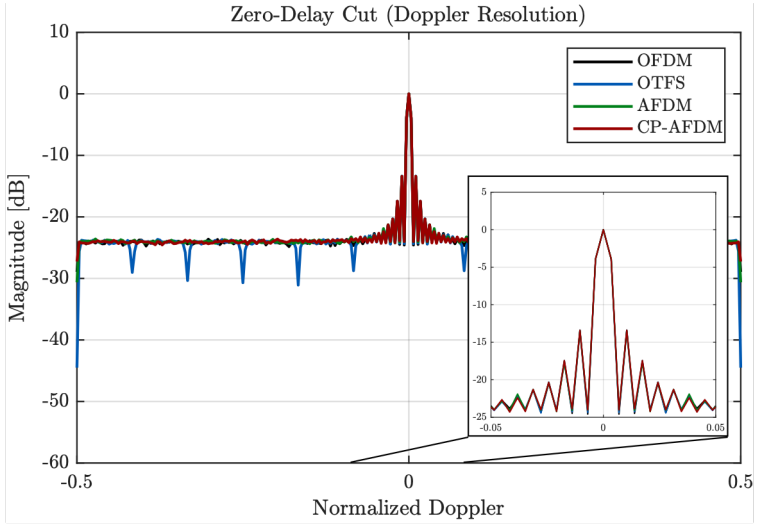

- Equivalence to **FMCW** under parametrisation
- Reusability of radar hardware/DSP chains (dechirp, range/Doppler FFT, CFAR)


- *J. Zhu et al., "ISAC With Affine Frequency Division Multiplexing: An FMCW-Based Signal Processing Perspective," in IEEE Trans. Wireless Commun., vol. 25, pp. 19725-19739, 2026*
- *H. S. Rou, G. T. F. de Abreu, ``Normalized Ambiguity Function Characteristics of OFDM, OTFS, AFDM, and CP-AFDM for ISAC,'' Proceedings of the IEEE ICC, 2026.*

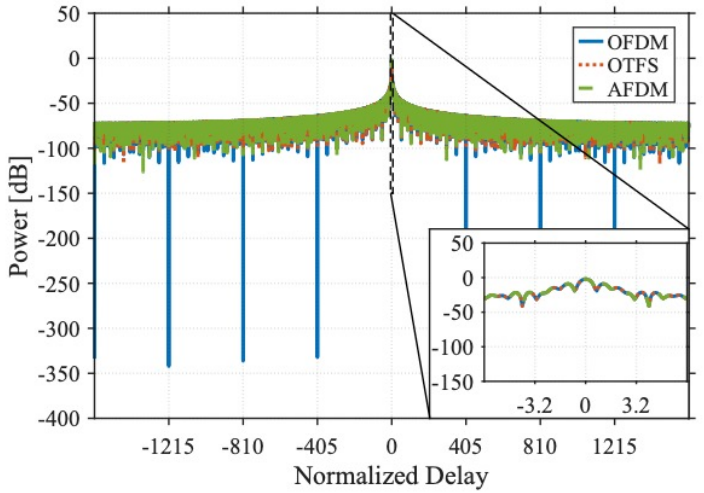
# Implications for ISAC in DD Channels

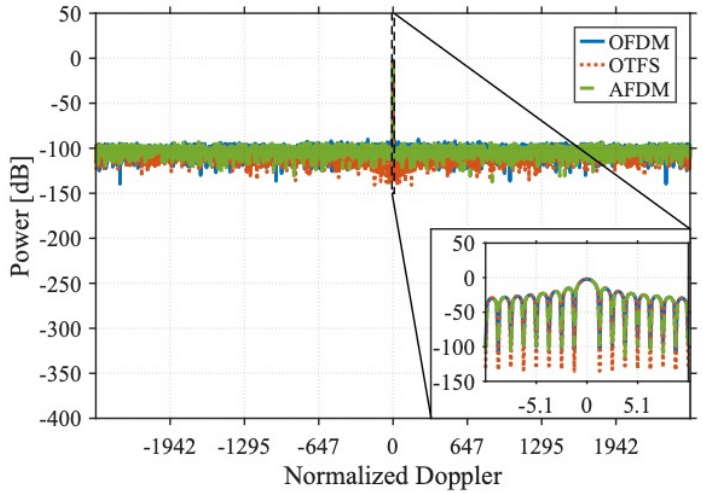
**Random symbol:** Resolutions are identical across waveforms




**Slow-time radar:**



# Implications for ISAC in DD Channels

## What exactly is AFDM's advantage for sensing?

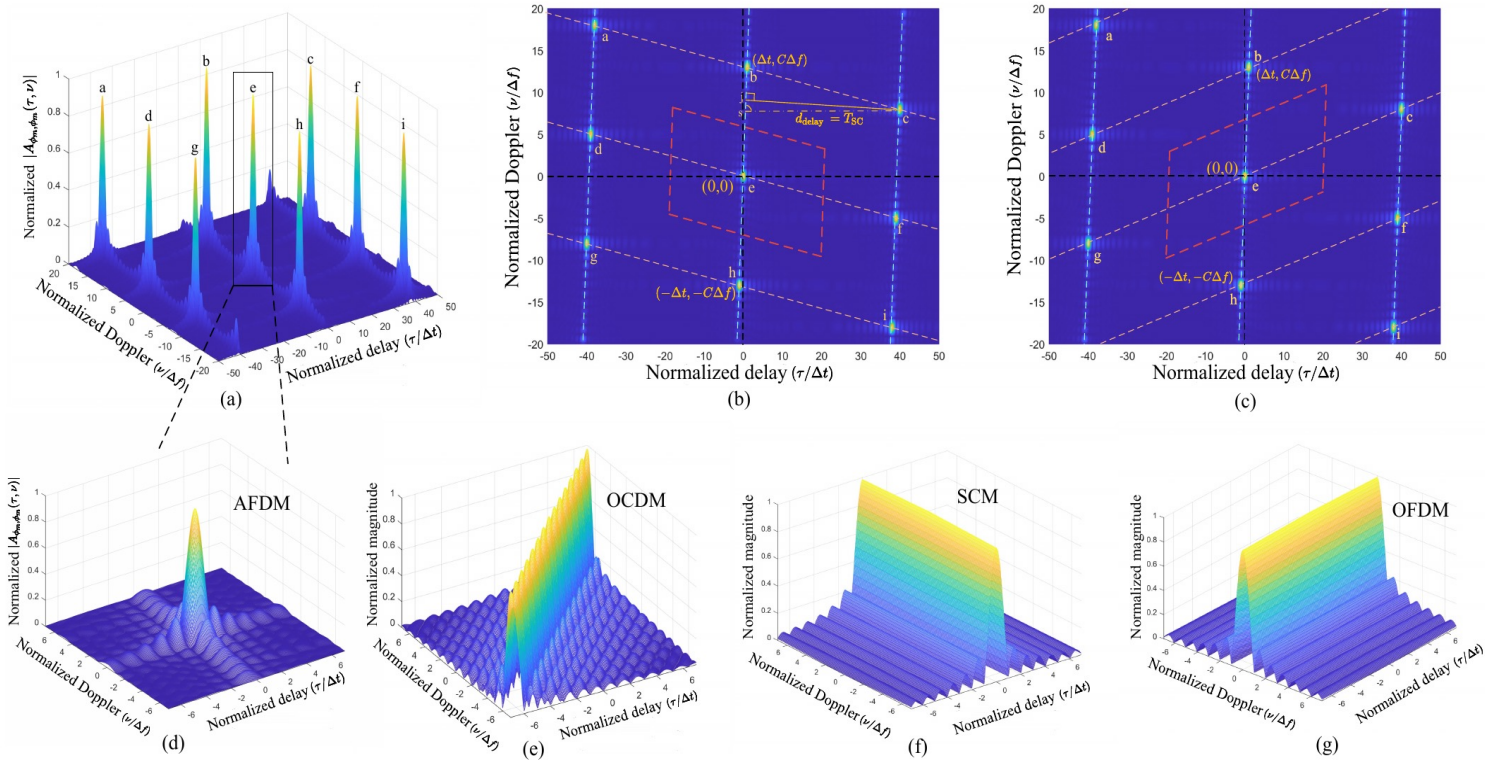
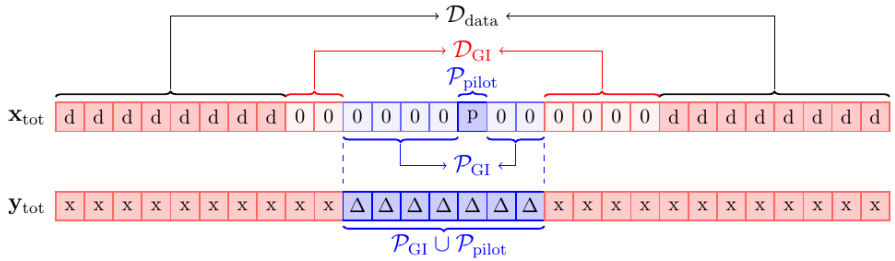



Fig. 3. Subcarrier/symbol AAFs of different waveforms, $N = 512$, $C = 13$, $m = 47$: (a) AAF of the AFDM chirp subcarrier; (b) Planform of the AFDM chirp subcarrier AAF; (c) Planform of the AFDM chirp subcarrier AAF with another type of unambiguity parallelogram; (d) Zoomed AAF of the AFDM chirp subcarrier; (e) AAF of the OCDM chirp subcarrier; (f) AAF of the SCM symbol; (g) AAF of the OFDM subcarrier.

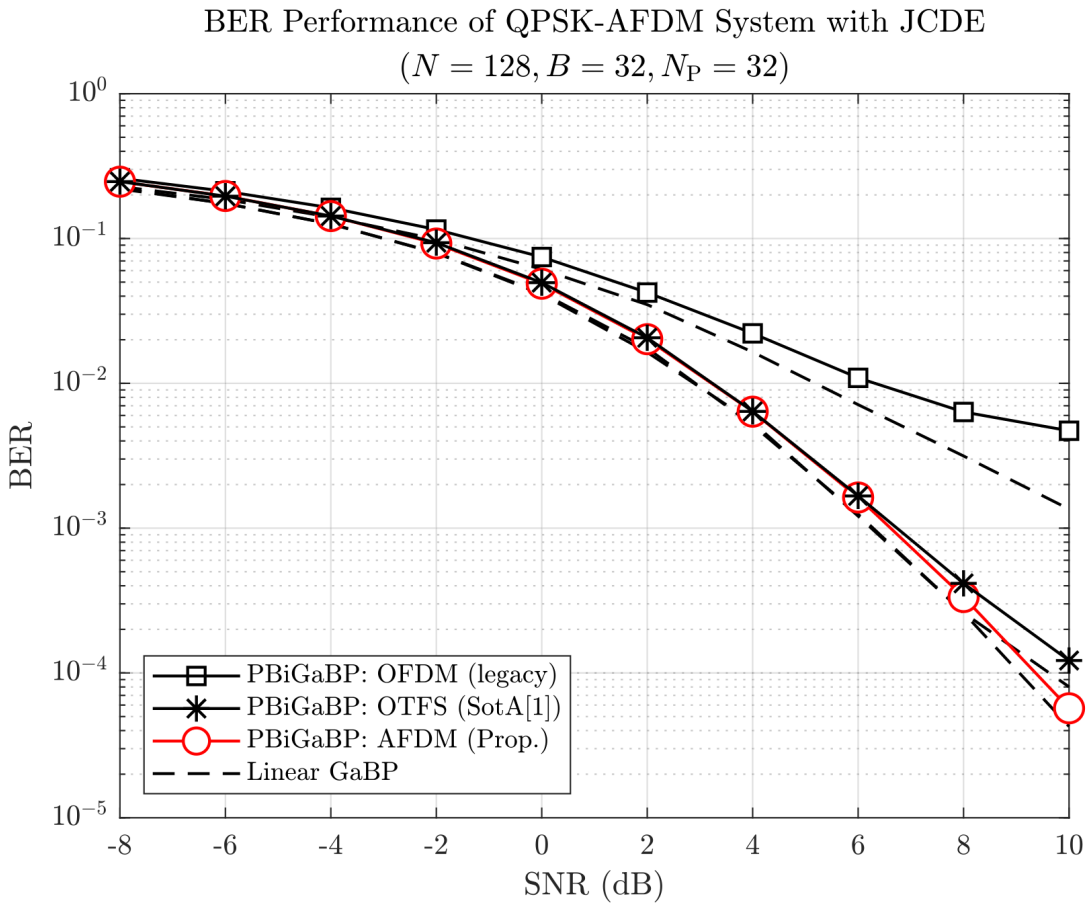


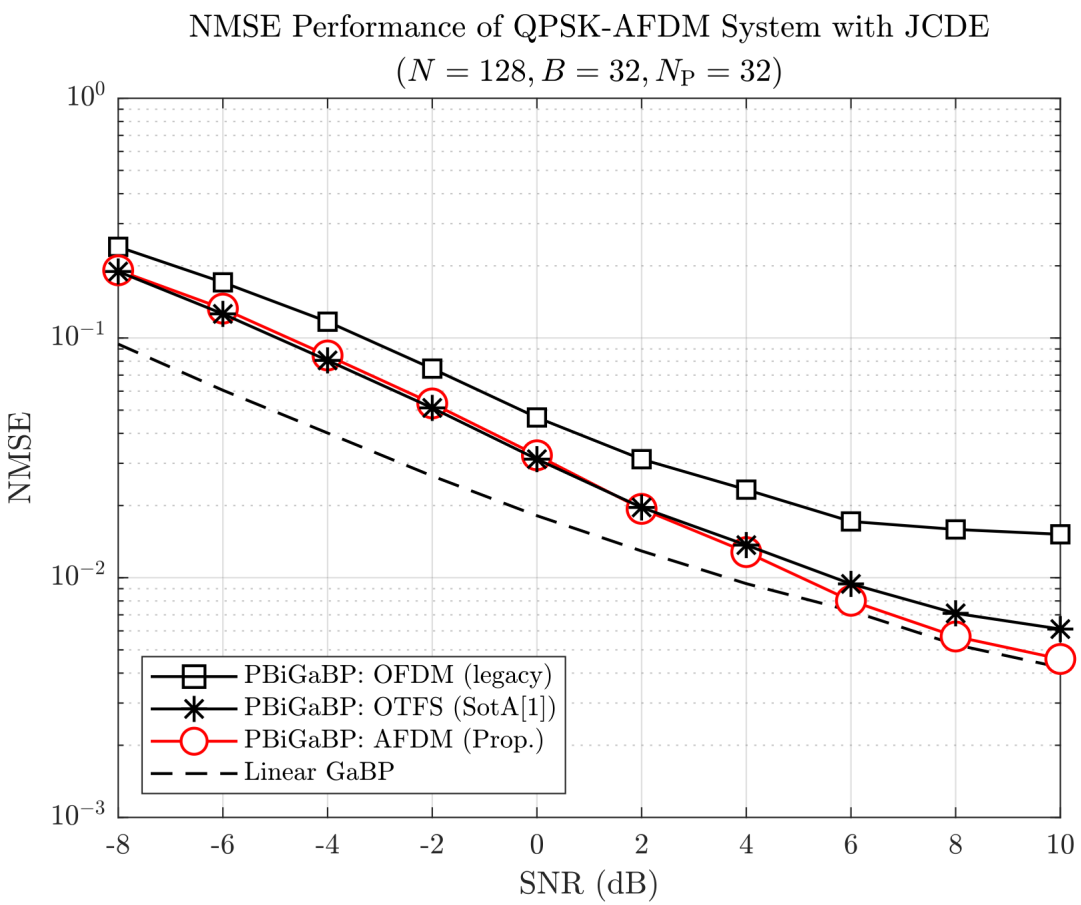

(a) Monostatic sensing with one pilot multiplexed with data


- *A. Bemani, N. Ksairi, and M. Kountouris, "Affine Frequency Division Multiplexing: From Communication to Sensing," arXiv:2511.04471, Nov. 2025.*
- *H. Yin et al., "Ambiguity Function Analysis of AFDM Signals for Integrated Sensing and Communications," IEEE J. Sel. Areas Commun., vol. 44, pp. 196–211, 2026.*

# Further Implications for ISAC in DD Channels

Not only ISAC, but also **joint channel estimation** and **radar parameter estimation** and **data detection:**



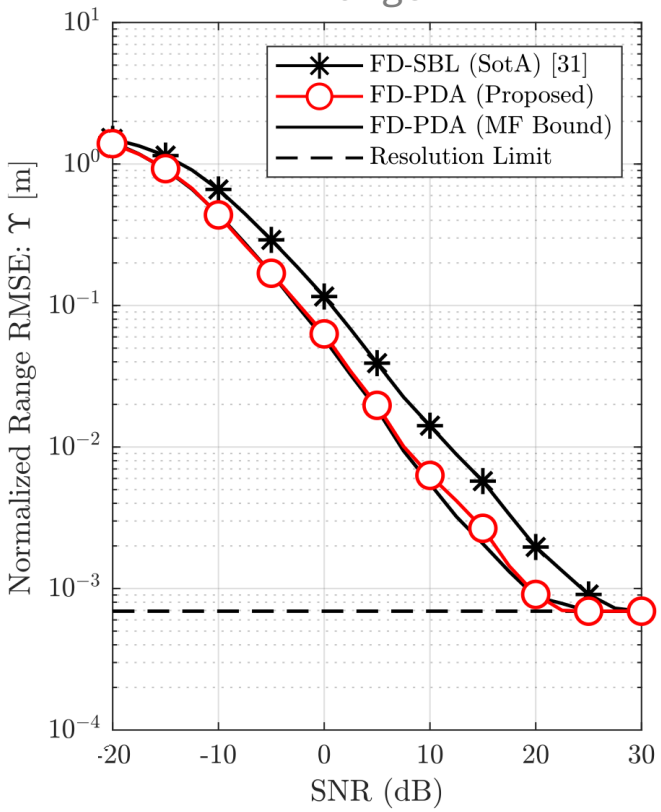


- A. Bemani, N. Ksairi and M. Kountouris, "Integrated Sensing and Communications With Affine Frequency Division Multiplexing," in IEEE Wireless Communications Letters, vol. 13, no. 5, pp. 1255-1259, May 2024
- K. R. R. Ranasinghe, K. Ando, H. S. Rou, G. T. F. de Abreu and A. Bathelt, "Blind Bistatic Radar Parameter Estimation in Doubly-Dispersive Channels," 2025 IEEE WCNC, Milan, Italy, 2025

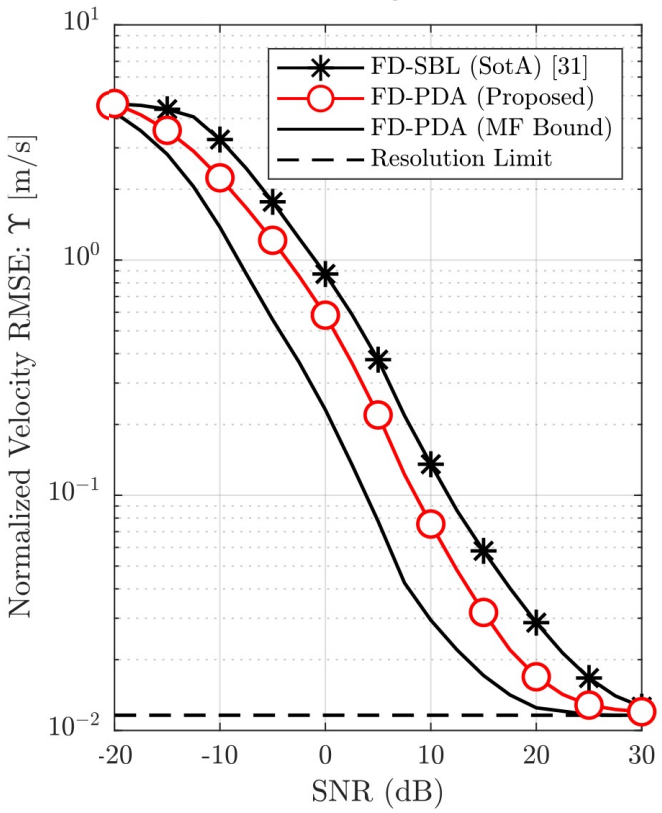
# Further Implications for ISAC in DD Channels

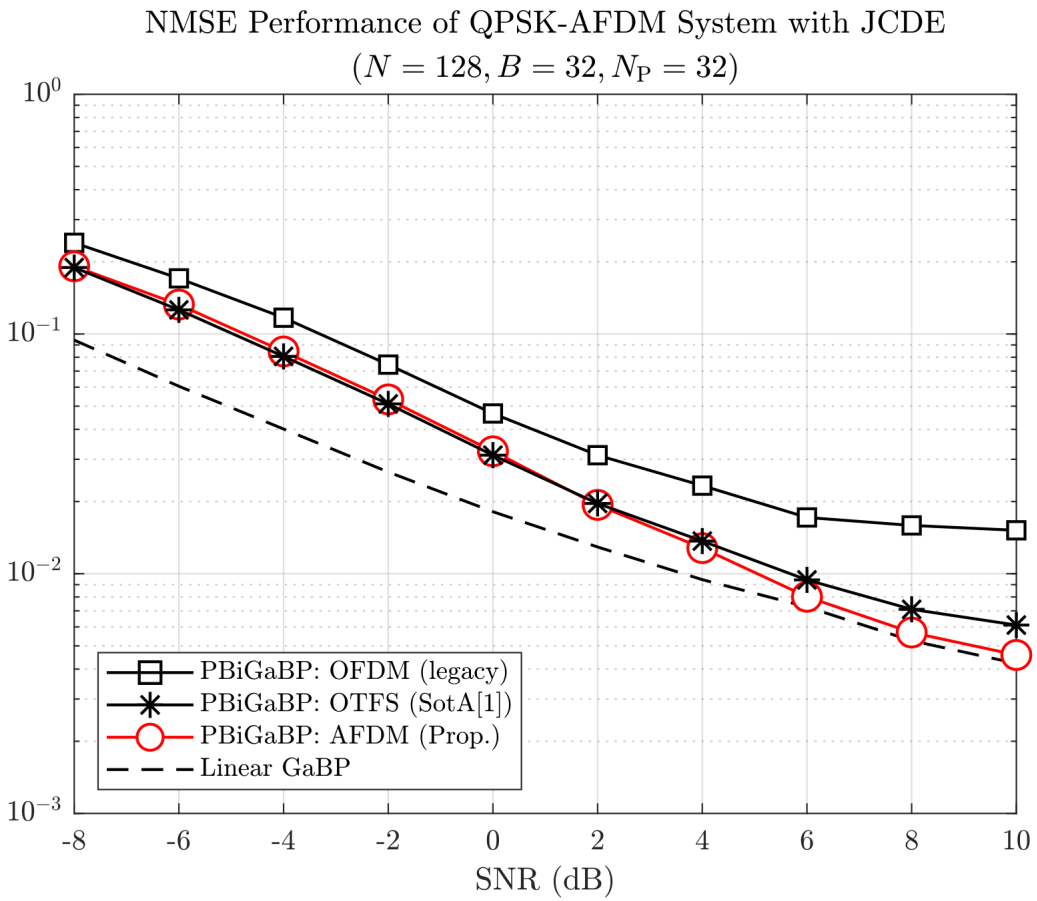
Not only ISAC, but also **joint channel estimation** and **radar parameter estimation** and **data detection:**

*Range*




*Velocity*



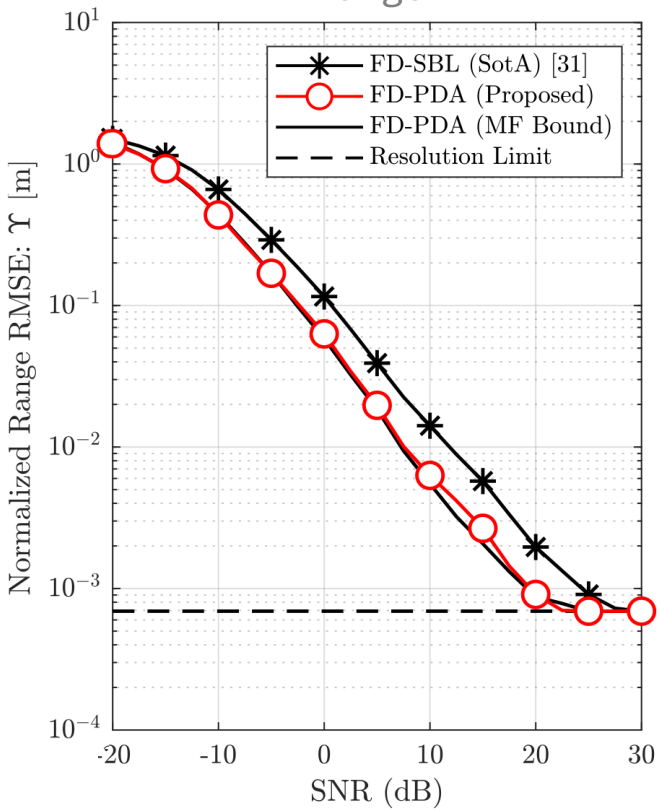


- A. Bemani, N. Ksairi and M. Kountouris, "Integrated Sensing and Communications With Affine Frequency Division Multiplexing," in IEEE Wireless Communications Letters, vol. 13, no. 5, pp. 1255-1259, May 2024
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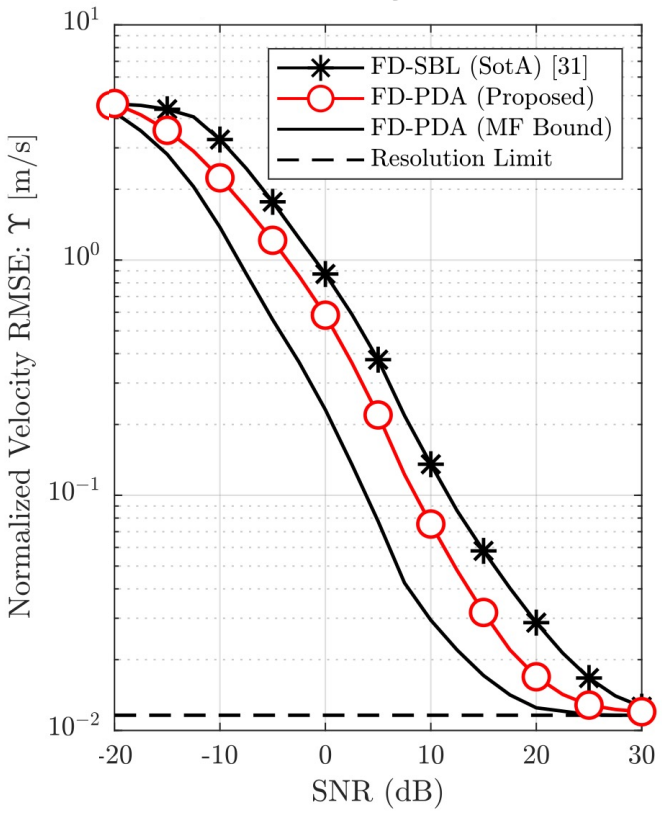
# Further Implications for ISAC in DD Channels

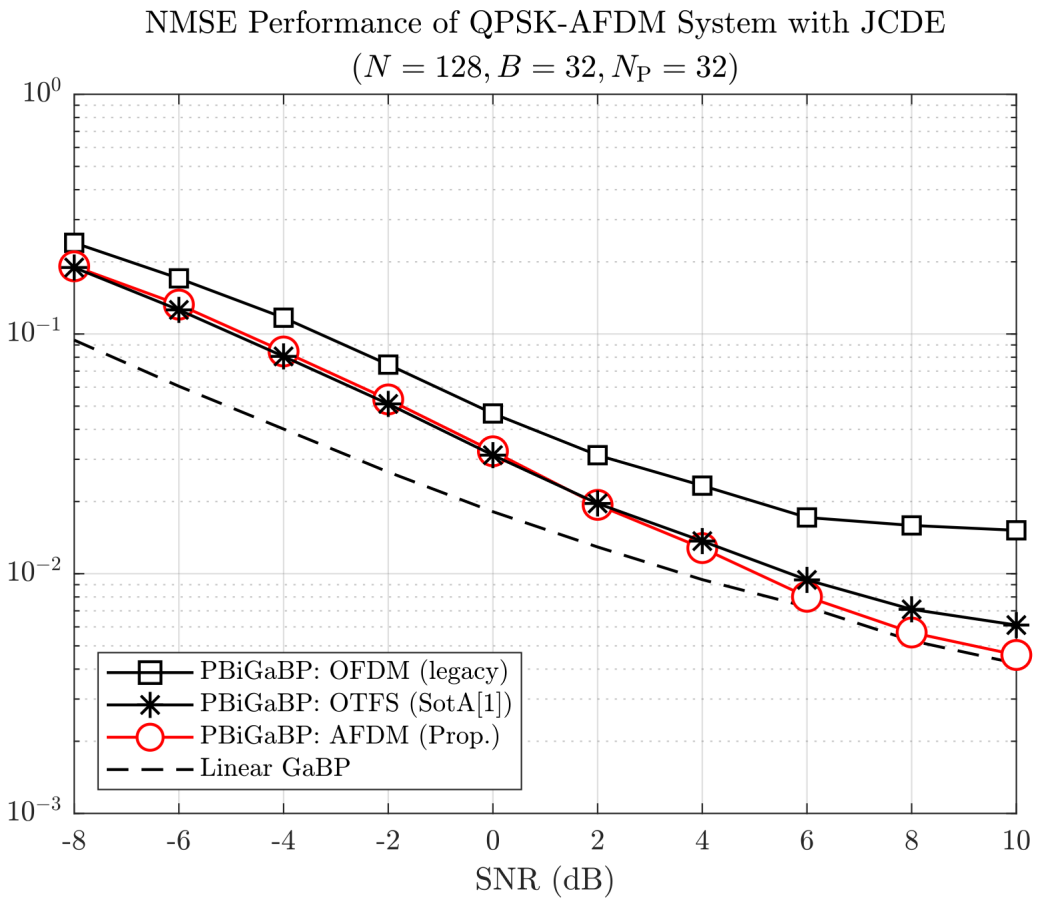
Not only ISAC, but also **joint channel estimation** and **radar parameter estimation** and **data detection:**

*Range*




*Velocity*



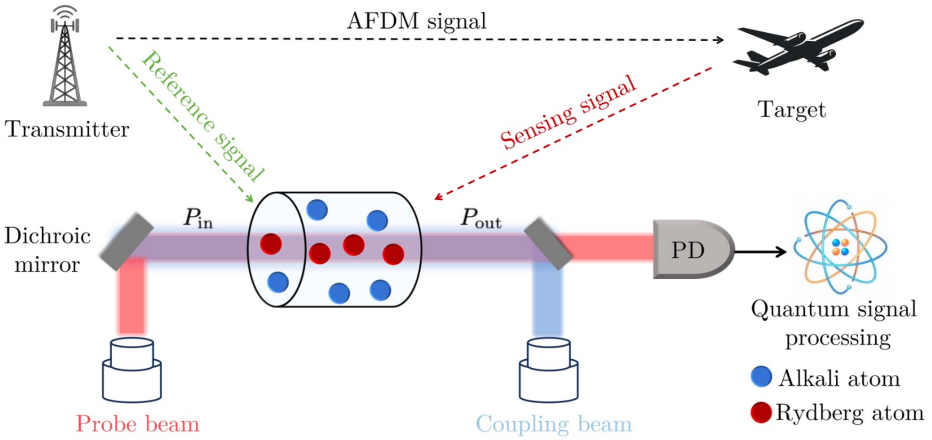



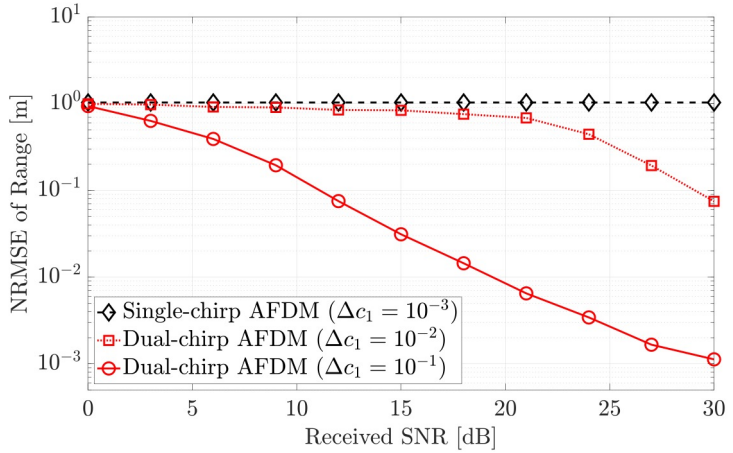


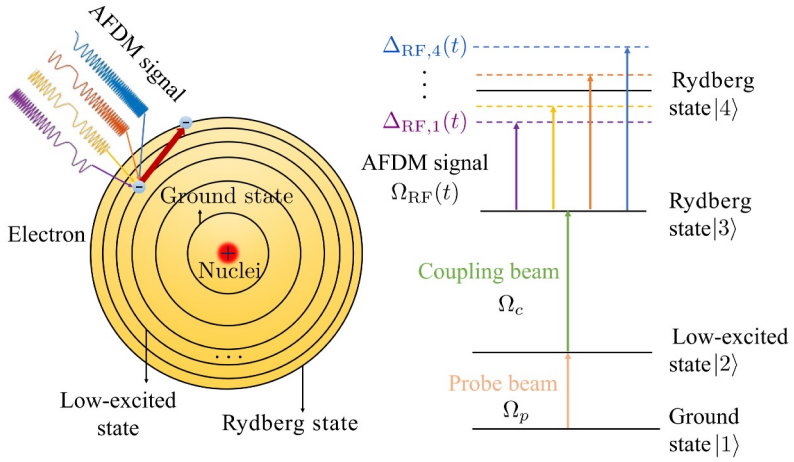
- A. Bemani, N. Ksairi and M. Kountouris, "Integrated Sensing and Communications With Affine Frequency Division Multiplexing," in IEEE Wireless Communications Letters, vol. 13, no. 5, pp. 1255-1259, May 2024
- K. R. R. Ranasinghe, K. Ando, H. S. Rou, G. T. F. de Abreu and A. Bathelt, "Blind Bistatic Radar Parameter Estimation in Doubly-Dispersive Channels," 2025 IEEE WCNC, Milan, Italy, 2025

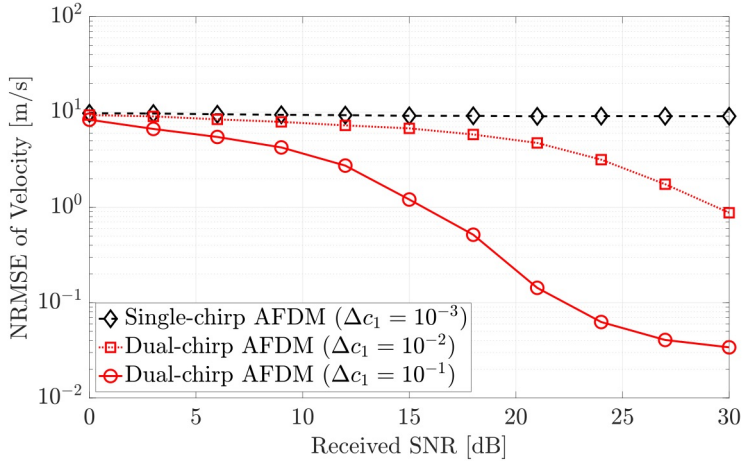
# Further Implications for ISAC in DD Channels

AFDM was shown to enable **joint delay-Doppler** estimation for **Quantum Rydberg Atomic Receiver-based ISAC!**



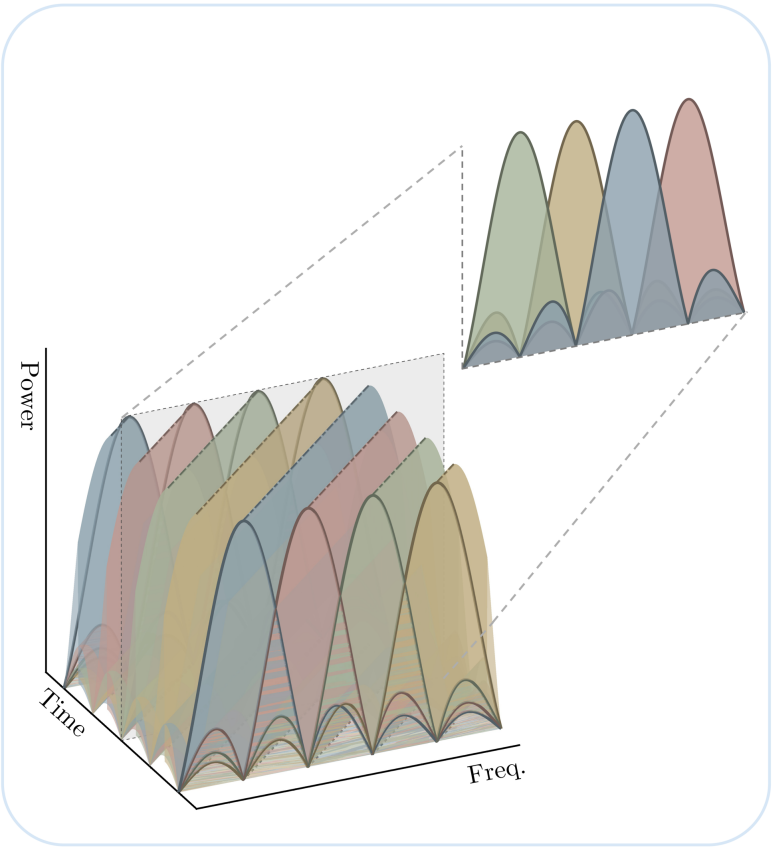


- H. Kim, H. S. Rou, K. Min, G. T. F. de Abreu and S. Kim, "Multi-Chirp AFDM for Rydberg Atomic Quantum Receivers: Waveform and Algorithm Design", submitted to IEEE Trans. Wireless Commun., 2026
- M. Chen et al., "New Paradigm for Integrated Sensing and Communication with Rydberg Atomic Receiver," in IEEE Communications Magazine, vol. 63, no. 12, pp. 104-111, December 2025

# Other Interesting Features of AFDM

In AFDM, the ***joint time-frequency domain spreading*** for **every subcarrier** also implies:

- Robustness against various **hardware impairments / non-idealities**:

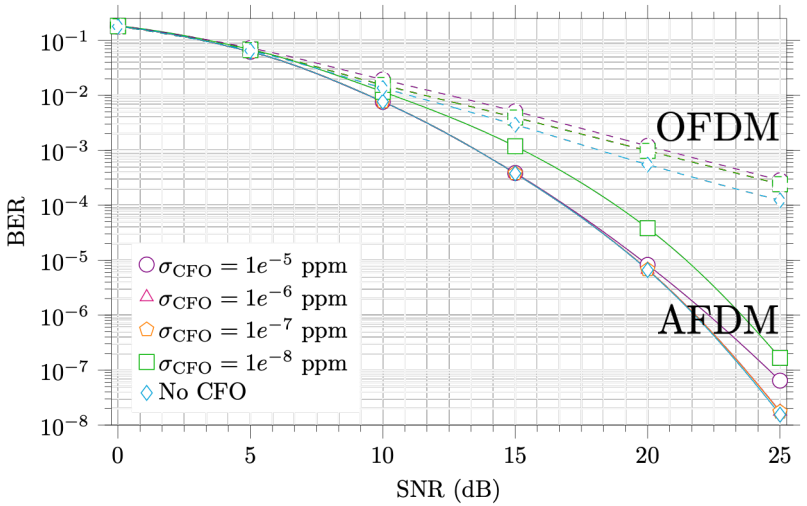


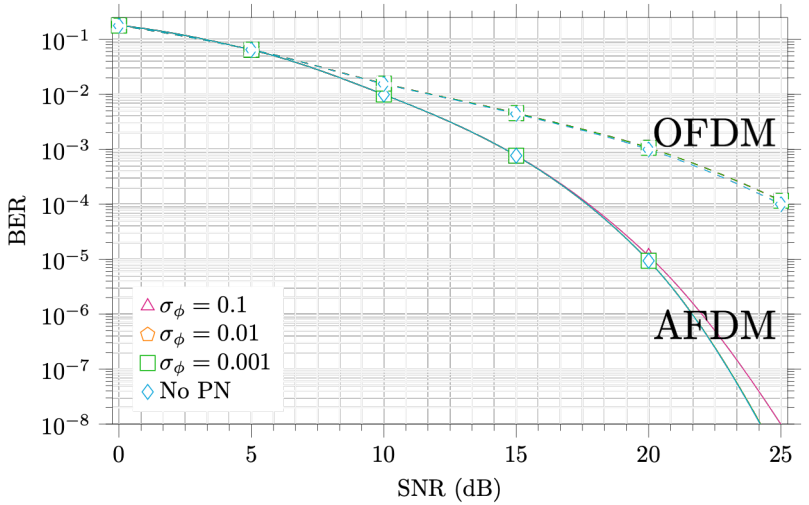

- M. Mirabella, H. S. Rou, P. Di Viesti, G. T. F. de Abreu, and G. M. Vitetta, "*Continuous-Time Analysis of AFDM: Pulse-Shaping, Fundamental Bounds and Impact of Hardware Impairments.*" *Submitted to the Journal on Selected Areas in Communications, preprint available arXiv:2602.20909 , 2026*
- *V. Savaux, S. Sawadogo, H. S. Rou and G. T. F. de Abreu, "On the Noise Robustness of Affine Frequency Division Multiplexing: Analysis and Applications," 2025 4th International Conference on 6G Networking (6GNet), Paris, France, 2025.*

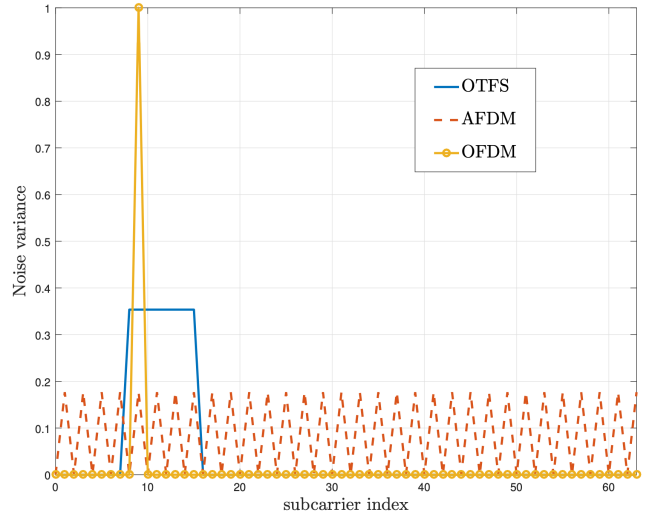
# Other Interesting Features of AFDM

In AFDM, the ***joint time-frequency domain spreading*** for **every subcarrier** also implies:

- Robustness against various **hardware impairments / non-idealities**:
  - ***Carrier frequency offset (CFO)***
  - ***Phase noise (PN)***
  - ***Sampling Jitter (SJ)***
  - ***Coloured and impulse noise***
  - ***Jamming***
  - ***etc..***



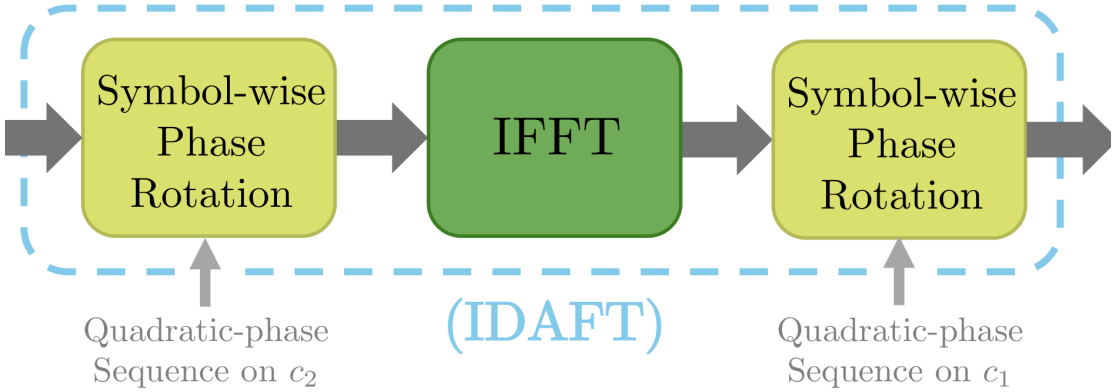


- M. Mirabella, H. S. Rou, P. Di Viesti, G. T. F. de Abreu, and G. M. Vitetta, "*Continuous-Time Analysis of AFDM: Pulse-Shaping, Fundamental Bounds and Impact of Hardware Impairments.*" *Submitted to the Journal on Selected Areas in Communications, preprint available arXiv:2602.20909 , 2026*
- *V. Savaux, S. Sawadogo, H. S. Rou and G. T. F. de Abreu, "On the Noise Robustness of Affine Frequency Division Multiplexing: Analysis and Applications," 2025 4th International Conference on 6G Networking (6GNet), Paris, France, 2025.*

# Other Interesting Features of AFDM

Based on the **chirp-parameter flexibility** a plethora of works have been proposed..

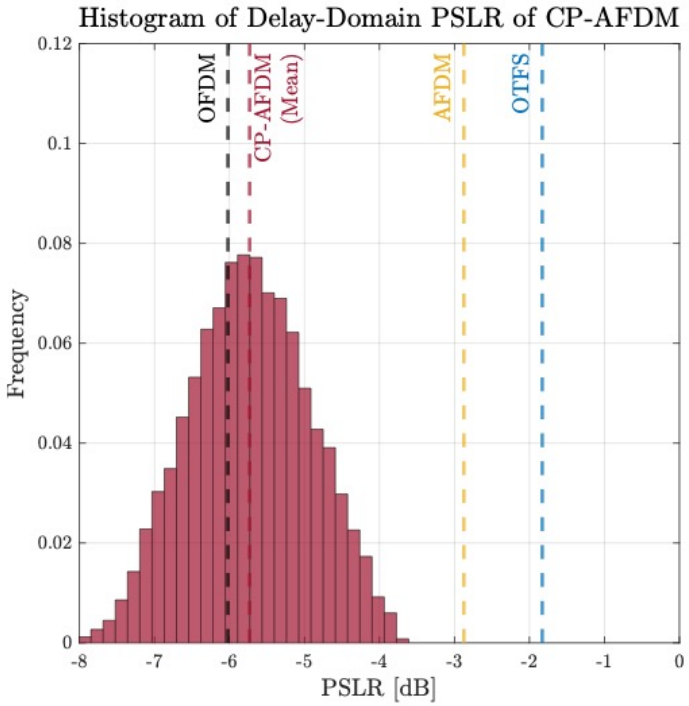
- ***Integrated sensing and communications (ISAC)***
- ***Physical layer security (PHY-Sec)***
- *Ambiguity function shaping*
- *PAPR reduction*
- *PSLR reduction*
- *Chirp-parameter hopping*
- *Index modulation*
- *Multiple access*
- *and more..*



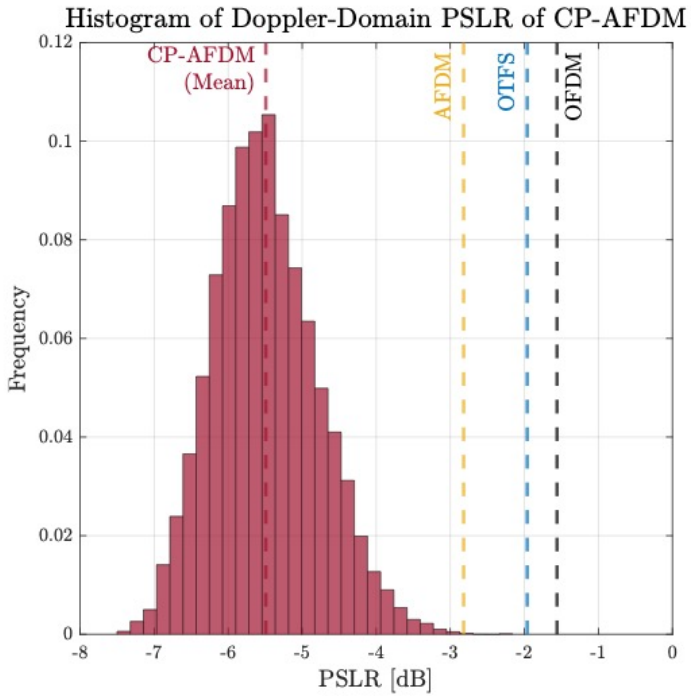

All of these secondary design space by simply tuning / modifying the chirp-rotation parameters **(two scalars!!!)**

# Example Variant: Chirp-permuted (CP) AFDM

For example, chirp-permuted (CP-AFDM) has been shown to enable PSLR improvements:




(a) Delay-domain PSLR.




(b) Doppler-domain PSLR.

*- H. S. Rou, H. Kim, G. T. F. de Abreu, S. Kim, "Chirp-Permuted AFDM: A Versatile Waveform Design for ISAC in 6G", submitted to IEEE Trans. Wireless Commun. 2026*

# Example Variant: Chirp-permuted (CP) AFDM

Notice that at the receiver, we must "demodulate" the signal via the ***N*-point DAFT**

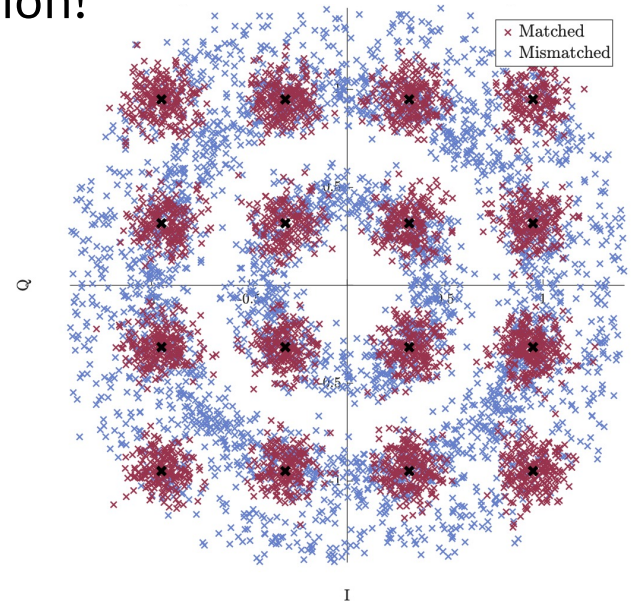
$$\mathbf{A}^{\mathsf{H}} = \mathrm{diag}(\boldsymbol{\lambda}_{c_1}^{*}) \cdot \mathbf{F}_N^{\mathsf{H}} \cdot \mathrm{diag}(\boldsymbol{\lambda}_{c_2}^{*}) \in \mathbb{C}^{N \times N}$$

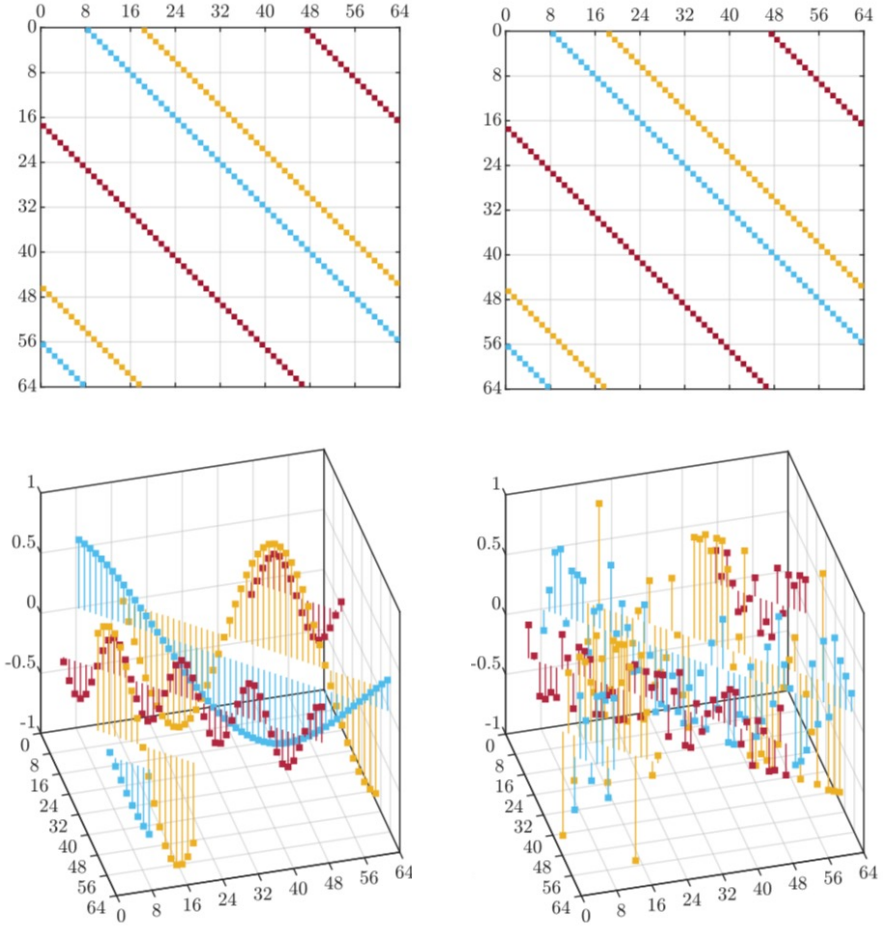
$$\mathbf{A} = \mathrm{diag}(\boldsymbol{\lambda}_{c_2}) \cdot \mathbf{F}_N \cdot \mathrm{diag}(\boldsymbol{\lambda}_{c_1}) \in \mathbb{C}^{N \times N}$$

The two chirp parameters must **match between Tx and Rx** to ensure correct demodulation!

If not, uncontrolled **erroneous phase** remnants appear!!
**(inherent PHY-sec)**

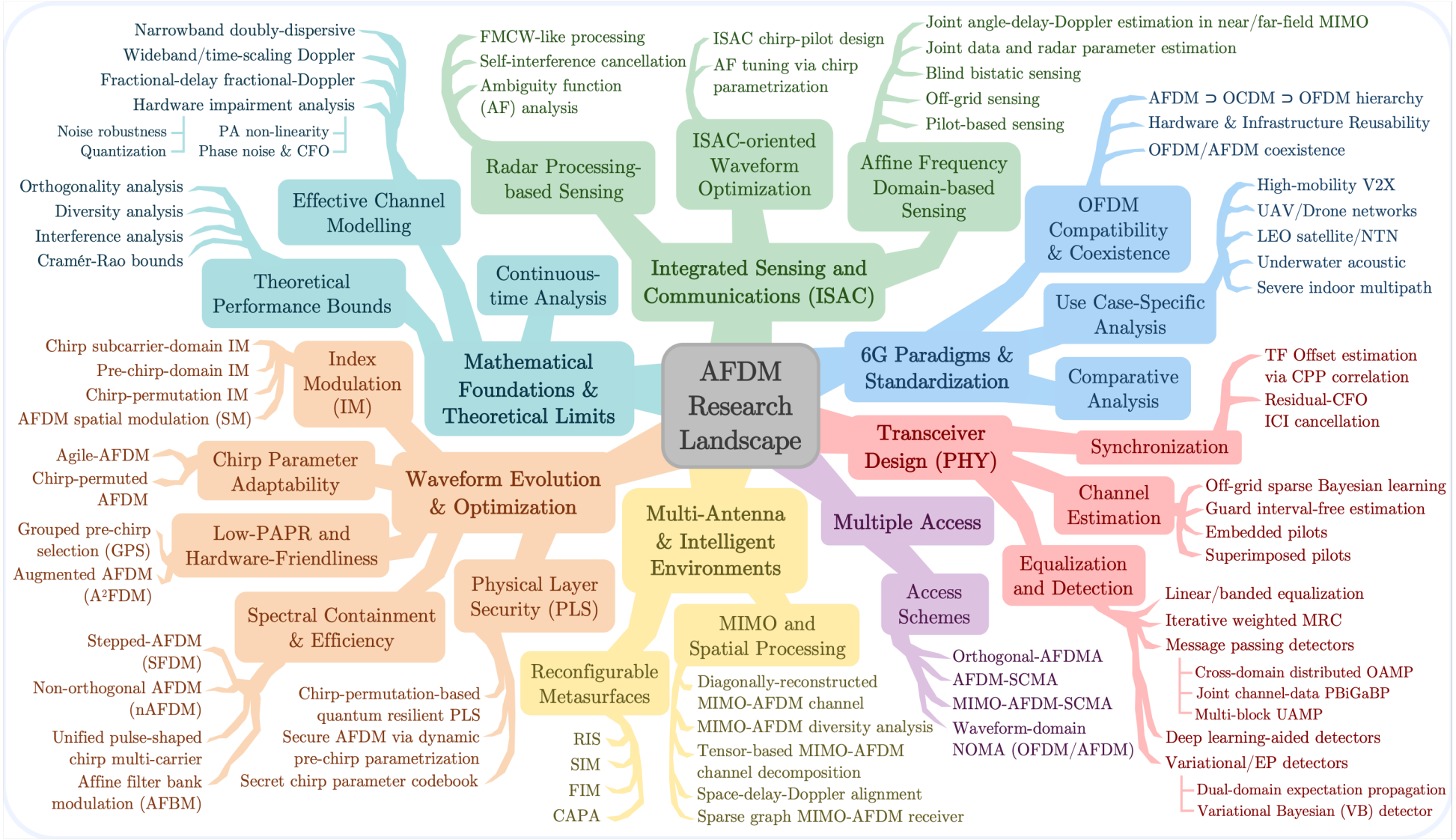
***(See live demo later!!)***



*- H. S. Rou and G. T. F. de Abreu, "Chirp-Permuted AFDM for Quantum-Resilient Physical-Layer Secure Communications," in IEEE Wireless Communications Letters, vol. 14, no. 8, pp. 2376-2380, Aug. 2025.*

# And Much More...




*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

# And Much More...

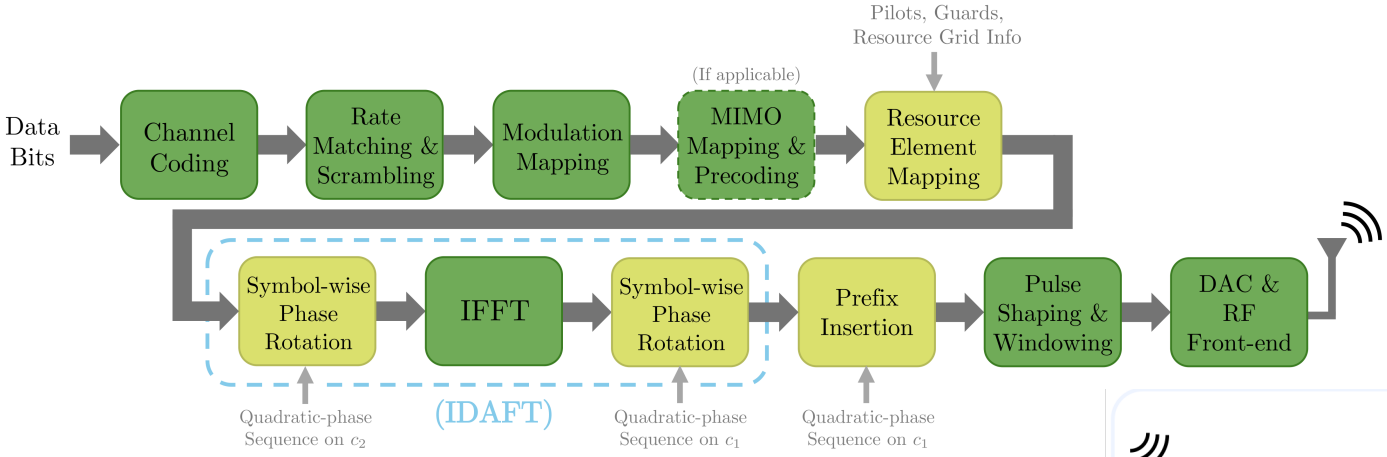


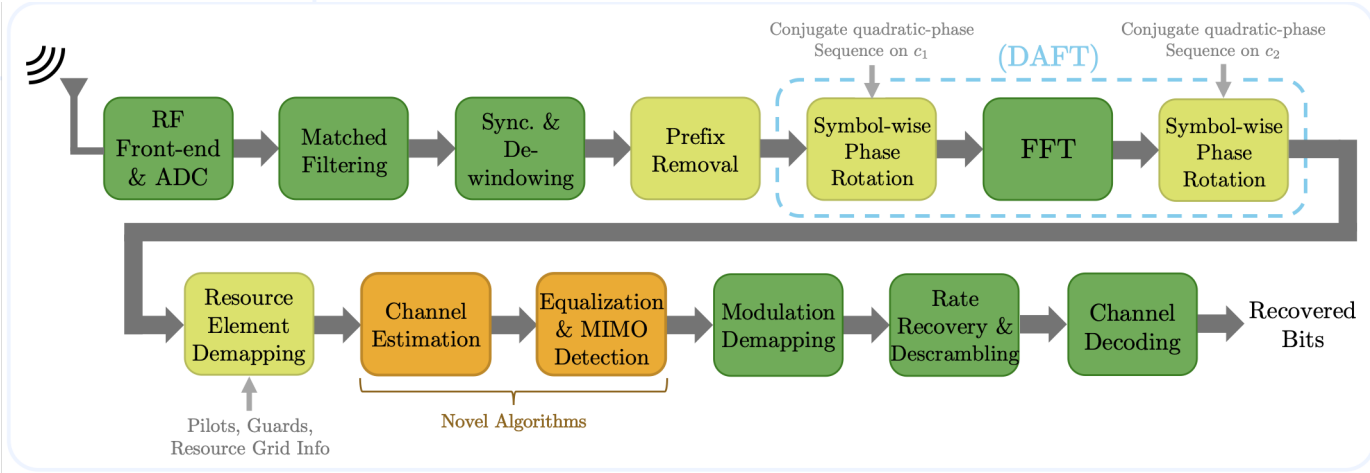

*- H. S. Rou, G. T. F. de Abreu, E. Björnson, S. Kim, and M. Kountouris, "The resurrection of spectrum spreading for 6G and beyond: From sinusoids to chirps," submitted to IEEE Wireless Commun. Mag., Apr. 2026. arXiv:2605.00249.*

# Implementation of AFDM

Aside from the “element-wise” rotations, essentially **everything else remains the same as OFDM!**

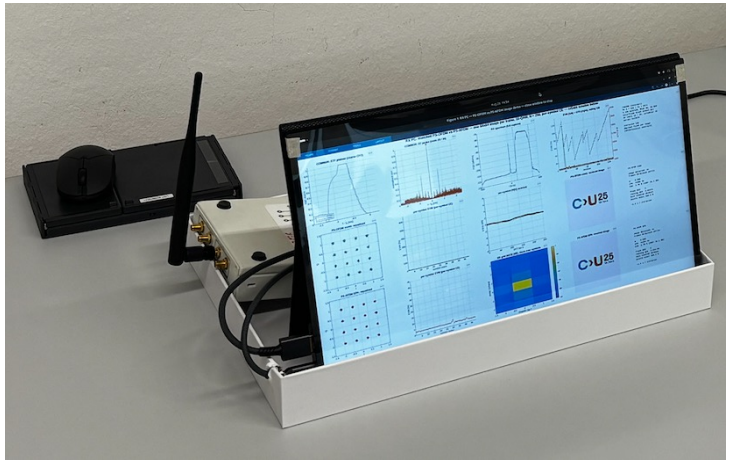
AFDM is a ***“software upgrade”*** of the OFDM legacy (efficient adoption & hardware)

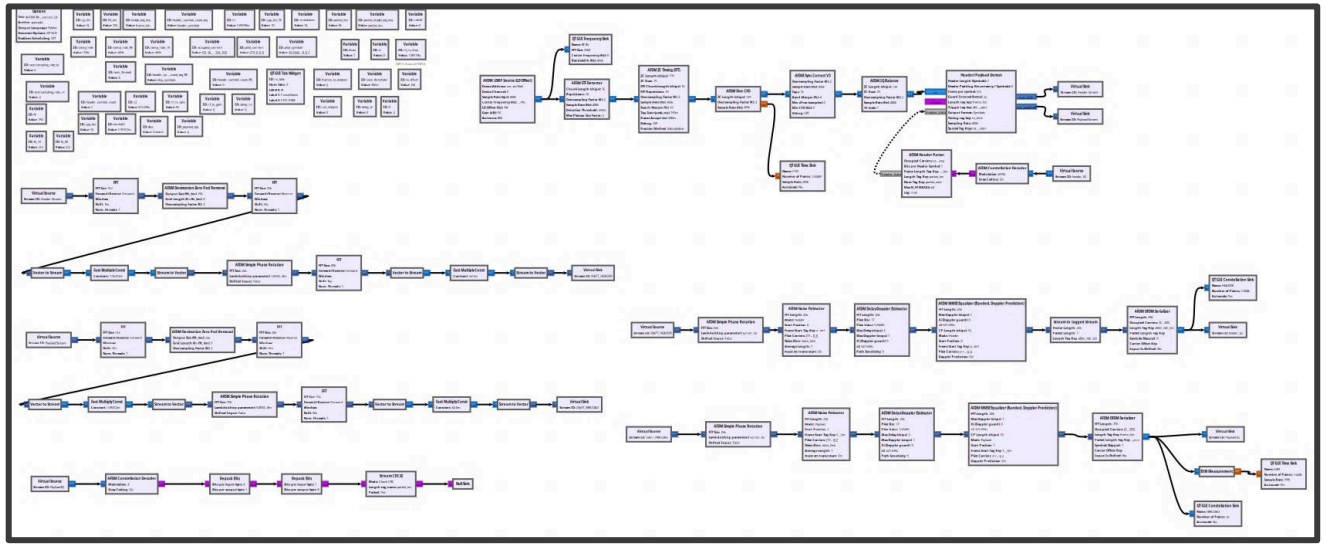


# Implementation of AFDM







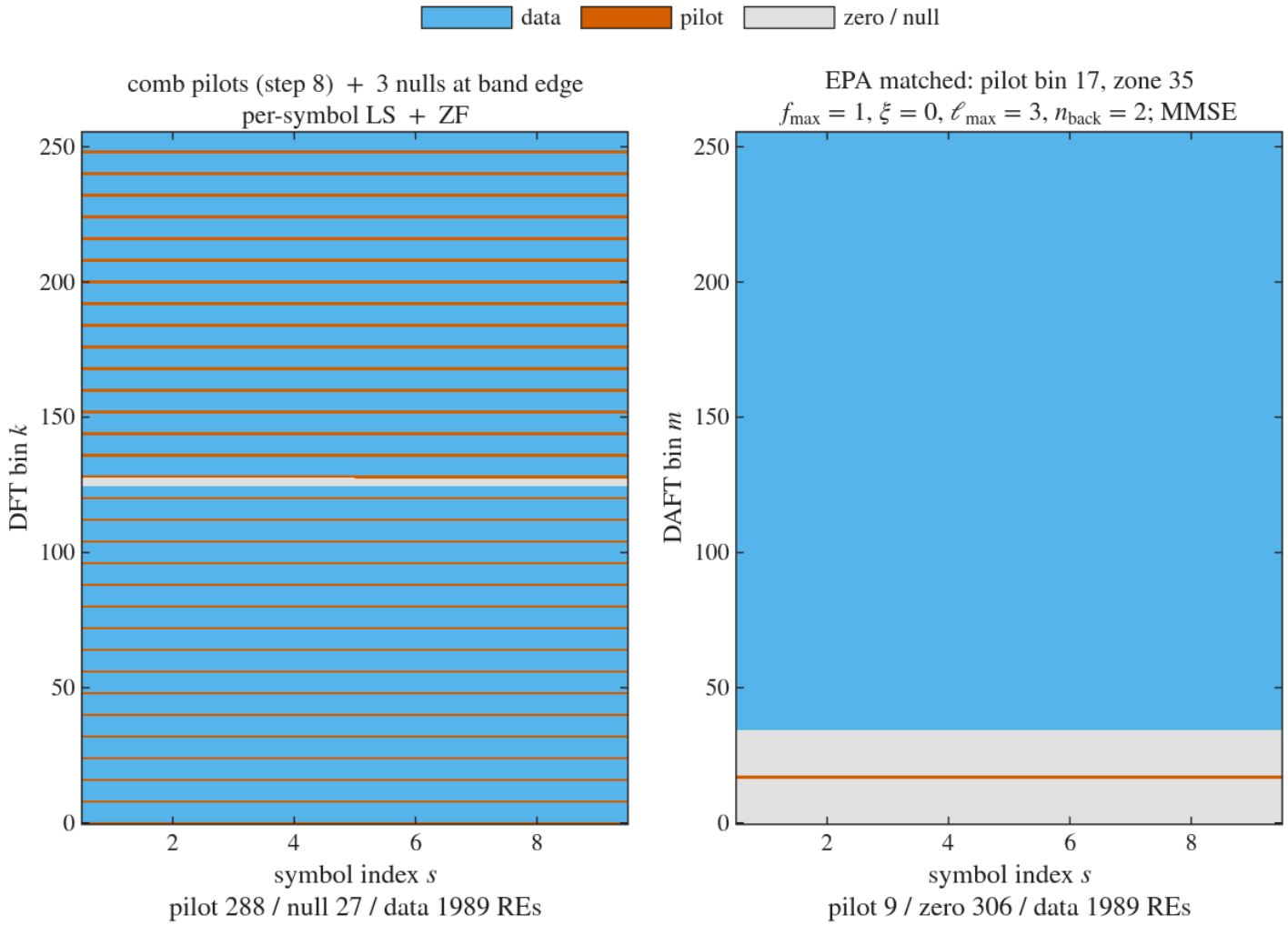
## Full Pulse-Shaped OFDM & AFDM

Running live on two independent USRPs

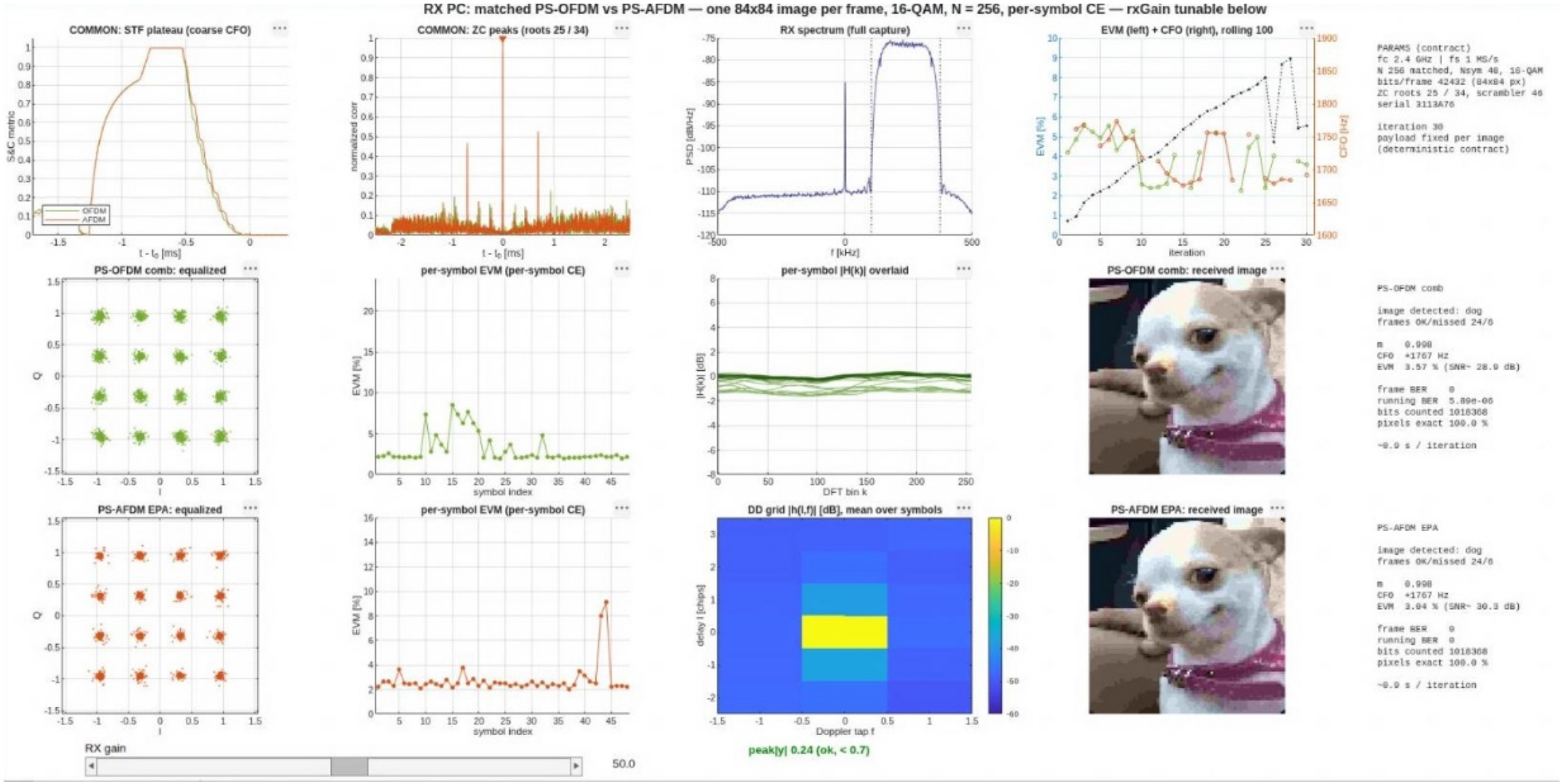
- Full synchronisation (STF + Zadoff-Chu)
- Full CFO correction
- Full channel estimation
- Full detection/demodulation

- *USRP B210s*
- *UHD library for USRP*
- *Matlab 2026 with Communications Toolbox, DSP System Toolbox and Signal Processing Toolbox, Communications Toolbox Support Package for USRP Radio*
- *GNURadio 3.10.12.0 (Python 3.14.4) with custom C++ blocks.*
- *GNURadio 200k chip/s, 200kHz BW,16QAM*

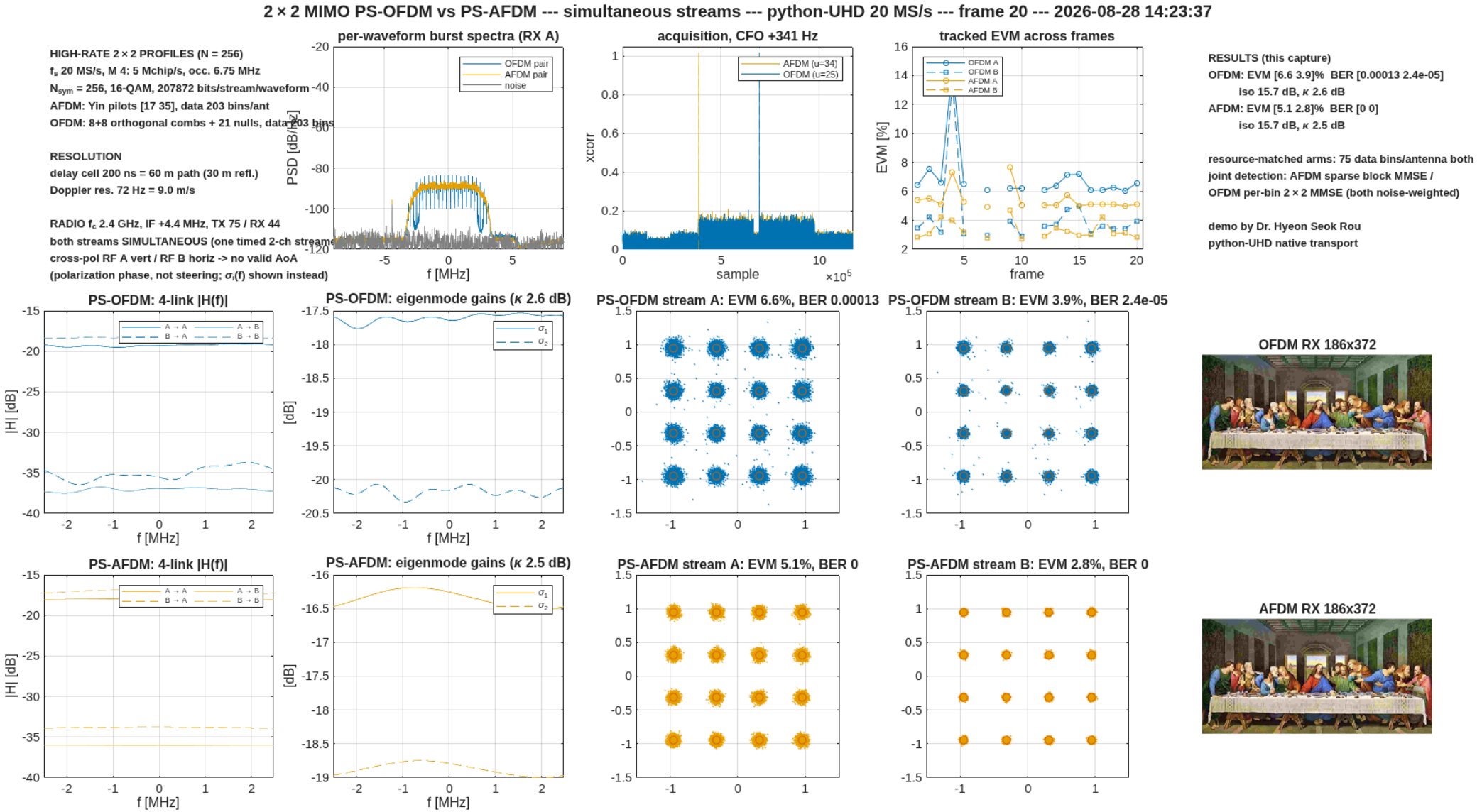
# Implementation of AFDM



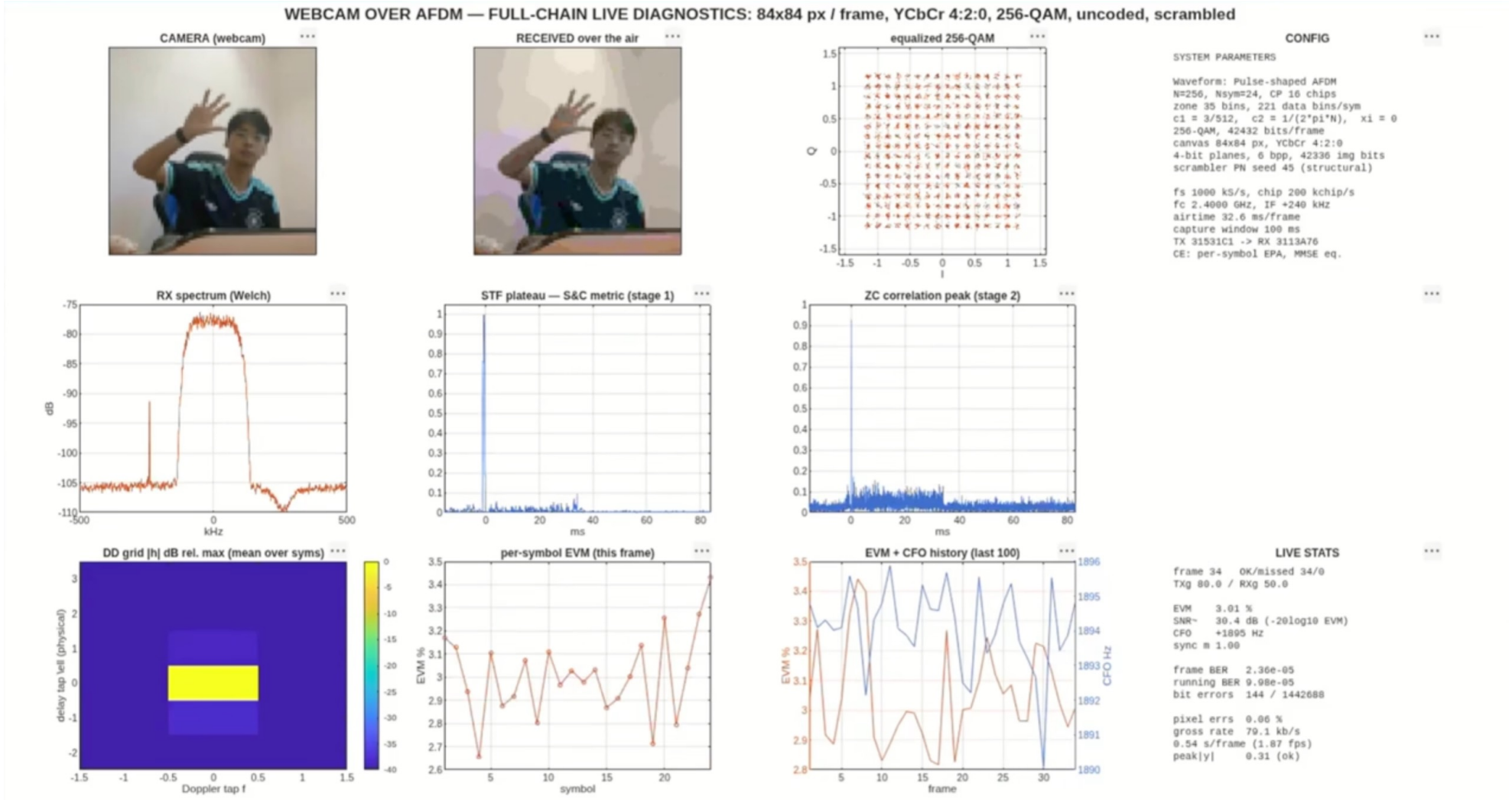
# Implementation of AFDM



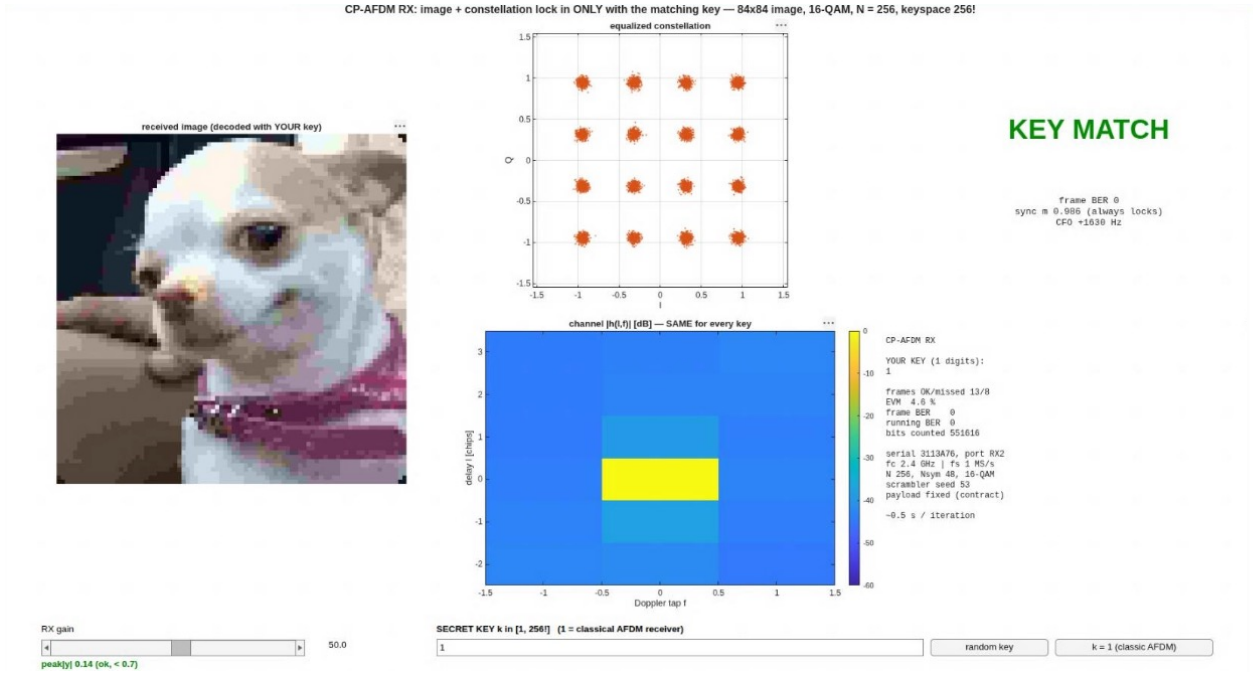
# Implementation of AFDM

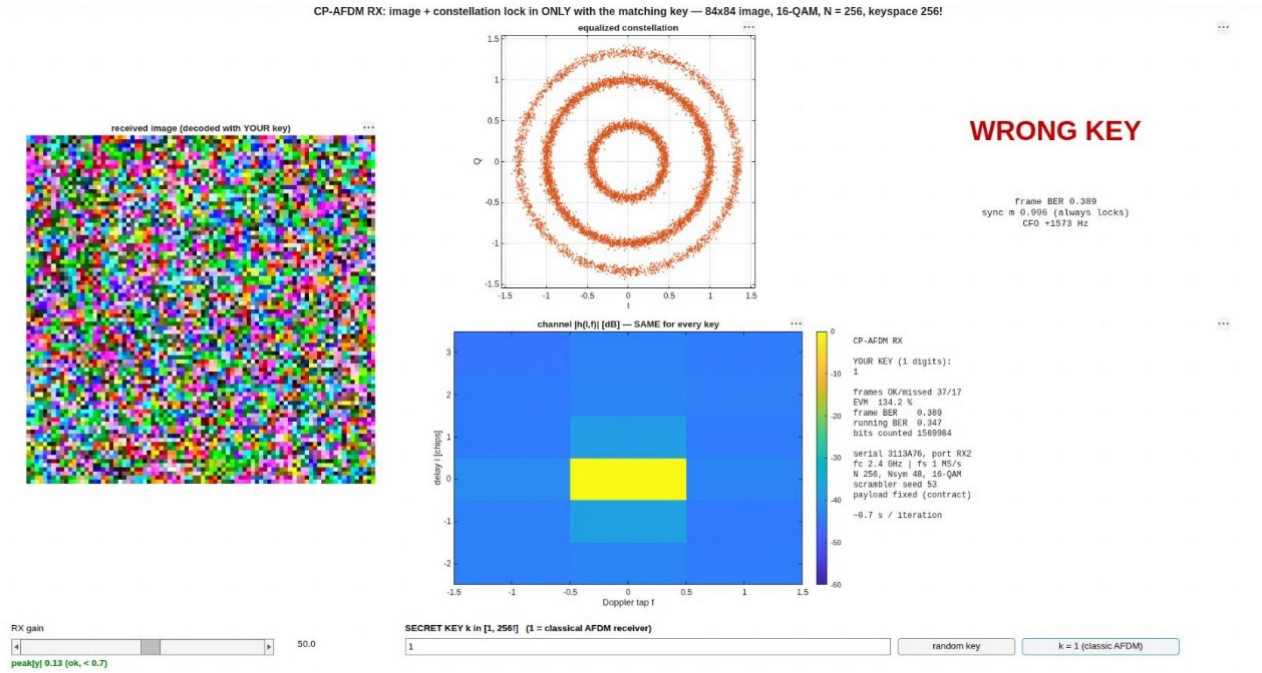


# Implementation of AFDM



# Implementation of AFDM



# Implementation of AFDM

**Let's see it in action!**

*(Interactive demo session)*

# Q&A

## Thank you!

Any questions?

## Next-Generation Chirp-based Waveforms for ISAC in 6G and Beyond:

*Fundamentals, Models, and Algorithms (with Live Demonstrations!)*




**Dr. Hyeon Seok Rou**
KAIST, South Korea



**Prof. Giuseppe T. F. de Abreu**
Constructor University, Germany



**Prof. Marios Kountouris**
University of Granada, Spain / EURECOM, France