Fading Multiaccess Channels in the Wideband Regime: the Impact of Delay Constraints

D. Tuninetti and G. Caire
Institut Eurécom
Sophia-Antipolis, France
e-mail: name@eurecom.fr

S. Verdú
Princeton University
Princeton, NJ, 08544 USA
e-mail: verdu@ee.princeton.edu

Abstract — **We consider a multiaccess Gaussian block fading channel where transmitters have causal state information. Variable-rate coding with input power constraint enforced on a per-codeword basis is examined. We find the average capacity region and average capacity region per unit energy. Moreover, we study the wideband slope of spectral efficiency vs. (E_b/N_0)dB, and we quantify the bandwidth expansion factor of TDMA over superposition coding in the wideband regime.**

I. Introduction and motivations

The literature on the capacity of fading channels has followed two distinct approaches to characterize power constraints: A) Power constraint on a per-symbol basis (averaged over the codebook); B) Power constraint on a per-codeword basis (averaged over the length of the codeword and the codebook).

Basic information theory results [1, 2] have shown that the laxer constraint B offers no advantage in unfaded channels or in fading channels where the transmitter does not know the channel. However, when the transmitter has instantaneous knowledge of the channel fading coefficients, constraint B leads to strictly larger capacity than A because it enables the use of "power control" which avoids wasting power at symbols where the channel undergoes deep fades. Under B, the optimum strategy as shown in [3] is water-filling in time. In this setting, the fading process is assumed to be stationary and ergodic and the codewords are long enough for the fading distribution to be revealed within the span of one codeword. If the fading dynamics are slow, this leads to intolerably long blocklength, and consequently delay. Furthermore, waterfilling power control leads to very large peak-to-average ratio of the transmitted waveform, in the low-SNR (or "wideband") regime.

In the high E_b/N_0 high spectral efficiency regime, constraints A and B, although leading to different optimum transmission strategies, achieve very similar single-user capacity. Only in conjunction with multiaccess and multiuser detection do optimum power control strategies lead to noticeable advantages in the high SNR regime [4]. On the other hand, in the low spectral efficiency regime, constraint B enables (for fading distributions with infinite support) reliable communication with energy per bit as small as desired, in stark contrast to constraint A which requires a minimum transmitted energy per bit that is bounded away from zero. Therefore, it is natural to focus the analysis of capacity under delay constraints in the wideband regime.

Incorporating delay constraints in Shannon theoretic settings is a perennial challenge. In fading channels, it is essential to specify: 1) the duration of a codeword with respect to the fading process coherence time, and 2) the time interval on which the average input power constraint is enforced. Although vanishing error probability is unattainable unless the number of degrees of freedom grows without bound, that number grows with the product of time duration and bandwidth. Thus, in the wideband regime, an asymptotic analysis is feasible even in a setting of fixed duration codebooks.

In [5] the concept of "delay-limited" capacity region for a multiaccess fading channel is introduced. In this setting, each codeword spans a single fading state (i.e., the fading coherence time is much longer than the codeword duration) but the input power constraint is even laxer than B given above: it is defined over an arbitrarily long sequence of codewords (we shall refer to such constraint as *long-term*). The delay-limited capacity region is the set of rates which can be achieved *for all* fading states (up to a set of measure zero), subject to the long-term input constraint. In other words: the coding rates are fixed while the transmit power fluctuates.

In this paper we take a somewhat complementary point of view: we assume a block fading model where a codeword spans a *finite* number of slots, with fading constant over each slot and varying independently from slot to slot, and the power constraint is enforced on a per-codeword basis (constraint B above). However, we allow *variable rate coding* so that users can coordinate their rates in order to be always inside the fading-dependent capacity region. Here, the transmit power is fixed while the coding rates are random. Consequently, we define the *long-term average* capacity region as the set of all achievable rates averaged over an arbitrarily long sequence of codewords. Finally, we assume that the fading states are revealed *causally* to the transmitters [6, 7].

In the rest of this paper, we borrow some terminology from current wireless cellular systems. A block-fading model (admittedly, highly idealized) is assumed [8], with $L \gg 1$ dimensions per fading state. Blocks of L input symbols spanning the same fading state are referred to as "slots". Codewords span groups of N slots, referred to as "frames". The model under investigation has several interesting features: by relaxing the frame-by-frame coding and decoding requirement, (but still enforcing the per-frame power constraint), the long-term average capacity region coincides with the standard *ergodic* capacity region under the per-frame power constraint. Furthermore, the parameter N governs the peak-to-average ratio of transmit power. For $N \to \infty$ the model is equivalent to the conventional setting B, where transmitters are allowed to blast power on good fading states and spend no power on bad fading states. For $N = 1$ constant power transmission is enforced and transmitter channel state information plays no role.

As shown recently in [9], information theoretic performance in the wideband regime is not only characterized by the minimum energy per bit but also by the "wideband slope" of the spectral-efficiency curve as a function of E_b/N_0 (dB) (b/s/Hz/3 dB). Minimum energy per bit alone is unable to give any indication about bandwidth requirements. Accordingly, our analysis focuses on both fundamental limits. We show that a "one-shot" power allocation policy that concentrates the whole transmit energy over one out of N slots, is optimal in terms of minimum energy per bit. In the single-user case, the same policy is also optimal with respect to the wideband slope. Since such slot must be chosen on the basis of causal feedback, the transmitter cannot simply choose the most favorable slot. Rather, the solution is obtained through dynamic programming and has the structure of a comparison with a decreasing threshold [6].

In the multiaccess setting, where users are subject to independent fading coefficients, we make use of the framework developed in [10] for the capacity-per-unit cost region for multiaccess channels as well as results on the region of achievable wideband slope we have obtained in [11]. The one-shot power allocation policy is decentralized: i.e., each user needs the knowledge of its own fading state sequence only. This policy in conjunction with TDMA suffices to achieve the capacity region per unit energy. However, we show that if the same power allocation policy is used in conjunction with superposition signaling and a successive interference cancellation receiver, wideband slopes generally larger than TDMA are achievable.

II. System model and basic definitions

We consider a block-fading Gaussian Multi-Access Channel (MAC) where K transmitters must deliver their message within N slots to the receiver by spending a fixed maximum energy. The number of complex dimensions per slot is $L = \lfloor WT \rfloor$, where T is the slot duration and W is the channel bandwidth. The baseband complex received vector in slot n is

$$\mathbf{y}_n = \sum_{k=1}^{K} c_{k,n}\mathbf{x}_{k,n} + \mathbf{z}_n \tag{1}$$

where $\mathbf{z}_n \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_L)$ [1] is an i.i.d. Gaussian noise vector with normalized unit variance per component, $\mathbf{x}_{k,n} \in \mathbb{C}^L$ is the signal of user k transmitted in slot n, $c_{k,n}$ is the complex fading coefficient for user k with power gain $\alpha_{k,n} \triangleq |c_{k,n}|^2$, assumed i.i.d. with *continuous* cdf $F_\alpha(x)$.

The receiver has perfect (*non-causal*) Channel State Information (CSI) while the transmitters have perfect *causal* CSI [6, 7], i.e., in slot n the transmitters know the channel state up to time n, defined by

$$\mathcal{S}_n \triangleq \{c_{k,i} : k = 1, \cdots, K,\ i = 1, \cdots, n\} \tag{2}$$

Each transmitter k is subject to the per-codeword input constraint (referred to as "short-term" power constraint).

$$\frac{1}{NL}\sum_{n=1}^{N} |\mathbf{x}_{k,n}|^2 \le \gamma_k \tag{3}$$

where γ_k is the transmitted energy per symbol, and because of the noise variance normalization adopted here it has the meaning of *transmit* Signal-to-Noise Ratio (SNR). In the following we will use the notation

$$\beta_{k,n} \triangleq \frac{1}{L}|\mathbf{x}_{k,n}|^2 \tag{4}$$

for the *instantaneous* SNR of transmitter k in slot n.

For finite N and L no positive rate is achievable. However, we can consider a sequence of channels indexed by the slot length L and study the achievable rates in the limit for $L \to \infty$ and fixed N. This is a standard mathematical abstraction in the study of the limit performance of block-fading channels [8] and it is motivated by the fact that, in many practical applications, the product WT is large and T is much smaller than the fading coherence time. Even in the limit of large L, the rate K-tuple at which reliable communication is possible over a frame of N slots is a random vector, because only a fixed number KN of fading coefficients affect each frame.

Consider a long sequence of frames, each composed of N slots. In this setting, it is meaningful to study the largest achievable *long-term average rate region*, subject to the short-term power constraint (3). Moreover, in the energy-limited case investigated here, a meaningful system design criterion is to look for the largest achievable *long-term average capacity per unit energy* (bit/joule). Next, in analogy with [12, 10], we characterize the long-term average capacity region and the long-term average capacity per unit energy for our system.

Variable-rate coding in our setting is essentially different from variable-rate coding in an ergodic setting, such as in [12, 3]. Here, we assume that each transmitter has an infinite "bit-reservoir" and, depending on the fading instantaneous realization, transmits a variable number of bits per frame. We model this setting by letting the message set size depend on the fading state. Because of space limitation, we omit formal definitions of variable-rate coding scheme, long-term average capacity region and long-term average capacity region per unit energy (see [13]).

III. Long-term average capacity region

We have the following:

Theorem 1. The long-term average capacity region is given by

$$C_{K,N}(\boldsymbol{\gamma}) = \bigcup_{\boldsymbol{\beta} \in \Gamma_{K,N}(\boldsymbol{\gamma})} C_{K,N}(\boldsymbol{\gamma}, \boldsymbol{\beta}) \tag{5}$$

where

$$\begin{aligned} C_{K,N}(\boldsymbol{\gamma}, \boldsymbol{\beta}) \triangleq \Big\{ \mathbf{R} \in \mathbb{R}_+^K : \textstyle\sum_{k \in \mathcal{A}} R_k \le \\ \le \mathrm{E}\left[\tfrac{1}{N}\textstyle\sum_{n=1}^{N} \log\left(1 + \sum_{k \in \mathcal{A}} \alpha_{k,n}\beta_{k,n}(\mathcal{S}_n)\right)\right], \\ \forall \mathcal{A} \subseteq \{1, \cdots, K\} \Big\} \end{aligned} \tag{6}$$

where expectation is with respect to the channel state $\mathcal{S}_N$ and where $\Gamma_{K,N}(\boldsymbol{\gamma})$ is the set of *feasible* causal power allocation policies $\boldsymbol{\beta} = \{\beta_{k,n} : k = 1, \cdots, K, n = 1, \cdots, N\}$ defined as

$$\Gamma_{K,N}(\boldsymbol{\gamma}) \triangleq \left\{ \boldsymbol{\beta} \in \mathbb{R}_+^{KN} : \frac{1}{N}\sum_{n=1}^{N} \beta_{k,n} \le \gamma_k,\ \beta_{k,n} = \beta_{k,n}(\mathcal{S}_n) \right\} \tag{7}$$

($\beta_{k,n} = \beta_{k,n}(\mathcal{S}_n)$ indicates the causality constraint, i.e., that $\beta_{k,n}$ is a function of the channel state at time n). □

The explicit characterization of the boundary of the long-term capacity region $C_{K,N}(\boldsymbol{\gamma})$ can be done by following the

[1] The notation $\mathbf{a} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}, \mathbf{R})$ indicates that $\mathbf{a}$ is a proper complex Gaussian random vector with mean $\mathrm{E}[\mathbf{a}] = \boldsymbol{\mu}$ and covariance $\mathrm{E}[(\mathbf{a}-\boldsymbol{\mu})(\mathbf{a}-\boldsymbol{\mu})^H] = \mathbf{R}$. $\mathbf{I}_L$ denotes the identity matrix of dimension $L \times L$.

approach of [12]. The boundary surface of $C_{K,N}(\boldsymbol{\gamma})$ is the closure of all K-tuples $\mathbf{R} \in \mathbb{R}_+^K$ that solve

$$\max_{\mathbf{R}\in C_{K,N}(\boldsymbol{\gamma})} \sum_{k=1}^{K} \mu_k R_k \tag{8}$$

for some $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_K) \in \mathbb{R}_+^K$. A closed form solution of (8) seems infeasible. In [6], the single-user long-term average capacity that, with a slight abuse of notation, will be denoted by

$$C_{1,N}(\gamma) \triangleq \sup_{\boldsymbol{\beta}\in\Gamma_{1,N}(\gamma)} \mathrm{E}\left[\frac{1}{N}\sum_{n=1}^{N} \log\left(1+\alpha_n\beta_n(\mathcal{S}_n)\right)\right] \tag{9}$$

was computed numerically by showing that the optimal power allocation policy

$$\widehat{\boldsymbol{\beta}} \triangleq \arg \sup_{\boldsymbol{\beta}\in\Gamma_{1,N}(\gamma)} \mathrm{E}\left[\frac{1}{N}\sum_{n=1}^{N} \log\left(1+\alpha_n\beta_n(\mathcal{S}_n)\right)\right] \tag{10}$$

is the solution of a dynamic programming problem:

Theorem 2[6]. Define the recursion

$$S_n(P) = \mathop{\mathrm{E}}_{\alpha}\left[\sup_{p\in[0,P]} \{\log(1+\alpha p) + S_{n-1}(P-p)\}\right] \tag{11}$$

for $n = 1, \ldots, N$, with initial condition $S_0(P) = 0$. The single user long-term average capacity is given by

$$C_{1,N}(\gamma) = \frac{1}{N} S_N(N\gamma) \tag{12}$$

□

Although we cannot characterize explicitly the boundary surface of $C_{K,N}(\boldsymbol{\gamma})$ for every finite N, we can prove the following limit theorem:

Theorem 3. In the limit for large N, the long-term average capacity region $C_{K,N}(\boldsymbol{\gamma})$ tends to $C_K^{(\mathrm{erg})}(\boldsymbol{\gamma})$,the ergodic capacity region given in [12]. □

IV. Long-term average capacity region per unit energy

A byproduct of the proof of Theorem 1 (see [13]) is that the long-term average capacity region coincides with the standard "ergodic" capacity region of the N-slot extension channel, which is frame-wise memoryless. The following theorem is an immediate consequence of this fact and of the general theory of capacity per unit cost [10]:

Theorem 4. The long-term average capacity region per unit energy is given by

$$U_{K,N} = \bigcup_{\boldsymbol{\gamma}\in\mathbb{R}_+^K} \left\{\mathbf{r} \in \mathbb{R}_+^K : (\gamma_1 r_1, \cdots, \gamma_K r_K) \in C_{K,N}(\boldsymbol{\gamma})\right\} \tag{13}$$

□

In analogy with [10], it is easy to show the following:

Theorem 5. The long-term average capacity region per unit energy is the hyper-cube

$$U_{K,N} = \left\{\mathbf{r} \in \mathbb{R}_+^K : r_k \leq U_{1,N}\right\} \tag{14}$$

where (with a slight abuse of notation)

$$U_{1,N} = \lim_{\gamma\to 0} \frac{1}{\gamma} \sup_{\boldsymbol{\beta}\in\Gamma_{1,N}(\gamma)} \mathrm{E}\left[\frac{1}{N}\sum_{n=1}^{N} \alpha_{k,n}\beta_{k,n}(\mathcal{S}_n)\right] \tag{15}$$

□

Theorem 6 [6]. Define the recursion

$$s_n = \mathop{\mathrm{E}}_{\alpha}[\max\{s_{n-1}, \alpha\}] \tag{16}$$

for $n = 1, \ldots, N$, with initial condition $s_0 = 0$. The single-user long-term average capacity per unit energy is given by $U_{1,N} = s_N$ and it is achieved by the "one-shot" power allocation policy defined by

$$\beta_n^\star = \begin{cases} N\gamma & \text{if } n = n^\star(\boldsymbol{\alpha}) \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where we define the "level-crossing" time

$$n^\star(\boldsymbol{\alpha}) = \min\{n \in \{1,\ldots,N\} \,:\, \alpha_n \geq s_{N-n}\} \tag{18}$$

□

The behavior of $U_{1,N}$ when N grows to infinity is given by the following:

Theorem 7. For large N, $U_{1,N}$ tends to the ergodic capacity region per unit energy defined in [10] and given explicitly by

$$\lim_{N\to\infty} U_{1,N} = \lim_{\gamma\to 0} \frac{\mathrm{d}C_1^{(\mathrm{erg})}(\gamma)}{\mathrm{d}\gamma} = \sup\{\alpha\} \tag{19}$$

($\sup\{\alpha\}$ denotes the supremum of the support of the probability distribution of α). □

V. Wideband performance

V.A Background

The optimality of a coding scheme in the wideband regime is defined and studied for several input-constrained additive noise channels in [9]. Let $C(\mathsf{SNR})$ be the capacity expressed in nat/dimension as a function of the (transmit) SNR, and let $\mathsf{C}(E_b/N_0)$ denote the corresponding spectral efficiency in bit/s/Hz as a function of the energy per bit vs. noise power spectral density, E_b/N_0, given implicitly by the parametric equation

$$\begin{cases} \frac{E_b}{N_0} = \frac{\mathsf{SNR}\log 2}{C(\mathsf{SNR})} \\ \mathsf{C}\left(\frac{E_b}{N_0}\right) = C(\mathsf{SNR})/\log 2 \end{cases} \tag{20}$$

The value $(E_b/N_0)_{\min}$ for which $\mathsf{C}(E_b/N_0) > 0 \Leftrightarrow E_b/N_0 > (E_b/N_0)_{\min}$, is given by [9]

$$\left(\frac{E_b}{N_0}\right)_{\min} = \lim_{\mathsf{SNR}\downarrow 0} \frac{\mathsf{SNR}\log 2}{C(\mathsf{SNR})} = \frac{\log 2}{\dot{C}(0)} \tag{21}$$

where $\dot{C}(0)$ is the derivative of the capacity function at $\mathsf{SNR} = 0$. From [10], we see immediately that the reciprocal of $(E_b/N_0)_{\min}$ is the capacity per unit energy (expressed in bit/joule) of the channel.

In the wideband (i.e., vanishing SNR) regime, the behavior of spectral efficiency in a (right) neighborhood of $(E_b/N_0)_{\min}$ is of great importance, as it is able to quantify the bandwidth requirement for a given desired data rate (see the detailed discussion in [9]). This behavior is captured by the slope of spectral efficiency in bit/s/Hz/(3 dB), at $(E_b/N_0)_{\min}$, given by (see [9, Theorem 6])

$$\mathcal{S}_0 = \frac{2\left(\dot{C}(0)\right)^2}{-\ddot{C}(0)} \tag{22}$$

where $\ddot{C}(0)$ denotes the second derivative of the capacity function at $\mathsf{SNR} = 0$. A signaling strategy is said to be *first-order optimal* if it achieves $(E_b/N_0)_{\min}$ and *second-order optimal* is it achieves $\mathcal{S}_0$ [9].

In a multiple-access channel, the individual user energy per bit over N_0 are defined by $E_k/N_0 \triangleq \gamma_k \log 2/R_k$, where γ_k is the transmit SNR (energy/symbol) and R_k is the rate (in nat/symbol) of user k. In [11], the achievable slope region for the standard 2-user Gaussian MAC is studied and its boundary is explicitly parameterized with respect to the ratio $\theta = R_1/R_2$.

In Section IV we have shown that the one-shot power allocation $\boldsymbol{\beta}^\star$ (in conjunction with Gaussian variable-rate coding) achieves the capacity region per unit energy, i.e., achieves $(E_b/N_0)_{\min}$ for all users for the block-fading MAC with causal transmitter CSI considered in this paper. Then, we conclude that the one-shot policy is first-order optimal for any number of users K. From the proof of Theorem 5 [13] it follows that first-order optimality can be obtained either by using superposition coding or by using TDMA inside each slot. Next, we shall study the wideband slope performance of $\boldsymbol{\beta}^\star$.

V.B Second-order optimality for $K = 1$

Even if we cannot give a closed form for either $C_{1,N}(\gamma)$ or the corresponding optimal power allocation policy $\widehat{\boldsymbol{\beta}}$ given in (10), the characterization of $C_{1,N}(\gamma)$ in the wideband regime and the second-order optimality of the one-shot policy $\boldsymbol{\beta}^\star$ are given by the following:

Theorem 8. $(E_b/N_0)_{\min}$ and $\mathcal{S}_0$ for the single-user block fading channel with causal transmitter CSI are given by

$$\begin{aligned} \left(\frac{E_b}{N_0}\right)_{\min} &= \frac{\log 2}{s_N} \\ \mathcal{S}_0 &= \frac{2\,(s_N)^2}{-N\,\ddot{S}_N(0)} \end{aligned} \tag{23}$$

where s_N is given in (16), where the function $S_N(P)$ is defined in (11) and where $\ddot{S}_N(0)$ denotes the second derivative of $S_N(P)$ at $P = 0$, given by the recursion

$$\begin{aligned} -\ddot{S}_n(0) &= \Pr(\alpha \geq s_{n-1})\,\mathrm{E}[\alpha^2|\alpha \geq s_{n-1}] \\ &\quad -\ddot{S}_{n-1}(0)\Pr(\alpha < s_{n-1}) \end{aligned} \tag{24}$$

for $n = 1, \ldots, N$, with $\ddot{S}_0(0) = 0$. Furthermore, the one-shot power allocation policy $\boldsymbol{\beta}^\star$ also achieves $(E_b/N_0)_{\min}$ and $\mathcal{S}_0$. □

V.C An achievable slope region and comparison with TDMA

We investigate the slopes of the user rates as functions of the individual E_k/N_0 for the particular choice of the one-shot power allocation policy $\boldsymbol{\beta}^\star$ defined in Theorem 6. Our approach follows that of [11]. We have the following:

Theorem 9. Fix a vectors $\boldsymbol{\theta} \in \mathbb{R}_+^K$ such that $\sum_{k=1}^K \theta_k = 1$. For vanishing user rates while keeping fixed user rate ratios $R_k/R_j = \theta_k/\theta_j$, under the one-shot policy and superposition coding the achievable slope region is given by the parametric form

$$\bigcup_{\boldsymbol{\lambda}} \prod_{k=1}^K \left\{ 0 \leq \mathcal{S}_0^{(k)} \leq \frac{\mathcal{S}_0}{1 + \mathcal{K}_0 \sum_{\boldsymbol{\pi}} \lambda_{\boldsymbol{\pi}} \sum_{j<\boldsymbol{\pi}^{-1}(k)} \frac{\theta_{\pi_j}}{\theta_k}} \right\} \tag{25}$$

where $\mathcal{S}_0$ is the single-user slope given in (23), where

$$\mathcal{K}_0 = \frac{2\sum_{n=1}^N \left(\mathrm{E}[\alpha_n 1\{n^\star(\boldsymbol{\alpha}) = n\}]\right)^2}{\sum_{n=1}^N \mathrm{E}\left[\alpha_n^2 1\{n^\star(\boldsymbol{\alpha}) = n\}\right]} \tag{26}$$

where $\sum_{\boldsymbol{\pi}}$ denotes the sum over all permutations of $\{1, \ldots, K\}$ and where $\boldsymbol{\lambda} = \{\lambda_{\boldsymbol{\pi}}\}$ are non-negative "time-sharing" coefficients (indexed by the permutations $\boldsymbol{\pi}$) such that $\sum_{\boldsymbol{\pi}} \lambda_{\boldsymbol{\pi}} = 1$. □

As a corollary of Theorem 9 we get that if the user rates are very imbalanced, more precisely, if θ_{π_j}/θ_k is vanishing for all $k = 1, \ldots, K$ and $j < \boldsymbol{\pi}^{-1}(k)$, for some permutation $\boldsymbol{\pi}$, then all users can achieve single user slopes. In fact, it is sufficient to choose the vertex corresponding to $\boldsymbol{\pi}$ and the denominator in (25) becomes 1 for all k. The condition for achieving single-user slopes for all users is that there exist a permutation $\boldsymbol{\pi}$ such that $R_{\pi_{k-1}} = o(R_{\pi_k})$ for all $k = 2, \ldots, K$.

As far as TDMA is concerned, because of second-order optimality of $\boldsymbol{\beta}^\star$ in the single-user case, the same one-shot policy achieves the maximum possible slopes under TDMA. This is given by the following:

Theorem 10. For any arbitrary rate ratios R_k/R_j, as the rates vanish, the best achievable slope region under TDMA is given by

$$\bigcup_{\boldsymbol{\tau}} \prod_{k=1}^K \left\{ 0 \leq \mathcal{S}_{0,\mathrm{tdma}}^{(k)} \leq \tau_k \mathcal{S}_0 \right\} \tag{27}$$

where $\boldsymbol{\tau} = \{\tau_k\}$ are non-negative "time-sharing" coefficients such that $\sum_{k=1}^K \tau_k = 1$. □

We can use Theorems 9 and 10 to determine the max-min slope of an equal-rate wideband system. For equal rates, $\theta_j/\theta_k = 1$ for all k, j, and the denominator of (25) becomes

$$\begin{aligned} 1 + \mathcal{K}_0 \sum_{\boldsymbol{\pi}} \lambda_{\boldsymbol{\pi}} \sum_{j<\boldsymbol{\pi}^{-1}(k)} 1 &= 1 + \mathcal{K}_0 \sum_{\boldsymbol{\pi}} \left(\lambda_{\boldsymbol{\pi}} \boldsymbol{\pi}^{-1}(k) - 1\right) \\ &= 1 - \mathcal{K}_0 + \mathcal{K}_0 \sum_{\boldsymbol{\pi}} \lambda_{\boldsymbol{\pi}} \boldsymbol{\pi}^{-1}(k) \end{aligned}$$

As $\boldsymbol{\pi}$ varies over all $K!$ permutations, $\boldsymbol{\pi}^{-1}(k)$ takes on each value $1, \ldots, K$ exactly $(K-1)!$ times. Because of symmetry, maximizing the minimum slope is achieved by letting $\mathcal{S}_0^{(k)} =$ const., i.e., $\lambda_{\boldsymbol{\pi}} = 1/K!$ for all $\boldsymbol{\pi}$. This yields to the max-min slope

$$\max \min_k \mathcal{S}_0^{(k)} = \frac{\mathcal{S}_0}{1 + \mathcal{K}_0(K-1)/2}$$

For TDMA, the max-min slope is obtained by letting $\tau_k = 1/K$, i.e., $\max\min_k \mathcal{S}_{0,\mathrm{tdma}}^{(k)} = \mathcal{S}_0/K$. For a desired user rate R_b (in bit/s) common to all users, and assuming that all users transmit with equal power, i.e., they have the same E_b/N_0 such that $(E_b/N_0)_{\mathrm{dB}} - ((E_b/N_0)_{\min})_{\mathrm{dB}} = \epsilon$, the system bandwidth is given approximately by [9]

$$W \approx \frac{R_b}{\min_k \mathcal{S}_0^{(k)} \epsilon}$$

Therefore, the bandwidth expansion factor of TDMA with respect to superposition coding is given by

$$\eta = \frac{K}{1 + \mathcal{K}_0(K-1)/2} \tag{28}$$

From (26) we have immediately that $\mathcal{K}_0 < 2$ (i.e., TDMA is strictly wideband-suboptimal) for any non-degenerate fading distribution. Notice also that the case of equal E_b/N_0 for all users is the most favorable for TDMA [11]. As already noticed, for a very imbalanced system the bandwidth expansion factor can be much larger than (28).

Next, we study in more detail the case $K = 2$. For superposition coding, by letting $\theta = \theta_1/\theta_2$, we have

$$\begin{aligned} \mathcal{S}_0^{(1)} &= \frac{\mathcal{S}_0}{1 + \mathcal{K}_0(1-\lambda)\frac{1}{\theta}} \\ \mathcal{S}_0^{(2)} &= \frac{\mathcal{S}_0}{1 + \mathcal{K}_0\lambda\theta} \end{aligned} \tag{29}$$

By eliminating the time sharing parameter λ we obtain the slope region boundary as

$$\left(\frac{1}{\mathcal{S}_0^{(1)}} - \frac{1}{\mathcal{S}_0}\right)\theta + \left(\frac{1}{\mathcal{S}_0^{(2)}} - \frac{1}{\mathcal{S}_0}\right)\frac{1}{\theta} = \frac{\mathcal{K}_0}{\mathcal{S}_0}, \quad 0 \le \mathcal{S}_0^{(k)} \le \mathcal{S}_0 \tag{30}$$

With TDMA we obtain the boundary $\mathcal{S}_{0,\text{tdma}}^{(1)} + \mathcal{S}_{0,\text{tdma}}^{(2)} = \mathcal{S}_0$.

We might wonder if for some θ TDMA achieves the same slope trade-off of superposition coding, i.e., if the two boundaries of the slope regions intersects at some point $(\mathcal{S}_0^{(1)}, \mathcal{S}_0^{(2)})$. By substituting in (30) $\mathcal{S}_0^{(1)} = \tau\mathcal{S}_0$ and $\mathcal{S}_0^{(2)} = (1-\tau)\mathcal{S}_0$ for $\tau \in [0,1]$, we find

$$\left(\frac{1}{\tau} - 1\right)\theta + \left(\frac{1}{1-\tau} - 1\right)\frac{1}{\theta} = \mathcal{K}_0$$

which yields

$$\tau = \frac{\theta}{2}\frac{2\theta + \mathcal{K}_0 \pm \sqrt{\mathcal{K}_0^2 - 4}}{\theta^2 + \mathcal{K}_0\theta + 1}$$

Again, for $\mathcal{K}_0 < 2$ (non-constant fading), TDMA is strictly suboptimal, for any choice of the rate ratio θ.

VI. Example with Rayleigh fading

In order to illustrate the results of previous sections we consider the case of i.i.d. Rayleigh fading. The channels gain law is $F_\alpha(x) = 1 - e^{-x}$ for $x \ge 0$. The one-shot policy is given by the threshold

$$s_n = s_{n-1} + e^{-s_{n-1}}, \quad n = 1, \dots, N \tag{31}$$

with $s_0 = 0$.

If we allow the input to depend on the whole CSI $\mathcal{S}_N$ in a non-causal way, the optimal power allocation would be waterfilling over the gains $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$. It is immediate to show that, under the noncausal policy,

$$\begin{aligned} \dot{C}_{1,N}^{(\text{noncausal})}(0) &= \sum_{n=1}^{N}\binom{N}{n}(-1)^{n+1}\frac{1}{n} \\ -\ddot{C}_{1,N}^{(\text{noncausal})}(0) &= N\sum_{n=1}^{N}\binom{N}{n}(-1)^{n+1}\frac{2!}{n^2} \end{aligned}$$

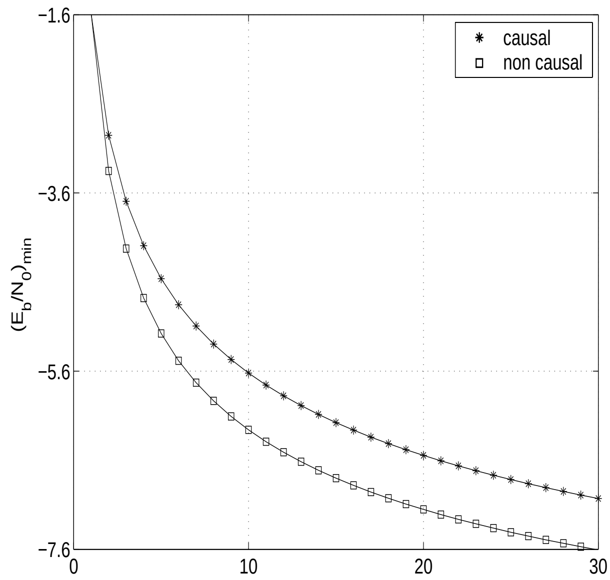

Figure 1: $(E_b/N_0)_{\min}$ (dB) vs. N for the Rayleigh fading channel.

Hence, the capacity per unit energy with noncausal transmitter CSI is

$$U_{1,N}^{(\text{noncausal})} = \sum_{n=1}^{N}\binom{N}{n}(-1)^{n+1}\frac{1}{n} \tag{32}$$

and the spectral efficiency slope is

$$\mathcal{S}_0^{(\text{noncausal})} = \frac{2\left(\sum_{n=1}^{N}\binom{N}{n}(-1)^{n+1}\frac{1}{n}\right)^2}{N\sum_{n=1}^{N}\binom{N}{n}(-1)^{n+1}\frac{2!}{n^2}} \tag{33}$$

Figs. 1 and 2 show $(E_b/N_0)_{\min}$ and $\mathcal{S}_0$ vs. N and for both the causal and noncausal knowledge of the channel.

Fig. 3 shows the bandwidth expansion factor η vs. K of TDMA with respect to superposition coding, for several values of N, for an equal-rate equal E_b/N_0 system. Finally, Fig. 4 shows the 2-user achievable slope region with the one-shot policy and superposition coding, for different rate ratios. The optimal region achievable by TDMA is shown for comparison. This figure clearly illustrates that even though TDMA achieves the capacity per unit energy, it is actually very suboptimal in the wideband regime, especially in a fading scenario.

References

[1] T.Cover and J.Thomas, *Elements of information theory*, Wiley, New York, 1991.

[2] R. Gallager, *Communications and Cryptography – Two sides of one tapestry*, chapter An inequality on the capacity region of multiaccess fading channels, pp. 129–139, Kluwer, Boston, MA, 1994.

[3] A.Goldsmith and P.P.Varaiya, "Capacity of fading channels with channel state information," *IEEE Trans. on Inform. Theory*, vol. 43, no. 6, pp. 1986–1992, November 1997.

[4] S.Shamai(Shitz) and S.Verdú, "The impact of flat fading on the spectral efficiency of CDMA," *IEEE Trans. on Inform. Theory*, vol. 47, no. 4, pp. 1302–1327, May 2001.

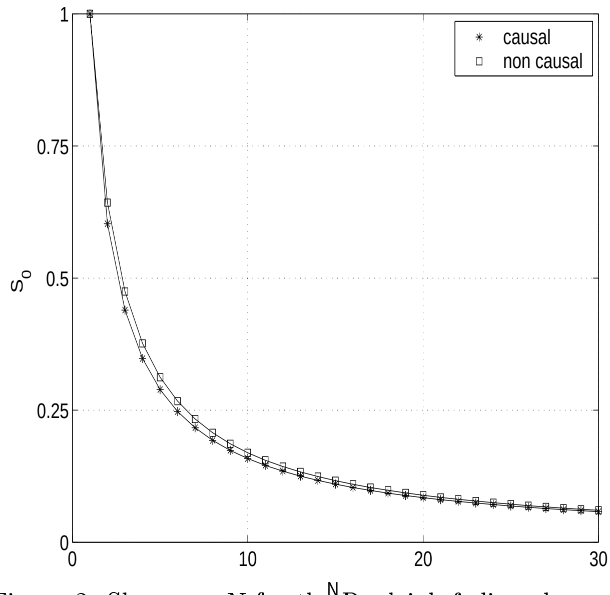

Figure 2: Slope vs. N for the Rayleigh fading channel.

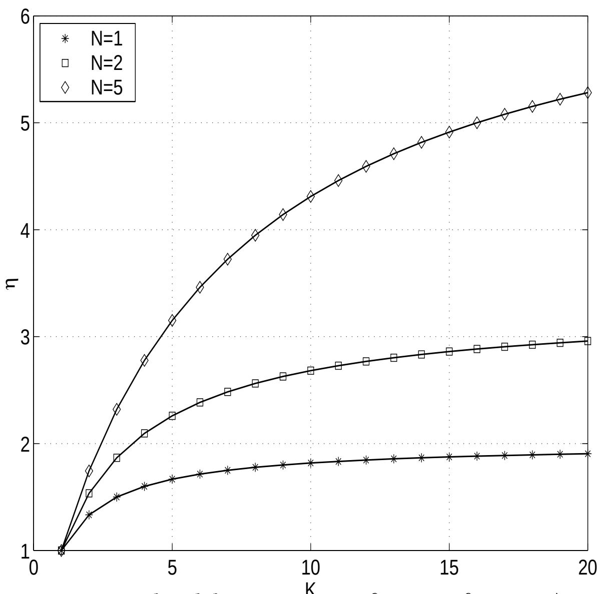

Figure 3: Bandwidth expansion factor of TDMA over superposition coding vs. the number of users K.

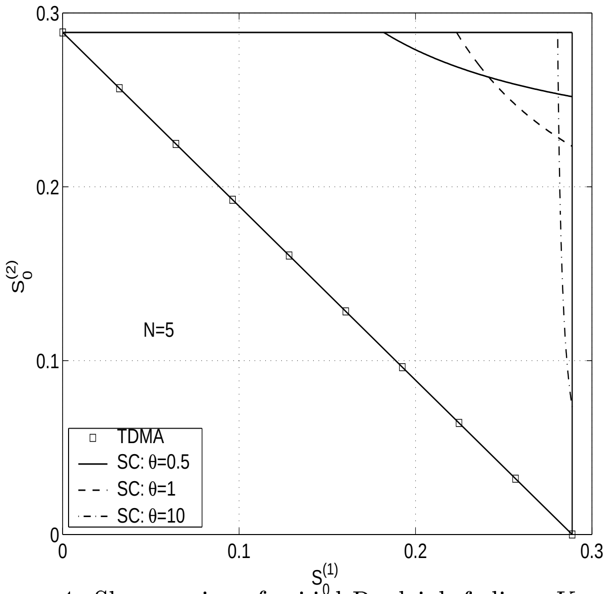

Figure 4: Slope regions for i.i.d Rayleigh fading, $K = 2$, $N = 5$ and different rate ratios.

[5] S.Hanly and D.Tse, "Multiaccess fading channels-Part II: Delay limited capacities," *IEEE Trans. on Inform. Theory*, vol. 44, no. 7, pp. 2816–2831, November 1998.

[6] R.Negi, M.Charikar, and J.Cioffi, "Transmission over fading channels with channel state information and delay constraint," in *Global Telecommunication Conference (GLOBECOM'99)*, 1999, pp. 2550–2554.

[7] R.Negi, M.Charikar, and J.Cioffi, "Minimum outage transmission over fading channels with delay constraint," in *IEEE International Conference on Communications 2000 (ICC 2000)*, 2000, pp. 282–286.

[8] L.Ozarow, S.Shamai, and A.D.Wyner, "Information theoretic considerations for cellular mobile radio," *IEEE Trans. on Vehic. Tech.*, vol. 43, no. 2, pp. 359–378, May 1994.

[9] S.Verdú, "Spectral efficiency in the wideband regime," *submitted to Trans. on Inform. Theory: special issue on Shannon Theory: perspective, trends and applications*, 2001.

[10] S.Verdú, "On channel capacity per unit cost," *IEEE Trans. on Inform. Theory*, vol. 36, no. 5, pp. 1019–1030, September 1990.

[11] S. Verdú, G. Caire, and D. Tuninetti, "Is TDMA optimal in the low power regime?," submitted to Intern. Symp. on Inform. Theory, ISIT 2002, 2001.

[12] D.Tse and S.Hanly, "Multiaccess fading channels-Part I: Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. on Inform. Theory*, vol. 44, no. 7, pp. 2796–2815, November 1998.

[13] D. Tuninetti, G. Caire, and S. Verdú, "Fading multiaccess in the wideband regime: the impact of delay constraints," Journal submission: in preparation, 2001.