Is TDMA Optimal in the Low Power Regime?

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Abstract— **We consider two-user additive Gaussian noise multiple-access and broadcast channels. Although the set of rate pairs achievable by time-division multiple-access (TDMA) is not equal to the capacity region (achieved by superposition), as the power decreases, the TDMA achievable region converges to the capacity region. Furthermore, TDMA achieves the same minimum energy per bit as superposition.**

Despite those features of TDMA, this paper answers the question in the title negatively except in two special cases: multiaccess channels where the users' energy per bit are identical and broadcast channels where the receivers have identical signal-to-noise ratios.

One of the simplest ways to engineer multipoint-to-point (multiaccess) or point-to-multipoint (broadcast) links is to use Time-Division Multiple Access (TDMA), by means of which each user is assigned nonoverlapping time slots during which they are the only active transmitters. This mutiaccess technology leads to very simple receiver design. However, multiuser information theory has shown that superposition strategies where users transmit simultaneously in time and frequency causing mutual interference offers in general higher capacity provided the inter-user interference is taken into account at the receiver. For example:

- The capacity region (set of achievable rates) of multiaccess channels and of broadcast channels are not achieved by TDMA [1].
- When users are affected by independent fading, they can achieve higher aggregate rate with superposition than with TDMA, as a simple consequence of the concavity of channel capacity as a function of signal-to-noise ratio [2].
- In cellular models where each base station neglects the structure of the out-of-cell interference, superposition coding (possibly coupled with so-called intercell time-sharing protocols, if the out-of-cell interference is sufficiently high) offers higher capacity than TDMA in the presence of fading [3].

Moreover, in practical implementations TDMA suffers from performance-limiting multiuser interference because of nonideal effects such as channel distortion and out-of-cell interference.

The most common practical embodiment of superposition multiaccess strategies is CDMA. Thus, in practice, superposition is particularly relevant in the wideband low-power regime where the received energy per information bit may not be far from its minimum value. Therefore, it is of considerable practical interest to compare the capabilities of TDMA to the capabilities of superposition in the low-power regime. To make the comparison as crisp as possible and in order not to incorporate features such as communication in a cellular environment or in the presence of fading, which as we saw before may tilt the comparison in favor of superposition strategies, we limit our analysis to additive white Gaussian noise channels not subject to fading. In this summary submitted to ISIT we include both multiple-access channels and broadcast channels in their simplest possible setting: the classical two-user scalar white Gaussian noise model. We have also obtained generalizations to K-user channels with fading.

I. The Multiple Access Channel

We consider the complex-valued multiple-access channel

$$Y = X_1 + X_2 + N \tag{1}$$

where N is Gaussian with independent real and imaginary components and $E[|N|^2] = \sigma^2$, $E[|X_1|^2] \le P_1$ and $E[|X_2|^2] \le P_2$. The capacity region is the Cover-Wyner pentagon [1]:

$$\begin{aligned} \{R_1 &\le \log_2\left(1+\frac{P_1}{\sigma^2}\right) \\ R_2 &\le \log_2\left(1+\frac{P_2}{\sigma^2}\right) \\ R_1+R_2 &\le \log_2\left(1+\frac{P_1+P_2}{\sigma^2}\right)\} \end{aligned} \tag{2}$$

In particular, we can conclude from (2) the celebrated result that the total capacity (maximum sum of rates) of the multiaccess channel is equal to the capacity of a single-user channel whose power is equal to the sum of the individual powers, namely

$$\log_2\left(1+\frac{P_1+P_2}{\sigma^2}\right).$$

As is well known, the boundary (or, more precisely, the Pareto-optimal points) of the capacity region is achieved by superposition. In contrast, TDMA achieves the region described as the union of rectangles:

$$\bigcup_{0\le\alpha\le 1} \begin{aligned} \{R_1 &\le \alpha\log_2\left(1+\frac{P_1}{\alpha\sigma^2}\right) \\ R_2 &\le (1-\alpha)\log_2\left(1+\frac{P_2}{(1-\alpha)\sigma^2}\right)\} \end{aligned} \tag{3}$$

where the parameter α is equal to the fraction of time that the first user is active. By letting the time sharing parameter be equal to

$$\alpha = \frac{P_1}{P_1+P_2},$$

we obtain that the total capacity achieved by (3) is also equal to (cf. Figure 1)

$$\log_2\left(1+\frac{P_1+P_2}{\sigma^2}\right).$$

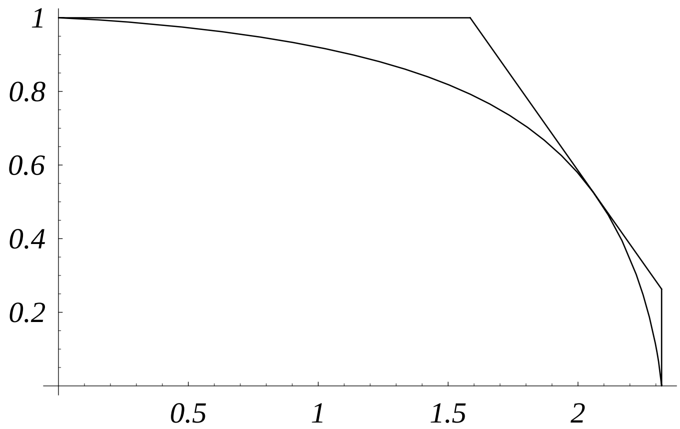

Fig. 1. Multiaccess channel capacity region and TDMA achievable region with with $P_1/\sigma^2 = 4$ and $P_2/\sigma^2 = 1$.

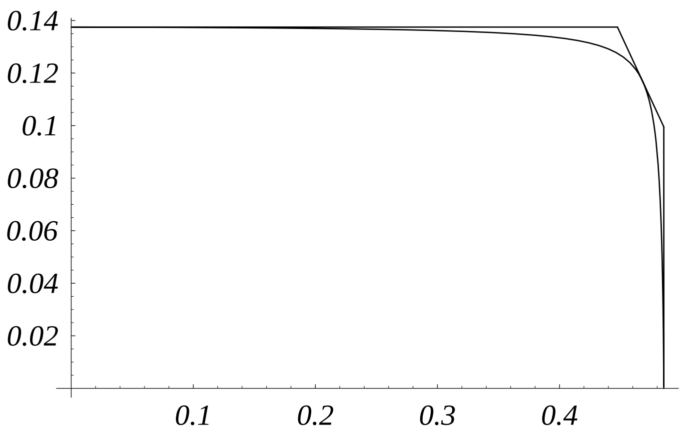

Fig. 2. Multiaccess channel capacity region and TDMA achievable region with with $P_1/\sigma^2 = 0.4$ and $P_2/\sigma^2 = 0.1$.

In particular, if $P_1 = P_2$ and $R_1 = R_2$, then TDMA is optimal.

Moreover, as σ grows, we operate predominantly in the linear region of the logarithm. Roughly speaking, as the background noise grows, the multiaccess interference becomes a secondary factor and the achievable rates become decoupled. This is illustrated by comparing Figures 1 and 2, where we see that the TDMA achievable rate region occupies an increasingly large fraction of the capacity region as σ^2 increases. This can be formalized by showing that the TDMA achievable region converges to the rectangle

$$\begin{aligned}\{R_1 &\le \log_2\left(1+\frac{P_1}{\sigma^2}\right)\\ R_2 &\le \log_2\left(1+\frac{P_2}{\sigma^2}\right)\}\end{aligned} \tag{4}$$

in the following sense:

$$\lim_{\sigma\to\infty}\frac{\alpha\log_2\left(1+\frac{P_1}{\alpha\sigma^2}\right)}{\log_2\left(1+\frac{P_1}{\sigma^2}\right)}+\frac{(1-\alpha)\log_2\left(1+\frac{P_2}{(1-\alpha)\sigma^2}\right)}{\log_2\left(1+\frac{P_2}{\sigma^2}\right)}=2. \tag{5}$$

Define the (received) energy per information bit relative to the noise spectral level of user $i = 1, 2$ as

$$\frac{E_i}{N_0}=\frac{P_i}{R_i\sigma^2}. \tag{6}$$

Note that sometimes a "system" energy per bit is considered instead of the individual per-user energies per bit defined in (6). For example, when all the per-symbol energies are identical, [4] uses a system energy per bit which is equal to the harmonic mean of the individual energies per bit.

One of the fundamental limits of interest in this paper is the minimum energy per information bit, which is obtained with asymptotically low power [5]. To that end, we can apply the general framework of capacity region per unit cost developed in [5]. However, in the particular case at hand it is instructive to give a self-contained derivation. Several of the performance measures we will encounter later depend on the ratio with which both rates go to 0. As the following result shows, this is not the case for the multiaccess minimum energy per bit.

Theorem 1: For all R_1/R_2, the minimum energies per information bit for the multiple-access channel are equal to

$$\frac{E_1}{N_0}=\frac{E_2}{N_0}=\log_e 2=-1.59dB. \tag{7}$$

Furthermore, (7) is achieved by TDMA.

Proof: For a single-user channel the minimum energy per bit is computed from the capacity-SNR function $C(\mathsf{SNR})$ (in bits):

$$\begin{aligned}\frac{E_b}{N_0}_{\min} &= \lim_{\mathsf{SNR}\to 0}\frac{\mathsf{SNR}}{C(\mathsf{SNR})} && (8)\\ &= \frac{\log_e 2}{\dot{C}(0)} && (9)\end{aligned}$$

where $\dot{C}(0)$ = derivative at 0 of $C(\mathsf{SNR})$ computed in nats.

Let us apply this framework to TDMA. First, consider a fixed time-sharing parameter $0 < \alpha < 1$. Using (3) we obtain

$$\frac{E_1}{N_0}=\lim_{P_1\to 0}\frac{P_1/\sigma^2}{\alpha\log_2\left(1+\frac{P_1}{\alpha\sigma^2}\right)}=\log_e 2. \tag{10}$$

and

$$\frac{E_2}{N_0}=\lim_{P_2\to 0}\frac{P_2/\sigma^2}{(1-\alpha)\log_2\left(1+\frac{P_2}{(1-\alpha)\sigma^2}\right)}=\log_e 2. \tag{11}$$

Since the convergence of the limits (10) and (11) is uniform over α, we can conclude that the result holds even if α is not held fixed and varies with the signal-to-noise ratio. (For example, in order to enforce a constraint on $R_1 = R_2$.)

Since TDMA achieves a subset of the capacity region, and since in noninterfering single-user channels the minimum energies per bit are also equal to (7), we conclude that for the multiaccess channel, TDMA achieves the same minimum energies per bit as superposition. □

From all the evidence we have seen so far, we would be justified to suspect that the purported advantage of superposition over TDMA may actually vanish in the low power regime. If this is the case, then the increase in receiver complexity required to realize the capacity achieved by superposition would be hardly justified unless some of the other factors mentioned above come into play. However, this is not the case. The minimum values of energy per bit are obtained in the limit of infinite bandwidth and therefore imply zero spectral efficiency. As explained in the recent work [6], the key performance measure in the wideband regime is the slope of the spectral efficiency vs $\frac{E_b}{N_0}$ curve (b/s/Hz/3 dB) at $\frac{E_b}{N_0}_{\min}$:

$$\begin{aligned}\mathcal{S}_0 &\stackrel{\text{def}}{=} \lim_{\frac{E_b}{N_0}\downarrow\frac{E_b}{N_0}_{\min}} \frac{\mathsf{C}(\frac{E_b}{N_0})}{10\log_{10}\frac{E_b}{N_0} - 10\log_{10}\frac{E_b}{N_0}_{\min}} 10\log_{10}2 \\ &= \frac{2\left[\dot{C}(0)\right]^2}{-\ddot{C}(0)} \qquad (12)\end{aligned}$$

A number of well-known conclusions made in the literature based on the (infinite bandwidth) analysis of $\frac{E_b}{N_0}_{\min}$ are shown in [6] to no longer apply in the wideband low-power regime where bandwidth is finite and the spectral efficiency is nonzero. In particular, unlike $\frac{E_b}{N_0}_{\min}$, the slope $\mathcal{S}_0$ gives an indication of the bandwidth requirements for a given data rate. Note that in the particular case of the single-user additive white Gaussian noise channel, $C(x) = \log(1+x)$ and $\mathcal{S}_0 = 2$.

Whereas the conventional capacity region supplies the tradeoff of rates for fixed powers, we can define a corresponding "slope region" that gives the tradeoff of individual user slopes for a fixed ratio with which the individual rates vanish. Although formula (12) applies to single-user channels it turns out to be sufficient for our analysis of both multiaccess and broadcast channels.

Theorem 2: For all R_1/R_2, the multiaccess slope region achieved by TDMA is:

$$\{(\mathcal{S}_1, \mathcal{S}_2):\ 0 \le \mathcal{S}_1,\ 0 \le \mathcal{S}_2,\ \mathcal{S}_1 + \mathcal{S}_2 \le 2\}.$$

Proof: Fix $0 \le \alpha \le 1$. Applying (12) to the individual rate constraint equations in (3),

$$\begin{aligned}C_1(P_1) &= \alpha \log_2\left(1 + \frac{P_1}{\alpha\sigma^2}\right) \\ C_2(P_2) &= (1-\alpha)\log_2\left(1 + \frac{P_2}{(1-\alpha)\sigma^2}\right)\} \qquad (13)\end{aligned}$$

we obtain

$$\begin{aligned}\dot{C}_1(0) &= \frac{1}{\sigma^2} \\ \dot{C}_2(0) &= \frac{1}{\sigma^2} \\ \ddot{C}_1(0) &= \frac{1}{\alpha\sigma^4} \\ \ddot{C}_2(0) &= \frac{1}{(1-\alpha)\sigma^4}\end{aligned} \qquad (14)$$

Thus, if the rate pair belongs to the boundary of the achievable region, then $\mathcal{S}_1 = 2\alpha$ and $\mathcal{S}_2 = 2 - 2\alpha$. □

Theorem 3: Let the rates vanish while keeping $R_1/R_2 = \theta$. The optimum multiaccess slope region (achieved by superposition) is:

$$\begin{aligned}\mathbf{S}(\theta) = \{(\mathcal{S}_1, \mathcal{S}_2) &:\ 0 \le \mathcal{S}_1 \le 2,\ 0 \le \mathcal{S}_2 \le 2, \\ &\frac{1}{2} \le \left(\frac{\theta}{1+\theta}\right)^2 \frac{1}{\mathcal{S}_1} + \left(\frac{1}{1+\theta}\right)^2 \frac{1}{\mathcal{S}_2}\}.\end{aligned} \qquad (15)$$

Furthermore,

$$\text{closure}\left\{\bigcup_{\theta>0} \mathbf{S}(\theta)\right\} = \{(\mathcal{S}_1, \mathcal{S}_2):\ 0 \le \mathcal{S}_1 \le 2,\ 0 \le \mathcal{S}_2 \le 2\}. \qquad (16)$$

Proof: As we showed in Theorem 1, as we let the powers and rates vanish, both energies per bit E_1 and E_2 approach the same value, and therefore (6) implies that in the limit

$$\frac{P_1}{P_2} = \frac{R_1}{R_2} = \theta.$$

In the rest of the proof we will assume that $P_2 = P_1/\theta$. While this is only required in the limit, we can handle the more general case invoking uniform convergence in the same way as in the proof of Theorem 1.

We can re-write (2) as

$$\begin{aligned}\bigcup_{0\le\alpha\le1}\{R_1 &\le \alpha\log_2\left(1+\frac{P_1}{\sigma^2}\right) + (1-\alpha)\log_2\left(1+\frac{P_1}{P_2+\sigma^2}\right) \\ R_2 &\le \alpha\log_2\left(1+\frac{P_2}{P_1+\sigma^2}\right) + (1-\alpha)\log_2\left(1+\frac{P_2}{\sigma^2}\right).\}\end{aligned} \qquad (17)$$

Let us apply (12) to the individual rate constraints of (17) (assuming we are operating in the Pareto-optimal segment of the Cover-Wyner pentagon). For fixed α and θ, the individual maximal achievable rates become

$$\begin{aligned}C_1(P_1) &= \alpha\log_2\left(1+\frac{P_1}{\sigma^2}\right) + (1-\alpha)\log_2\left(1+\frac{P_1}{P_1/\theta+\sigma^2}\right) \\ C_2(P_2) &= \alpha\log_2\left(1+\frac{P_2}{\theta P_2+\sigma^2}\right) + (1-\alpha)\log_2\left(1+\frac{P_2}{\sigma^2}\right).\}\end{aligned} \qquad (18)$$

The first and second derivatives are equal to (in the limit as $P_1 \to 0$):

$$\dot{C}_1(0) = \dot{C}_2(0) = \frac{1}{\sigma^2},$$

$$\ddot{C}_1(0) = -\frac{1}{\sigma^4}\left(1+\frac{2(1-\alpha)}{\theta}\right).$$

$$\ddot{C}_2(0) = -\frac{1}{\sigma^4}\left(2\theta\alpha+1\right).$$

Plugging these results into (12) we obtain

$$\mathcal{S}_1 = \frac{2\theta}{2-2\alpha+\theta} \quad (19)$$

$$\mathcal{S}_2 = \frac{2}{1+2\alpha\theta}. \quad (20)$$

We can solve for α in (19) and (20) and subtract the resulting equations in order to obtain:

$$1 = \frac{\theta}{\mathcal{S}_1} - \frac{\theta}{\mathcal{S}_2} + \frac{1}{\theta\mathcal{S}_2} - \frac{1}{2\theta}, \quad (21)$$

which is equivalent to the boundary condition in (15). The conditions $\mathcal{S}_1 \leq 2$, $\mathcal{S}_2 \leq 2$ follow immediately from the fact that the existence of an interferer cannot improve the rate. Moreover, the points at which the lines $\mathcal{S}_1 = 2$ and $\mathcal{S}_2 = 2$ intersect with (21) correspond to $\alpha = 1$ and $\alpha = 0$, respectively, i.e. to the vertices to the Cover-Wyner pentagon.

To show (16) note that as either $\theta \to 0$ or $\theta \to \infty$ the third constraint in (15) becomes redundant. □

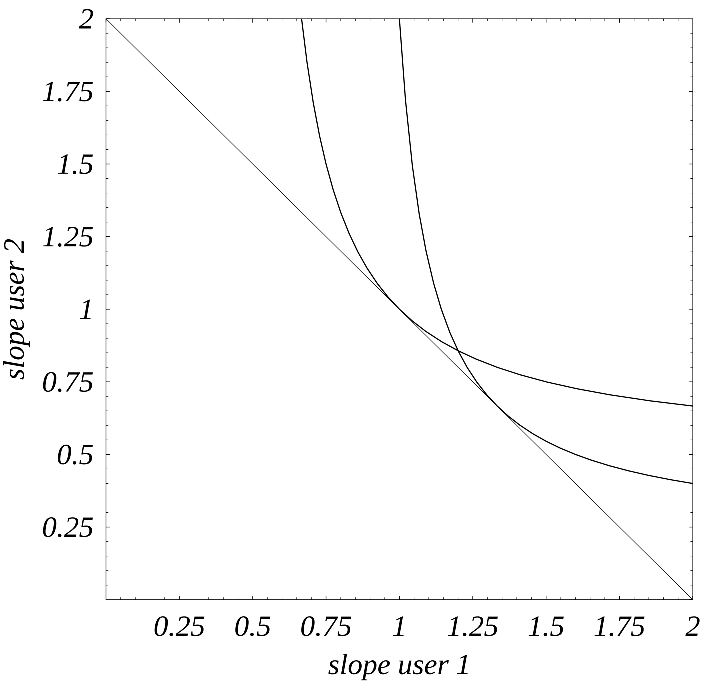

Fig. 3. Slope regions in multiaccess channel with TDMA and superposition $\theta = 1$ and $\theta = 2$.

Theorem 3 shows that *both* users can achieve slopes that are arbitrarily close to the single-user slopes provided they use superposition, optimum decoding, and their powers are sufficiently unbalanced. Therefore, even in the simple setting of the two-user additive Gaussian multiaccess channel the low-power capabilities of TDMA are markedly suboptimal. As a concrete example, suppose we constrain user 1 to have a small rate $R_1 = \epsilon$ and

$$\frac{E_1}{N_0} = (3.01\epsilon - 1.59)dB$$

whereas

$$\frac{E_2}{N_0} = (3.01\frac{\epsilon}{4} - 1.59)dB,$$

then the highest rate achievable by TDMA is

$$R_2^{\mathsf{T}} = \frac{\epsilon}{4}$$

whereas superposition achieves

$$R_2 = \frac{\epsilon}{2},$$

operating at the point $\theta = 2$, $\mathcal{S}_1 = 1$, $\mathcal{S}_2 = 2$ (Figure 3).

Theorem 4: If both users are constrained to have the same energy per bit, then TDMA achieves optimum slopes.

Proof: Identical energies per bit imply that

$$\frac{\mathcal{S}_1}{\mathcal{S}_2} = \frac{R_1}{R_2} = \theta$$

which, when substituting the values found in (20), requires $\alpha = 1/2$. On the other hand, for every value of θ the superposition slope region "touches" the TDMA region at one point (Figure 3), which corresponds to the mid point $\alpha = 1/2$ in the Pareto-optimal segment of the Cover-Wyner pentagon. To see this, note that $\alpha = 1/2$ achieves the minimum sum (equal to 2) of the slopes in (20):

$$\frac{2\theta}{2-2\alpha+\theta} + \frac{2}{1+2\alpha\theta}.$$

□

II. Broadcast Channels

We consider the simple complex-valued two-user broadcast Gaussian channel where users 1 and 2 receive the same signal from the transmitter embedded in independent Gaussian noise with different powers:

$$\begin{aligned} Y_1 &= X + N_1 \\ Y_2 &= X + N_2 \end{aligned} \quad (22)$$

where $E[|X|^2] \leq P$, $E[|N_i|^2] \leq \sigma_i^2$. We will assume $\sigma_1^2 < \sigma_2^2$ as in the case $\sigma_1^2 = \sigma_2^2$ TDMA is trivially optimal. The capacity region of this channel (achieved by superposition and stripping) is equal to [1]

$$\bigcup_{0\leq\alpha\leq1} \{R_1 \leq \log_2\left(1+\frac{\alpha P}{\sigma_1^2}\right)$$
$$R_2 \leq \log_2\left(1+\frac{(1-\alpha)P}{\alpha P+\sigma_2^2}\right)\} \quad (23)$$

whereas the region achievable by TDMA is

$$\bigcup_{0\le\alpha\le1}\left\{\begin{aligned} R_1 &\le \alpha\log_2\left(1+\frac{P}{\sigma_1^2}\right)\\ R_2 &\le (1-\alpha)\log_2\left(1+\frac{P}{\sigma_2^2}\right)\end{aligned}\right\} \tag{24}$$

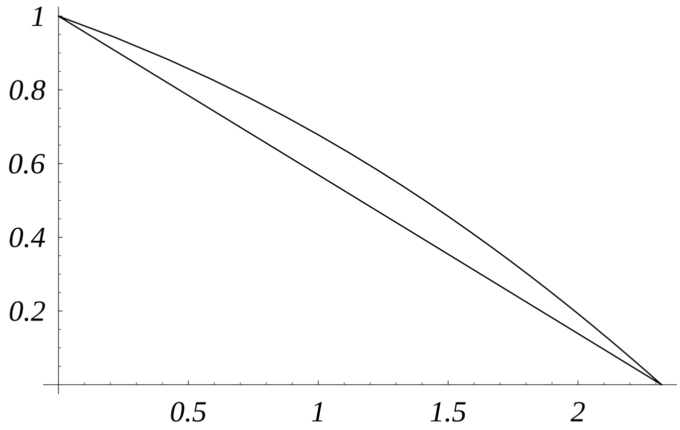

Fig. 4. Capacity region and TDMA-achievable rate region of broadcast channel with $P/\sigma_1^2 = 4$ and $P/\sigma_2^2 = 1$.

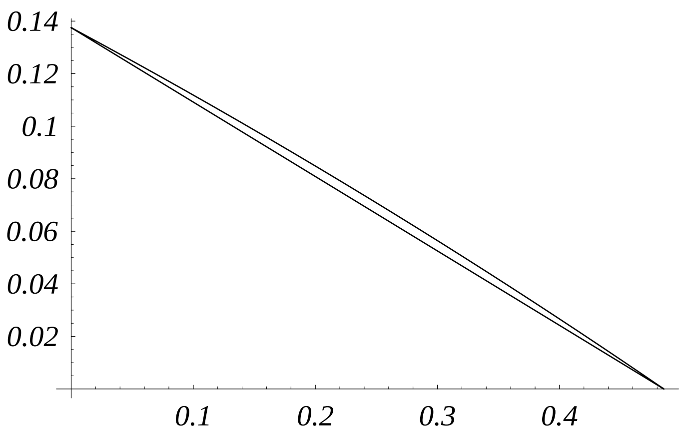

Fig. 5. Capacity region and TDMA-achievable rate region of broadcast channel with $P/\sigma_1^2 = 0.4$ and $P/\sigma_2^2 = 0.1$.

As for multiaccess channels, it appears from Figures 4 and 5 that as the power decreases, the TDMA-achievable region occupies a larger fraction of the capacity region. This fact has been formalized in [7] showing that for every pair (R_1, R_2) in the broadcast capacity region,

$$\limsup_{P\to0} \frac{R_1}{\log_2\left(1+\frac{P}{\sigma_1^2}\right)} + \frac{R_2}{\log_2\left(1+\frac{P}{\sigma_2^2}\right)} \le 1. \tag{25}$$

Analogously to (6) we define

$$\frac{E_i}{N_0} = \frac{P}{R_i\sigma_i^2}. \tag{26}$$

Theorem 5: Suppose that $R_1/R_2 = \theta$. Then, the minimum energies per bit achieved by both TDMA and superposition are:

$$\frac{E_1}{N_0} = \left(1+\frac{\sigma_2^2}{\sigma_1^2\theta}\right)\log_e 2 \tag{27}$$

$$\frac{E_2}{N_0} = \left(1+\frac{\theta\sigma_1^2}{\sigma_2^2}\right)\log_e 2 \tag{28}$$

Proof: Let us start with TDMA. Enforcing the constraint on the ratio of the rates in (24), pins down the value of the time-sharing parameter and we obtain that the rate achieved by user 1 is

$$R_1 = \frac{\theta\log_2\left(1+\frac{P}{\sigma_1^2}\right)\log_2\left(1+\frac{P}{\sigma_2^2}\right)}{\log_2\left(1+\frac{P}{\sigma_1^2}\right)+\theta\log_2\left(1+\frac{P}{\sigma_2^2}\right)}. \tag{29}$$

The reciprocal of the derivative of (29) with respect to P at $P = 0$ is equal to $\theta/(\theta\sigma_1^2+\sigma_2^2)$ and (27) follows. Formula (28) is obtained in an entirely analogous way or simply by noticing from (26) that

$$\frac{\frac{E_1}{N_0}}{\frac{E_2}{N_0}} = \frac{\sigma_2^2}{\theta\sigma_1^2}. \tag{30}$$

Let us analyze now the capacity region (23). Define $\alpha_\theta(P)$ to be the solution to

$$\log_2\left(1+\frac{\alpha_\theta(P)P}{\sigma_1^2}\right) = \theta\log_2\left(1+\frac{(1-\alpha_\theta(P))P}{\alpha_\theta(P)P+\sigma_2^2}\right). \tag{31}$$

Although we are unable to find an explicit solution for $\alpha_\theta(P)$, we will be able to compute the derivative with respect to P at $P = 0$. Equating the derivatives of both sides of (31), we obtain

$$\frac{\frac{\alpha_\theta(P)}{\sigma_1^2}+\frac{P\dot\alpha_\theta(P)}{\sigma_1^2}}{1+\frac{P\alpha_\theta(P)}{\sigma_1^2}} = \frac{\theta\left(\frac{1-\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} - \frac{P\dot\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} - \frac{P(1-\alpha_\theta(P))(\alpha_\theta(P)+P\dot\alpha_\theta(P))}{(\sigma_2^2+P\alpha_\theta(P))^2}\right)}{1+\frac{P(1-\alpha_\theta(P))}{\sigma_2^2+P\alpha_\theta(P)}} \tag{32}$$

Letting $P = 0$ in (32) we get that the derivative at 0 is equal to $\alpha_\theta(0)\sigma_1^2$ and

$$\alpha_\theta(0) = \frac{\theta\sigma_1^2}{\sigma_2^2+\theta\sigma_1^2}. \tag{33}$$

Taking the reciprocal of the derivative and multiplying by $\log_e 2/\sigma_1^2$, (27) follows, and so does (28) by applying (30). □

Let us direct our attention to the analysis of the slopes.

Theorem 6: Let the rates vanish while keeping $R_1/R_2 = \theta$. The broadcast slope region achieved by TDMA is:

$$\{(\mathcal{S}_1,\mathcal{S}_2):\ 0\le\mathcal{S}_1\le\frac{2\theta}{1+\theta},\ 0\le\mathcal{S}_2\le\frac{2}{1+\theta}\}. \tag{34}$$

Proof: As in the proof of Theorem 5, enforcing the constraint $R_1 = \theta R_2$, we obtain the value of the time sharing parameter and the value of the individual rates. Applying formula (12) to (29) and after considerable algebra (omitted in this summary) we obtain the desired result. The fact that if we operate on the boundary of the capacity region we get $\mathcal{S}_1 = \theta \mathcal{S}_2$ can be readily seen from the fact that the numerator in the definition of slope has a factor of θ because $R_1 = R_2\theta$, whereas the denominators are identical: the $\frac{E_b}{N_0}$'s differ by a multiplicative constant (30) which is immaterial in the definition of slope (left side of (12)). □

Theorem 7: Let the rates vanish while keeping $R_1/R_2 = \theta$. The optimum broadcast slope region (achieved by superposition) is:

$$\begin{aligned}\{(\mathcal{S}_1, \mathcal{S}_2):\ 0 \ &\leq\ \mathcal{S}_1 \leq \frac{2\theta\,(\theta + \sigma_2^2/\sigma_1^2)}{\theta^2 + 2\theta + \sigma_2^2/\sigma_1^2},\\ 0 \ &\leq\ \mathcal{S}_2 \leq \frac{2\,(\theta + \sigma_2^2/\sigma_1^2)}{\theta^2 + 2\theta + \sigma_2^2/\sigma_1^2}\}. \end{aligned} \tag{35}$$

Proof: It is sufficient to justify the slope of user 1 because as we saw above $\mathcal{S}_1 = \theta\mathcal{S}_2$. As in the proof of Theorem 5, we need to work with an expression for the achievable rate (31) which depends on an implicitly-defined function $\alpha_\theta(P)$. Equating the second derivatives of both sides in (31) we get

$$-\frac{1}{\theta}\left(\frac{\left(\frac{\alpha_\theta(P)}{\sigma_1^2} + \frac{P\dot\alpha_\theta(P)}{\sigma_1^2}\right)^2}{\left(1 + \frac{P\alpha_\theta(P)}{\sigma_1^2}\right)^2} + \frac{\frac{2\dot\alpha_\theta(P)}{\sigma_1^2} + \frac{P\ddot\alpha_\theta(P)}{\sigma_1^2}}{1 + \frac{P\alpha_\theta(P)}{\sigma_1^2}}\right)$$

$$=$$

$$-\frac{\left(\frac{1-\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} - \frac{P\dot\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} - \frac{P(1-\alpha_\theta(P))(\alpha_\theta(P)+P\dot\alpha_\theta(P))}{(\sigma_2^2+P\alpha_\theta(P))^2}\right)^2}{\left(1+\frac{P(1-\alpha_\theta(P))}{\sigma_2^2+P\alpha_\theta(P)}\right)^2} +$$

$$\left(\frac{1}{1+\frac{P(1-\alpha_\theta(P))}{\sigma_2^2+P\alpha_\theta(P)}}\right)\left(-\frac{2\dot\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} -\right.$$

$$\frac{2(1-\alpha_\theta(P))(\alpha_\theta(P)+P\dot\alpha_\theta(P))}{\left(\sigma_2^2+P\alpha_\theta(P)\right)^2} +$$

$$\frac{2P\dot\alpha_\theta(P)(\alpha_\theta(P)+P\dot\alpha_\theta(P))}{\left(\sigma_2^2+P\alpha_\theta(P)\right)^2} +$$

$$\frac{2P(1-\alpha_\theta(P))(\alpha_\theta(P)+P\dot\alpha_\theta(P))^2}{\left(\sigma_2^2+P\alpha_\theta(P)\right)^3} -$$

$$\frac{P\ddot\alpha_\theta(P)}{\sigma_2^2+P\alpha_\theta(P)} -$$

$$\left.\frac{P(1-\alpha_\theta(P))(2\dot\alpha_\theta(P)+P\ddot\alpha_\theta(P))}{\left(\sigma_2^2+P\alpha_\theta(P)\right)^2}\right)$$

Substituting the value of $\alpha_\theta(0)$ found in (33), and letting $P = 0$, we obtain

$$\dot\alpha_\theta(0) = -\frac{\theta\sigma_1^2(2\theta\sigma_1^2 + \sigma_2^2 - \theta\sigma_2^2)}{2(\theta\sigma_1^2+\sigma_2^2)^3}. \tag{36}$$

Therefore, the first and second derivatives are now computable explicitly. Applying (12), we obtain the formula in (35). □

Comparing the results of Theorems 6 and 7 we see that unless $\sigma_1^2 = \sigma_2^2$ (in which case TDMA is optimal), TDMA is wasteful of channel resources.

Figure 6 plots the ratio between the slopes achieved with superposition and TDMA as a function of θ. This quantifies the bandwidth expansion factor that TDMA requires in the low-power wideband regime.

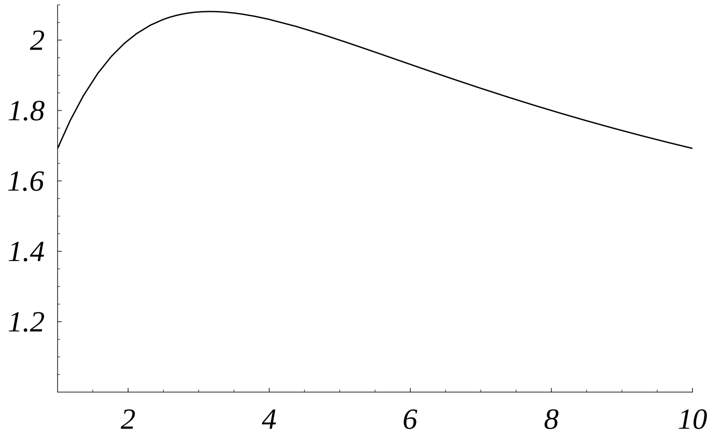

Fig. 6. Bandwidth factor penalty incurred by TDMA as a function of $R_1/R_2 = \theta$ for $\sigma_2^2 = 10\sigma_1^2$

Reference [7] devoted to an analysis of the broadcast channel in the wideband regime states "Our results thus suggest that for high bandwidth channels, time-sharing may be very close to optimal, and that for such channels, very little is to be gained from using the complicated coding schemes that are usually needed to beat the simple time-sharing strategy." This is another example to add to those in reference [6] where it was shown that infinite-bandwidth analyses may lead to misleading conclusions in the wideband regime.

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