Iterative Multiuser Joint Decoding in DS-CDMA: Unified Framework and Performance Analysis

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Abstract— **We present a framework for iterative multiuser joint decoding based on the application of the sum-product algorithm to the factor graph representation of the *a posteriori* joint probability mass function of the users information bits. Several low-complexity algorithms previously proposed based on parallel and serial hard and soft Interference Cancellation (IC) are derived in a simple and unified way. A wide class of these algorithms is analyzed by using the approach of *density evolution on graphs* combined with results from the theory of *large random matrices*.**

I. System Model

Notation conventions: Let $\mathbf{A}$ be a matrix, then $\mathbf{a}_n, \mathbf{a}^k$ and $a_{k,n}$ denote the n-th column, the k-th row and the (k,n)-th element of $\mathbf{A}$; $\mathbf{z} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ indicates that the random vector $\mathbf{z}$ is complex circularly-symmetric jointly Gaussian with mean $E[\mathbf{z}] = \boldsymbol{\mu}$ and covariance $E[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^H] = \boldsymbol{\Sigma}$; $A \propto B$ indicates that A and B differ by a multiplicative term.

We consider the uplink of a coded direct-sequence CDMA system with synchronous transmission (at the chip, symbol and frame level) over frequency-non-selective channels. The system is frame-oriented, i.e., encoding and decoding is performed frame-by-frame. In each frame, the complex baseband equivalent discrete-time signal is given by [1]

$$\mathbf{Y} = \mathbf{S}\mathbf{W}\mathbf{X} + \mathbf{N} \tag{1}$$

where: $\mathbf{Y} \in \mathbb{C}^{L\times N}$ is the array of received signal samples; $\mathbf{N} \in \mathbb{C}^{L\times N}$ is the corresponding array of noise samples, assumed complex circularly-symmetric Gaussian i.i.d. $\sim \mathcal{N}_{\mathbb{C}}(0,1)$; $\mathbf{S} \in \mathbb{C}^{L\times K}$ contains the user spreading sequences by columns; $\mathbf{W} = \mathrm{diag}(w_1, \ldots, w_K)$ contains the user complex amplitudes w_k; $\mathbf{X} \in \mathbb{C}^{K\times N}$ is the array of transmitted code symbols $x_{k,n} \in \mathcal{A}$ (where $\mathcal{A}$ is a unit-energy modulation alphabet, e.g., M-PSK); N, T, L and K denote the code block length and the training sequence length (in symbols), the spreading factor (number of chips per symbol) and the number of users, respectively. With the normalization $|\mathbf{s}_k|^2 = 1$, the signal-to-noise ratio (SNR) of user k is given by $\gamma_k = |w_k|^2$. At each frame, each user encodes a sequence of B_k information bits $\mathbf{b}_k$ into a code word $\mathbf{x}^k = \phi_k(\mathbf{b}_k)$ where $\phi_k : \{0,1\}^{B_k} \to \mathcal{C}_k$ is the encoding function, $\mathcal{C}_k$ is the code book of user k, and $R_k = B_k/N$ is the coding rate in bit/symbol. In this paper we consider non-systematic non-recursive convolutional codes (CC) mapped onto BPSK or Gray-mapped onto QPSK. Each code word is independently interleaved before transmission. The system spectral efficiency is given by $\rho = \alpha \sum_{k=1}^{K} R_k$ where $\alpha = K/L$ is the *channel load* (users per chip).

II. Joint decoding: graph representation and iterative algorithms

Factor graph. The *a posteriori* pmf of the user information bits is given by

$$\Pr(\mathbf{b}_1, \ldots, \mathbf{b}_K | \mathbf{Y}) \propto \prod_{n=1}^{N} q_n(\mathbf{x}_n) \prod_{k=1}^{K} p_k(\mathbf{x}^k, \mathbf{b}_k) \tag{2}$$

where we define the *code constraint functions* $p_k(\mathbf{x}, \mathbf{b}) = 1\{\mathbf{x} = \phi_k(\mathbf{b})\}$ and the *channel transition functions* $q_n(\mathbf{x}) = \exp\left(-|\mathbf{y}_n - \mathbf{S}\mathbf{W}\mathbf{x}|^2\right)$. Fig. 1 represents the factor graph for $\Pr(\mathbf{b}_1, \ldots, \mathbf{b}_K|\mathbf{Y})$ induced by the factorization (2) (black circles denote information bits, white circles denote coded symbols).

The sum-product algorithm. The optimal

MAP detection rule minimizing the average BER for each user is based on the posterior marginal pmfs of the user information bits. In general, the marginalization of $\Pr(\mathbf{b}_1, \ldots, \mathbf{b}_K|\mathbf{Y})$ has complexity exponential in K [1]. A general method for approximating the posterior marginals consists of applying the sum-product algorithm [2] to the factor graph. In the sum-product algorithm the factor graph nodes exchange "messages" along the graph edges. In our case, messages are in the form of real-valued functions defined over $\mathcal{A}$. We let $Q_{k,n}(a)$ and $P_{k,n}(a)$ denote the messages calculated at the n-th channel transition function node q_n and at the k-th code constraint function node p_k, respectively, and sent to the variable node $x_{k,n}$ ($a \in \mathcal{A}$ denotes a dummy variable). The basic computation steps of the algorithm are given as follows.

Computation at the channel transition function nodes:

$$Q_{k,n}(a) \propto \sum_{\substack{\mathbf{a} \in \mathcal{A}^K \\ a_k = a}} \exp\left(-|\mathbf{y}_n - \mathbf{S}\mathbf{W}\mathbf{a}|^2\right) \prod_{j \neq k} P_{j,n}(a_j) \quad \text{for } a \in \mathcal{A} \tag{3}$$

Computation at the code constraint function nodes:

$$P_{k,n}(a) \propto \sum_{\substack{\mathbf{a} \in \mathcal{C}_k \\ a_n = a}} \prod_{j \neq n} Q_{k,j}(a_j) \quad \text{for } a \in \mathcal{A} \tag{4}$$

In some works (see [3] and references therein), the quantities defined in (4) is referred to as the "extrinsic information" (EXT) of the decoder and the calculation (4) is often referred to as *soft-in soft-out* (SISO) decoding. Unfortunately, the CDMA channel has no particular structure enabling efficient evaluation of (3) and this computation has still complexity exponential in K.

III. Low-complexity approximations: IC schemes

We notice that (3) consists of computing the *a posteriori* pmf of $x_{k,n}$ given the observation $\mathbf{y}_n$, assuming that the interfering symbols $x_{j,n}$ are statistically independent with marginal pmf $P_{j,n}(a)$. By artificially modifying the marginal pmfs of the interfering symbols several low-complexity algorithms can be derived in a unified way.

Hard IC. By replacing $P_{j,n}(a)$ with its single mass point approximation $\widehat{P}_{j,n}(a) = 1$ if $a = \arg\max_{a \in \mathcal{A}} P_{j,n}(a)$ (3) reduces to $Q_{k,n}(a) \propto \exp\left(-\gamma_k |z_{k,n} - a|^2\right)$ where

$$z_{k,n} = \frac{1}{w_k} \mathbf{s}_k^H \left(\mathbf{y}_n - \sum_{j \neq k} \mathbf{s}_j w_j \widehat{x}_{j,n}\right) \tag{5}$$

and where $\widehat{x}_{j,n}$ are hard decisions based on the EXT pmfs $P_{j,n}(a)$.

SUMF-based soft IC. By replacing $P_{j,n}(a)$ with the complex circularly-symmetric Gaussian pdf with the same mean $\widetilde{x}_{j,n} = \sum_{a \in \mathcal{A}} a P_{j,n}(a)$ and variance $\xi_{j,n} = 1 - |\widetilde{x}_{j,n}|^2$, $\mathbf{y}_n$ can be treated as a Gaussian vector conditionally on $x_{k,n}$. With the additional simplifying assumption that $\mathbf{y}_n$ is conditionally white, after some trivial algebra (3) reduces to $Q_{k,n}(a) \propto \exp\left(-\delta_{k,n} |z_{k,n} - a|^2\right)$ where we let $\delta_{k,n} = \frac{\gamma_k}{1 + \frac{1}{L}\sum_{j \neq k} \gamma_j \xi_{j,n}}$ and

$$z_{k,n} = \frac{1}{w_k} \mathbf{s}_k^H \left(\mathbf{y}_n - \sum_{j \neq k} \mathbf{s}_j w_j \widetilde{x}_{j,n}\right) \tag{6}$$

LMMSE-based soft IC. We make the same conditional Gaussian approximation but we take into account the resulting covariance matrix $\widetilde{\mathbf{\Sigma}}_{k,n} = \sum_{j \neq k} \mathbf{s}_j \mathbf{s}_j^H |w_j|^2 \xi_{j,n} + \mathbf{I}$. Then, (3) reduces to $Q_{k,n}(a) \propto \exp\left(-\beta_{k,n} |z_{k,n} - a|^2\right)$ where we let $\beta_{k,n} = \gamma_k \mathbf{s}_k^H \widetilde{\mathbf{\Sigma}}_{k,n}^{-1} \mathbf{s}_k$ and

$$z_{k,n} = \frac{w_k^*}{\beta_{k,n}} \mathbf{s}_k^H \widetilde{\mathbf{\Sigma}}_{k,n}^{-1} \left(\mathbf{y}_n - \sum_{j \neq k} \mathbf{s}_j w_j \widetilde{x}_{j,n}\right) \tag{7}$$

Remark: on the use of APPs versus extrinsic pmfs. In several papers (e.g., [4], [5], [6], [7], [8]) similar algorithms were derived from different arguments but $\widehat{x}_{j,n}$ and $\widetilde{x}_{j,n}$ are calculated based on the APPs and not on the EXT pmfs. As a consequence, the residual interference plus noise at the decoder input, $\zeta_{k,n} = z_{k,n} - x_{k,n}$, is biased conditionally on $x_{k,n}$ [9], [10], i.e., $E[\zeta_{k,n}|x_{k,n}] = -\mu_{k,n} x_{k,n}$ where $\mu_{k,n} \geq 0$. In other words, the use of APPs induces a term which *reduces* the useful signal while trying to cancel interference. On the contrary, if EXT probabilities are used (assuming large random interleavers) $\zeta_{k,n}$ is conditionally unbiased and behaves as an additive noise term. It turns out that the APP based algorithms provide a maximum spectral efficiency considerably lower than their EXT-based counterparts derived here [11].

IV. Asymptotic performance analysis

There are two main obstacles to the performance analysis of the IC algorithms derived in the previous section: 1) The factor graph contains cycles; 2) The output statistics of $z_{k,n}$ produced by IC and filtering depends in a complicated way on the interferers residual power and spreading sequences. The first obstacle is removed by assuming $N \to \infty$. The second obstacle is removed by assuming random spreading and a *large system* regime, i.e., $K \to \infty$ while the load K/L converges to a given value α (see [12] and references therein).

Our analysis is based on the probability *density evolution* (DE) [13]. Briefly, the messages $\{P_{k,i}(a) : a \in \mathcal{A}\}$ and $\{Q_{k,n}(a) : a \in \mathcal{A}\}$ are random vectors and their pdfs evolve with the iterations by propagating along the factor graph. The probability measure on the messages is induced by the joint probability measure of the noise $\mathbf{N}$ and of the code words $\mathbf{X}$ (uniformly distributed over the user codes). Both the IC steps and the SISO steps can be characterized in terms of input-output SNR (by making a Gaussian approximation of log-likelihood ratios at the SISO outputs). The details involve several technicalities and are given in [10]. Notice that the APP-based algortihms of [4], [5], [6], [7], [8] cannot be analyzed in this way since APPs violate the basic fact that *what is propagated forward nevers comes back*, which is indeed satisfied by the sum-product (EXT-based) algorithms.

Let $\eta^{(m)}$ denote the *Asymptotic Multiuser Efficiency* (AME) at the input of the decoder of any user (in a large system this is independent of the particular user). Then, for each of the IC schemes derived before it is possible to write a recursion for $\eta \in [0, 1]$ in the form

$$\eta^{(m+1)} = f(F(\gamma), \eta^{(m)})$$

where $F(\gamma)$ is the limit of the empirical distribution of the user SNRs and the functional f depends on the particular IC scheme considered. Fig. 2 shows the characteristic curves of the above recursion for CC of rate $R = 1/2$ and generators $(5, 7)_8$ mapped onto BPSK, equal power users with SNR $\gamma = 3$dB (corresponding to $E_b/N_0 = 6$ dB), and $\alpha = 2.5$ users/chip. Stable fixed points corresponds to the intersections of the "S" shaped curve with the diagonal, with slope less than 1. For load α less than a certain threshold which depend on the SNR and on the IC scheme considered, a single stable fixed point exists for which η is very close to 1. This corresponds to the (quasi) single-user performance, i.e., the iterative decoder is able to remove completely interference and all users achieve the single-user BER bound. If another fixed point appears, then the system gets stuck at a very high BER. For example, in the figure, the hard IC scheme does not reach the single-user performance while the soft SUMF and MMSE scheme do. Then, for a given target BER the system spectral efficiency is determined by $\rho = \alpha^* R$ where α^* is the maximum load at which the single-user performance (equal to the target BER) is attained.

Fig. 3 shows the spectral efficiency achievable by optimal binary CCs of rate 1/2 and 4,8,16,32 and 64 states concatenated with BPSK and QPSK with various receivers (hard and soft IC) and infinite iterations. For the sake of comparison, the spectral efficiency achievable by optimal joint decoding, linear MMSE and SUMF with single-user decoding and Gaussian inputs is also shown (see [14]). The code complexity increases with E_b/N_0 and decreases with the spectral efficiency ρ.

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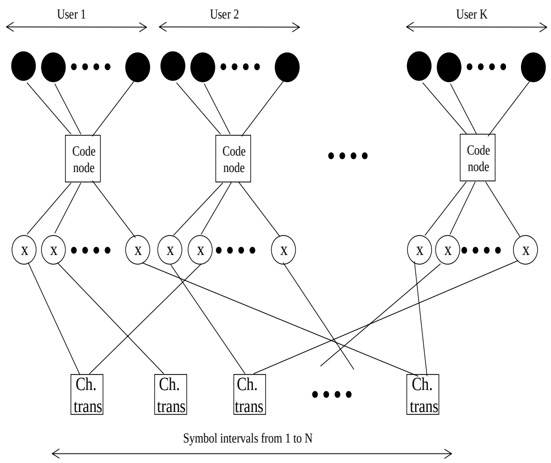

Fig. 1. Factor graph of the CDMA coded channel.

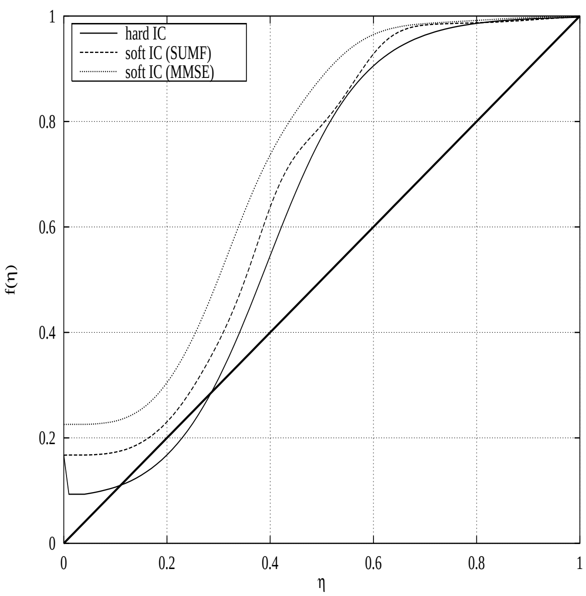

Fig. 2. Density evolution characteristic curves of IC iterative schemes.

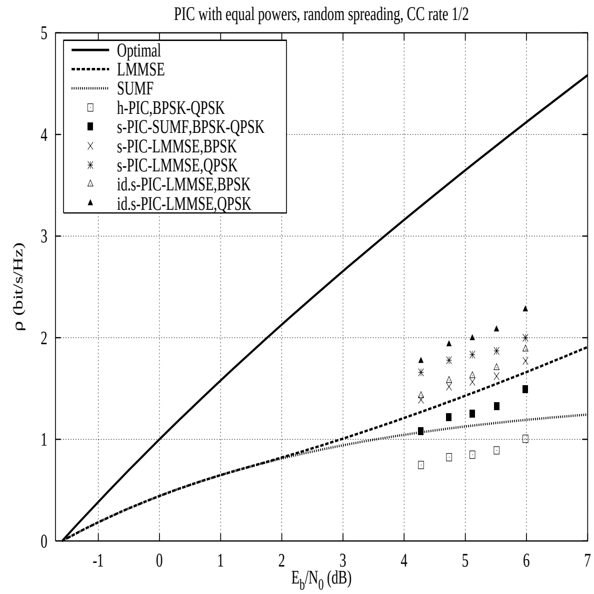

Fig. 3. Spectral efficiency achieved by CCs of rate 1/2.

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