The Optimal Received Power Distribution for IC-based Iterative Multiuser Joint Decoders

Giuseppe Caire
Eurecom Institute
Sophia-Antipolis, France
caire@eurecom.fr

Ralf Müller
FTW
Vienna, Austria
mueller@ftw.at

Abstract

We use the large-system Density Evolution (DE) analysis of iterative Interference-Cancellation (IC) multiuser detection/decoding as a tool for optimizing the received SNR distribution. We show that the SNR distribution achieving maximum possible spectral efficiency for a given code and target bit-error rate (BER) can be obtained by linear programming. We show that large spectral efficiencies and low BER are achievable by very simple standard convolutional codes and QPSK modulation, provided that the SNR distribution is appropriately optimized.

1 Introduction

Iterative IC coupled with single-user soft-input soft-output (SISO) decoders is a powerful and well-known technique for joint detection in CDMA (see [1] and references therein). In [1], iterative IC-based multiuser joint decoding was shown to be an instance of the the sum-product (belief-propagation) algorithm [2], with some simplifications, and, as such, rigorous asymptotic performance analysis can be obtained by applying the general technique of density evolution (DE), previously developed in [3] for the analysis of message-passing decoders.

In this paper, we make use of this analysis as a design tool in order to shape the *received* user SNR distribution for maximizing the system spectral efficiency. We restrict our treatment to synchronous CDMA, binary convolutional codes and Gray-labeled QPSK. In this simple but relevant case, the optimal SNR distribution is obtained through linear programming (a striking analogy with the optimization of degree sequences of Low-Density Parity-Check codes [3]).

We show that large spectral efficiencies and low BER are achievable by very simple standard convolutional codes and QPSK modulation, provided that the SNR distribution is appropriately shaped. This has an interesting consequence in terms of power-control strategy for the uplink of CDMA systems: traditional power control induces at the receiver a uniform SNR distribution. On the contrary, if IC-MUD is used, it should induce the optimized SNR distribution that maximizes the total (cell) spectral efficiency. By exploiting the fact that users are naturally received at non-constant SNR levels, this new strategy might yield important savings in the overall power consumption, and reduce further the inter-cell interference in a multicell CDMA system.

2 Synchronous CDMA system model

We consider the complex baseband discrete-time channel model

$$\mathbf{Y} = \mathbf{SWX} + \mathbf{N} \tag{1}$$

originated by sampling at the chip-rate a synchronous CDMA system [4], where: 1) $\mathbf{Y}, \mathbf{N} \in \mathbb{C}^{L\times N}$, are the matrix of received chip-rate samples and the corresponding AWGN samples $\sim \mathcal{N}_{\mathbb{C}}(0,1)$; 2) $\mathbf{S} \in \mathbb{C}^{L\times K}$ contains the user spreading sequences by columns; 3) $\mathbf{W} = \mathrm{diag}(w_1, \dots, w_K)$ contains the user complex amplitudes; 4) $\mathbf{X} \in \mathcal{A}^{K\times N}$ is the matrix of user modulation symbols, where $\mathcal{A} \subseteq \mathbb{C}$ is a complex signal set (in our case, QPSK). The row $\mathbf{x}^k$ of $\mathbf{X}$ is the code word transmitted by user k. The column $\mathbf{x}_n$ of $\mathbf{X}$ is the vector of symbols transmitted by all users at the same time (n-th symbol interval); 5) L, K and N denote the spreading factor, the number of users and the code block length, respectively.

Spreading sequences are random with i.i.d. elements with mean 0, variance $1/L$ and finite fourth order moment, the average energy per symbol is 1, and the k-th user *received* SNR is $\gamma_k = |w_k|^2$. Users send information messages in the form of binary uniformly distributed vectors $\mathbf{b}_k \in \mathbb{F}_2^B$, and make use of Bit-Interleaved Coded Modulation (BICM) [5]. Namely, the code word $\mathbf{x}^k$ is obtained by applying the Gray labeling $\mu : \{0,1\}^2 \to \mathcal{A}$ to the binary code word $\mathbf{c}^k = \phi_k(\mathbf{b}_k)$ of length $2N$, where $\phi_k : \mathbb{F}_2^B \to \mathcal{C}_k \subseteq \mathbb{F}_2^{2N}$ denotes the encoding function of user k code (including the bit-interleaving) and $\mathcal{C}_k$ denotes the code book of user k. For the sake of simplicity, we assume that all user codes are based on the same binary convolutional code, and differ only by the bit-interleavers. The user coding rate is given by $R = B/N$ bit/symbol and the system spectral efficiency is given by $\rho = \alpha R$, where $\alpha = K/L$ (users per chip) is the *channel load.*

3 IC-based iterative decoding algorithms

For a binary variable c with pmf $(\Pr(c = 0), \Pr(c = 1))$ we define its log-ratio by

$$\mathcal{L} \triangleq \log \frac{\Pr(c=0)}{\Pr(c=1)}$$

All the decoding algorithms considered here compute iteratively the log-ratios $\mathcal{L}_{k,j}^{\mathrm{bit}}$ for the information bits according to the sum-product algorithm applied to the factor-graph of the a posteriori joint pmf $\Pr(\mathbf{b}_1, \dots, \mathbf{b}_K|\mathbf{Y})$. IC follows as a simplification of the basic sum-product computation, as shown in [1]. After a given number of iterations, an approximated MAP symbol-by-symbol decision is made according to the threshold rule $\widehat{b}_{k,j} = 1\{\mathcal{L}_{k,j}^{\mathrm{bit}} < 0\}$.

The main blocks exchanging messages in the iterative decoder are the SISO decoders and the IC multiuser detectors (IC-MUD). SISO decoding is formally given by

$$\mathcal{L}_{k,\ell}^{\mathrm{dec}} = \log \frac{\sum_{\mathbf{c}\in\mathcal{C}_k : c_\ell = 0} \exp\left(\sum_{j\neq\ell} c_j \mathcal{L}_{k,j}^{\mathrm{dem}}\right)}{\sum_{\mathbf{c}\in\mathcal{C}_k : c_\ell = 1} \exp\left(\sum_{j\neq\ell} c_j \mathcal{L}_{k,j}^{\mathrm{dem}}\right)} \tag{2}$$

where $\mathcal{L}_{k,j}^{\mathrm{mud}}$ is the message sent by the IC-MUD for user k relative to coded symbol $c_{k,j}$ and $\mathcal{L}_{k,\ell}^{\mathrm{dec}}$ is the so called decoder "extrinsic information" For convolutional codes,

(2) is efficiently implemented by the forward-backward algorithm. Obviously, the same forward-backward algorithm can compute the messages $\{\mathcal{L}_{k,j}^{\rm bit} : j = 1, \dots, B\}$ for the information bit nodes while computing (2).

IC-MUD consists of forming "conditional mean" estimates

$$\widetilde{x}_{k,n} = \frac{1}{\sqrt{2}} \tanh(\mathcal{L}_{k,\ell_1}^{\rm dec}/2) + j \frac{1}{\sqrt{2}} \tanh(\mathcal{L}_{k,\ell_2}^{\rm dec}/2)$$

of $x_{k,n}$ (where we assume that c_{k,ℓ_1} and c_{k,ℓ_2} are the symbols mapped onto the I and Q components of modulation symbol $x_{k,n}$), and compute the observation $z_{k,n}$ for $x_{k,n}$ in the form of IC followed by filtering, i.e.,

$$z_{k,n} = \mathbf{h}_{k,n}^H \left(\mathbf{y}_n - \sum_{j \neq k} w_j \mathbf{s}_j \widetilde{x}_{j,n} \right) \tag{3}$$

Here, $\mathbf{h}_{k,n}$ is a suitable linear filter which may vary from iteration to iteration. Assuming $z_{k,n} = x_{k,n} + \nu_{k,n}$, where $\nu_{k,n} \sim \mathcal{N}_{\mathbb{C}}(0, 1/\beta_{k,n})$ and $\beta_{k,n}$ is the SINR at the output of the IC-MUD filter for user k, the messages sent back to the SISO decoder are given by

$$\mathcal{L}_{k,\ell_i}^{\rm mud} = \left\{ \begin{array}{ll} 2\sqrt{2}\beta_{k,n} {\rm Re}\{z_{k,n}\} & i = 1 \\ 2\sqrt{2}\beta_{k,n} {\rm Im}\{z_{k,n}\} & i = 2 \end{array} \right. \tag{4}$$

Choice of the IC filter. We shall consider the following choices: 1) Single-user matched filter (SUMF) [6] $\mathbf{h}_{k,n} = \frac{1}{w_k}\mathbf{s}_k$; 2) Unconditional Linear-MMSE filter (Unc.-MMSE), chosen to minimize $E\left[|x_{k,n} - z_{k,n}|^2\right]$. Assuming $N \to \infty$ and random interleaver, it can be shown that at any finite number of iterations the messages $\{\mathcal{L}_{k,n}^{\rm dec} : k = 1, \dots, K\}$ are mutually independent and independent of the noise at time n. This yields

$$\mathbf{h}_{k,n} = \frac{w_k}{\beta_{k,n}} \left[\mathbf{I} + \sum_{j \neq k} \gamma_j E[|x_{j,n} - \widetilde{x}_{j,n}|^2] \mathbf{s}_j \mathbf{s}_j^H \right]^{-1} \mathbf{s}_k \tag{5}$$

where

$$\beta_{k,n} = \gamma_k \mathbf{s}_k^H \left[\mathbf{I} + \sum_{j \neq k} \gamma_j E[|x_{j,n} - \widetilde{x}_{j,n}|^2] \mathbf{s}_j \mathbf{s}_j^H \right]^{-1} \mathbf{s}_k \tag{6}$$

3) Conditional Linear-MMSE filter (Cond.-MMSE) [7, 8, 9], chosen to minimize $E\left[|x_{k,n} - z_{k,n}|^2 | \{P_{j,n}(a) : a \in \mathcal{A}\}, j \neq k\right]$. Under the same assumptions, we obtain

$$\mathbf{h}_{k,n} = \frac{w_k}{\beta_{k,n}} \left[\mathbf{I} + \sum_{j \neq k} \gamma_j (1 - |\widetilde{x}_{j,n}|^2) \mathbf{s}_j \mathbf{s}_j^H \right]^{-1} \mathbf{s}_k \tag{7}$$

where

$$\beta_{k,n} = \gamma_k \mathbf{s}_k^H \left[\mathbf{I} + \sum_{j \neq k} \gamma_j (1 - |\widetilde{x}_{j,n}|^2) \mathbf{s}_j \mathbf{s}_j^H \right]^{-1} \mathbf{s}_k \tag{8}$$

The SUMF is invariant with the symbol index n and with the iterations. The Unc.-MMSE filter must be computed at each iteration, but is invariant with n, while the Cond.-MMSE filter must be computed for all $n = 1, \dots, N$ and each iteration, thus, its practical relevance is limited to systems with very small K and L.

4 Density evolution and Gaussian approximation

The messages $\mathcal{L}^{\text{mud}}_{k,n}$ and $\mathcal{L}^{\text{dec}}_{k,\ell}$ are random variables whose joint pdf is induced by the joint probability measure of the users information bits, of the channel noise, of the user spreading sequences, of the user carrier phases $\theta_k = \arg w_k$, assumed i.i.d. uniformly distributed over $[0, 2\pi]$, and of the random bit-interleavers in the BICM user codes.

The DE approach to the analysis of message-passing iterative decoding algorithms consists of propagating from iteration to iteration the pdf of the messages [3]. The bit-error probability performance of the decoder can be derived by the limiting pdf of the messages after a large number of iterations. Under mild conditions, as $N \to \infty$ a general concentration result [3] ensures that the messages arriving at each node are mutually statistically independent, and their marginal pdfs converge in probability to the marginal pdfs computed on a cycle-free average graph, where in our case "averaging" is with respect to the graph structure defined by the bit-interleavers.

In order to remove the randomness due to the random selection of the users spreading sequences, information messages and carrier phases, we study the iterative IC joint decoder in the large-system limit [10, 11, 12], i.e., we let $L \to \infty$ with fixed ratio $K/L = \alpha$ (notice the order of the limits: first we let $N \to \infty$ and then $L, K \to \infty$). We assume that the users are grouped into a finite number J of classes of cardinality $K_1, \ldots, K_J$, where $K = \sum_{j=1}^{J} K_j$, with received SNR levels $g_1, \ldots, g_J$, i.e., $\gamma_k = g_j$ if user k belongs to class j. We assume also that the underlying convolutional code has finite *unisotropy degree* [1]. [1] Finally, we assume that the ratio $\alpha_j = K_j/L$ remains fixed for all j, as $L \to \infty$.

Let $U_{k,n}$ denote the residual symbol variance after IC for the n-th symbol of the k-th user, at decoder iteration m. Under the above assumptions, we can show that the empirical cdf of the residual symbol variance of users in class j converges to a given deterministic cdf $F^{(m)}_j(u)$. This limit depends only on the user class j and on the iteration index m, not on k and n. Hence, known results from large-system analysis of randomly spread CDMA [12, 14] apply and we get that the real and imaginary parts of the residual interference plus noise at the output of the IC-MUD for a user in class j converge in distribution to a real zero-mean Gaussian random variable with variance $1/(2g_j\eta^{(m)})$, where $\eta^{(m)}$ can be interpreted as the *Asymptotic Multiuser Efficiency* (AME) [4] of the decoder at iteration m. For the SUMF we obtain [12]

$$\eta^{(m)} = \left[1 + \sum_{j=1}^{J} \alpha_j g_j \int u dF^{(m)}_j(u)\right]^{-1} \tag{9}$$

For the Unc.-MMSE and Cond.-MMSE IC-MUD, $\eta^{(m)}$ is the unique non-negative solution of the fixed point equation [12]

$$\eta = \left[1 + \sum_{j=1}^{J} \alpha_j \int \frac{g_j u}{1 + \eta g_j u} dF^{(m)}_j(u)\right]^{-1} \tag{10}$$

This implies that, in the large system limit, the conditional distribution of $\mathcal{L}^{\text{dem}}_{k,\ell}$ given

[1] Briefly, the unisotropy degree of a code is defined as the number of different marginal pdfs of the output messages of the SISO decoder, when the decoder input is the observation provided by a binary-input symmetric-output memoryless channel. A time-invariant linear convolutional code of rate k/n has unisotropy degree at most n [13].

$c_{k,\ell} = 0$, for user k in class j, is $\mathcal{N}(2g_j\eta^{(m)}, 4g_j\eta^{(m)})$. Hence, the DE is completely expressed by the evolution of the single parameter $\eta^{(m)}$, for $m = 0, 1, 2, \ldots$.

For general linear convolutional codes, the SISO decoder is too complicated for the computation in closed form of the pdf of $\mathcal{L}^{\rm dec}_{k,\ell}$ from the pdf of $\mathcal{L}_{k,\ell}$. A semianalytic technique to the DE consists of approximating the pdf by a Monte Carlo generated histogram, obtained directly by the forward-backward algorithm applied to randomly generated i.i.d. input log-ratios $\mathcal{L}^{\rm mud}_{k,\ell} \sim \mathcal{N}(2g_j\eta^{(m)}, 4g_j\eta^{(m)})$. A simpler approach consists of a Gaussian Approximation (GA) of the SISO output messages [15, 16, 17, 1]. Here, we make use of the "Gaussian tail matching" approximation of [1]. Let ϵ denote the symbol-error rate (SER) at the SISO decoder output, given by [2] $\epsilon = \Pr\left(1\left\{\mathcal{L}^{\rm dec}_{k,\ell} < 0\right\} \neq c_{k,\ell}\right)$. Assuming $\mathcal{L}^{\rm dec}_{k,\ell} \sim \mathcal{N}(\mu, 2\mu)$ then $\epsilon = Q(\sqrt{\mu/2})$. For a given convolutional code over AWGN, the SER ϵ is a known function of the decoder input SNR, which in our case is given by $g_j\eta^{(m)}$ for a user in class j at iteration m. Hence, the pdf of the log-ratios at the output of its SISO decoder (under the GA assumption) is uniquely identified by the single parameter

$$\mu_j^{(m)} = 2\left[Q^{-1}(\epsilon(g_j\eta^{(m)}))\right]^2 \tag{11}$$

We shall denote by $p_j^{(m)}(\lambda)$ the pdf of $\mathcal{L}^{\rm dec}_{k,\ell}$ given $c_{k,\ell} = 0$, for a user in class j. Now, in order to obtain the full DE-GA analysis, we need to express $\eta^{(m+1)}$ as a function of $\eta^{(m)}$.

DE-GA for the SUMF-based iterative IC joint decoder. In the large-system limit, $U_{k,n} = 1 - E[|\widetilde{x}_{k,n}|^2]$ converges to the deterministic quantity $v_j = E[u(\mathcal{L}_1, \mathcal{L}_2)]$, where $\mathcal{L}_1, \mathcal{L}_2$ are i.i.d. and distributed as $\mathcal{L}^{\rm dec}_{k,\ell}$, and where we define

$$u_j(\lambda_1, \lambda_2) \triangleq \frac{1}{2}\left(1 - |\tanh(\lambda_1/2)|^2\right) + \frac{1}{2}\left(1 - |\tanh(\lambda_2/2)|^2\right) \tag{12}$$

Thus, by using the GA developed above and the the symmetry property $p_j^{(m)}(\lambda)e^{-\lambda} = p_j^{(m)}(-\lambda)$, valid for any log-ratio pdf [3], we obtain

$$v_j = \int \frac{4}{(e^\lambda + 1)^2} p_j^{(m)}(\lambda) d\lambda \tag{13}$$

Clearly, v_j is a function of $g_j\eta^{(m)}$, and will be denoted by $v_j = V(g_j\eta^{(m)})$. Finally, plugging it into (9) the full DE-GA of the iterative decoder with SUMF IC-MUD is the trajectory of the homogeneous non-linear one-dimensional dynamic system

$$\eta^{m+1} = \Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta^{(m)}) \tag{14}$$

initialized by $\eta^{(0)} = 0$, where the mapping function of the system is defined as

$$\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta) \triangleq \left[1 + \sum_{j=1}^{J} \alpha_j g_j V(g_j\eta)\right]^{-1} \tag{15}$$

and where the vectors $\mathbf{g} \triangleq (g_1, \ldots, g_J)$ and $\boldsymbol{\alpha} \triangleq (\alpha_1, \ldots, \alpha_J)$ define the received SNR distribution.

[2] Notice that SER refers to decisions based on extrinsic information, not on a posteriori probabilities of the SISO decoders.

DE-GA for the Unc-MMSE-based iterative IC joint decoder. Also in this case, the DE-GA is given by the trajectory of a homogeneous non-linear one-dimensional dynamic system initialized by $\eta^{(0)} = 0$, whose mapping function is given implicitly by the solution $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta)$ of the fixed-point equation

$$\Psi = \left[1 + \sum_{j=1}^{J} \alpha_j \frac{g_j V(g_j \eta)}{1 + \Psi g_j V(g_j \eta)} \right]^{-1} \tag{16}$$

DE-GA for the Cond-MMSE-based iterative IC joint decoder. Again, the DE-GA is given by the trajectory of a homogeneous non-linear one-dimensional dynamic system initialized by $\eta^{(0)} = 0$, whose mapping function is given implicitly by the solution $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta)$ of the fixed-point equation

$$\Psi = \left[1 + \sum_{j=1}^{J} \alpha_j E \left[\frac{g_j u(\mathcal{L}_1, \mathcal{L}_2)}{1 + \Psi g_j u(\mathcal{L}_1, \mathcal{L}_2)} \right] \right]^{-1} \tag{17}$$

By using the symmetry properties of log-ratio pdfs, the expectation in (17) takes on the form

$$\int \frac{4(e^{\lambda_2} + 1)^2}{\frac{1}{g_j}(e^{\lambda_1} + 1)^2 (e^{\lambda_2} + 1)^2 + 2\Psi[e^{\lambda_1}(e^{\lambda_2} + 1)^2 + e^{\lambda_2}(e^{\lambda_1} + 1)^2]} p_j(\lambda_1) p_j(\lambda_2) d\lambda_1 d\lambda_2 \tag{18}$$

where $p_j(\lambda)$ denotes $p_j^{(m)}(\lambda)$ evaluated at $\eta^{(m)} = \eta$. Notice that in this case the dependency of $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta)$ on η is given through the pdf $p_j(\lambda)$.

5 Optimal received power distribution

Within the limits of the assumptions made in order to obtain the DE-GA, the decoder performance is completely characterized by the fixed points of the system defined by the mapping function $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta)$. This is continuous and non-decreasing in η, with $\Psi(\mathbf{g}, \boldsymbol{\alpha}, 0) > 0$ and $\Psi(\mathbf{g}, \boldsymbol{\alpha}, 1) \leq 1$. Then, the trajectory with initial condition $\eta^{(0)} = 0$ converges to the fixed point given by the smallest solution of the equation

$$\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta) = \eta, \quad \eta \in [0, 1] \tag{19}$$

Next, we optimize the system spectral efficiency with respect to the received power distribution, defined by $(\mathbf{g}, \boldsymbol{\alpha})$. We fix a target maximum BER, to be achieved by all users in the system. This implies that for all users, after the iterative decoder has converged to a stationary point, the SINR at the SISO decoder inputs must be not smaller than a given threshold value SINR_{th}, which depends on the code and on the target BER. We discretize the SNR values such that $g_1 < g_2 < \cdots < g_J$, for some integer J, we select a desired channel load α, a *constraint interval* $[\delta_1, \delta_2] \subseteq [0, 1]$ and a margin $\epsilon > 0$. Then, we solve the following constrained optimization problem with respect to $\boldsymbol{\alpha}$

$$\left\{ \begin{array}{ll} \text{minimize} & \sum_{j=1}^{J} \alpha_j g_j \\ \text{subject to} & \Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta) \geq \eta + \epsilon, \quad \forall \ \eta \in [\delta_1, \delta_2] \\ & \sum_{j=1}^{J} \alpha_j = \alpha, \\ & \alpha_j \geq 0, \quad \forall \ j \end{array} \right. \tag{20}$$

The solution $\boldsymbol{\alpha}^\star$ can be accepted if the fixed point $\eta^\star$, i.e., the smallest solution of the equation $\Psi(\mathbf{g}, \boldsymbol{\alpha}^\star, \eta) = \eta$ for $\eta \in [0,1]$, is such that $g_1\eta^\star \geq \text{SINR}_{\text{th}}$. Otherwise, the program is run again by changing the SNR values $\mathbf{g}$ and the design parameters $\delta_1, \delta_2, \epsilon$.

The program (20) is indeed linear, as we show in the following. The objective function, the non-negativity constraint and the equality constraint are obviously linear. The inequality constraint $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta) \geq \eta + \epsilon, \quad \forall \;\; \eta \in [\delta_1, \delta_2]$ is also linear in $\boldsymbol{\alpha}$. Consider first the SUMF IC-MUD algortihm, for which the mapping function is given explicitly. By using (15) we get that the inequality constraint can be written as

$$\sum_{j=1}^{J} \alpha_j g_j V(g_j \eta) \leq \frac{1}{\eta + \epsilon} - 1 \tag{21}$$

for all $\eta \in [\delta_1, \delta_2]$.

For the Unc.-MMSE and Cond.-MMSE cases, we do not have an explicit expression for $\Psi(\mathbf{g}, \boldsymbol{\alpha}, \eta)$. However, we can use the following property of the fixed-point equation (10) [12]. Let $\Psi = G_\eta(\Psi)$ be the fixed point equation yielding Ψ as a function of η, and let $\Psi^\star$ be the unique non-negative solution, for η fixed. Then, the following implication holds: $\Psi \leq \Psi^\star \Leftrightarrow \Psi \leq G_\eta(\Psi)$. Hence, we conclude that the inequality $\Psi^\star \geq \eta$ is equivalent to $\eta \leq G_\eta(\eta)$, which yields

$$\sum_{j=1}^{J} \alpha_j \int \frac{g_j u}{1 + \eta g_j u} dF_j(u) \leq \frac{1}{\eta + \epsilon} - 1 \tag{22}$$

6 Results

We define the *system* E_b/N_0 as $(E_b/N_0)_{\text{sys}} \triangleq \frac{\sum_{j=1}^{J} \alpha_j g_j}{\rho}$. with the following operational meaning: if any user is assigned to class j for a fraction α_j/α of the time, then the average received E_b/N_0 of each user is precisely $(E_b/N_0)_{\text{sys}}$. Clearly, a reference system with equal rate and power for all users has $(E_b/N_0)_{\text{sys}}$ equal to the conventional definition of E_b/N_0 for each user. By fixing the user coding rate R and the load α (i.e., by fixing the spectral efficiency $\rho = \alpha R$), the optimization problem (20) finds the minimum $(E_b/N_0)_{\text{sys}}$ at which the target BER can be achieved by all users for given user codes.

Fig. 1 shows the map Ψ for the optimal 4-state and 64-state codes of rate $1/2$, $\alpha = 2.2$, Unc.MMSE, and equal SNR distribution (denoted by "Eq.") corresponding to $E_b/N_0 = 6$ dB. We notice that the 4-state code achieves almost single-user performance ($\eta^\star \approx 1$) while the 64-state code has a fixed point at $\eta^\star \approx 0.16$, meaning that the system in this case converges to very large BER for all users. The curve denoted by "Opt." corresponds to the 64-state code with optimized SNR distribution (we let $\epsilon = 0.01$, $\delta_1 = 0$ and $\delta_2 = 0.7$). In this case, the system converges to BER $\leq 10^{-5}$ for all users, at $(E_b/N_0)_{\text{sys}} = 5.2$ dB. As a general remark, we observe that better codes do not necessarily provide better performance with equal SNR distribution, while they provide always better performance with optimized SNR distribution.

Fig. 2 shows the trajectory of $\eta^{(m)}$ vs. the number of decoder iterations for the system of Fig. 1 with optimized SNR distribution and the 64-state code. The equal power case is also shown, and we notice that its AME does not improve with iterations. We show the trajectories given by the GA, DE (with Monte Carlo generation of the SISO output pdf.) and a snapshot simulation of a finite-dimensional system with $L = 32$, $K = 70$ and $N = 2000$, obtained by quantizing the optimized infinite-dimensional system. The GA yields slightly optimistic results. However, for the purpose of system design, it provides an efficient tool.

Figs. 3 and 4 show the spectral efficiency vs. $(E_b/N_0)_{\rm sys}$ for SUMF IC-MUD and Unc.MMSE IC-MUD, with the optimal codes of rate 1/4, 1/3 and 1/2 and 64 states, at BER $\leq 10^{-5}$. The maximum achievable spectral efficiency of random CDMA with optimal joint decoding, and linear receiver (MMSE and SUMF) followed by non-iterative single-user decoding, with Gaussian inputs and single-user capacity achieving codes is shown for comparison.

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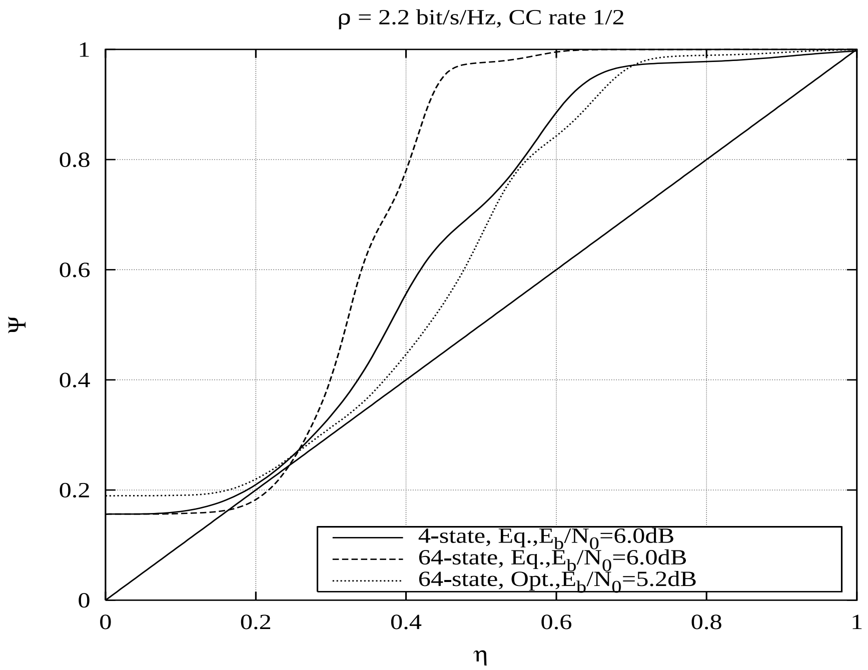

Figure 1: System mapping function with equal and optimized SNR distribution for Unc.MMSE.

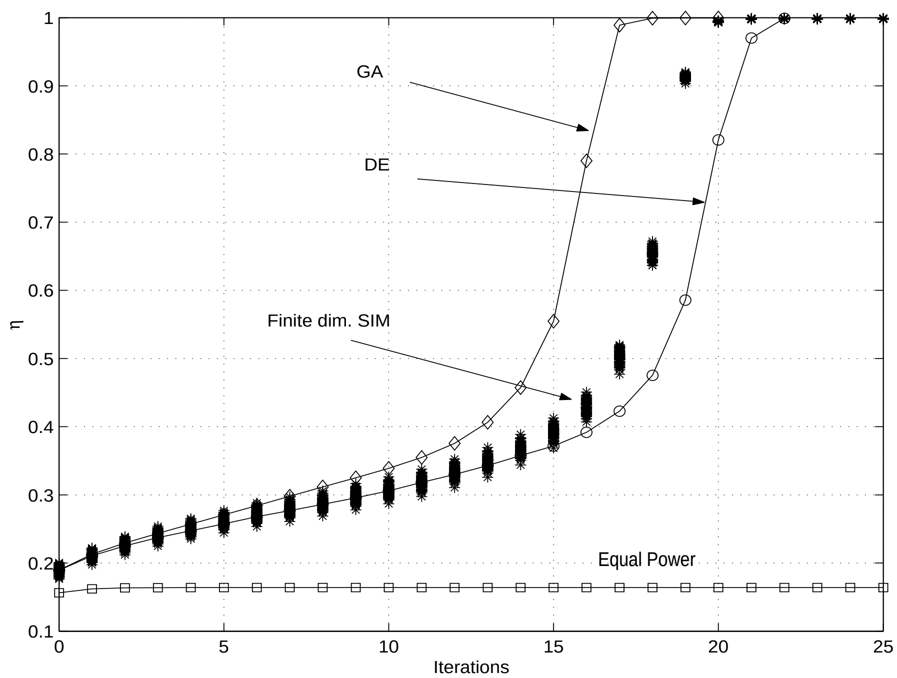

Figure 2: AME evolution vs. number of iterations for the system of Fig. 1.

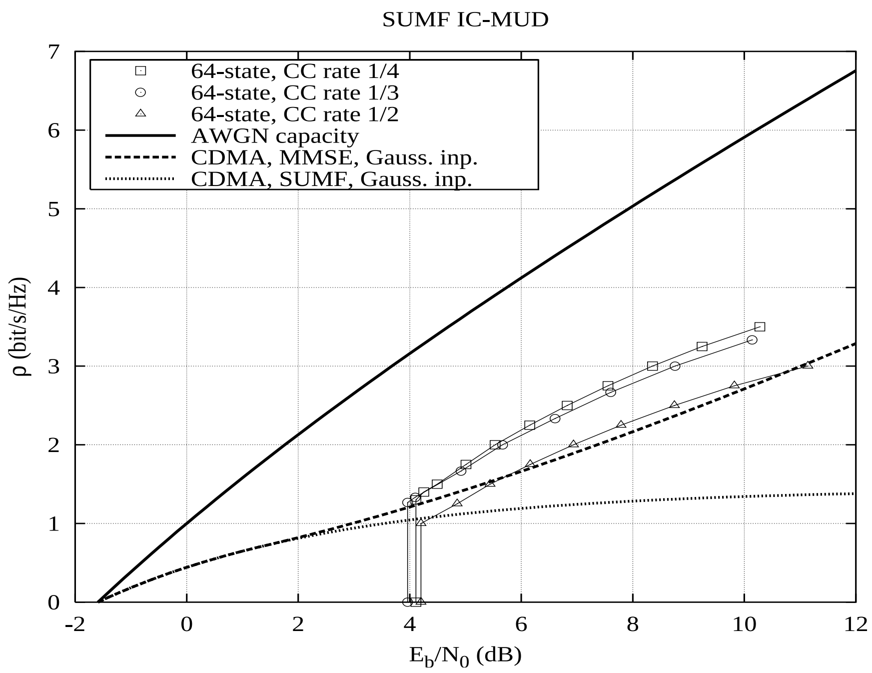

Figure 3: Spectral efficiency of 64-state codes with SUMF IC-MUD and optimal SNR distribution.

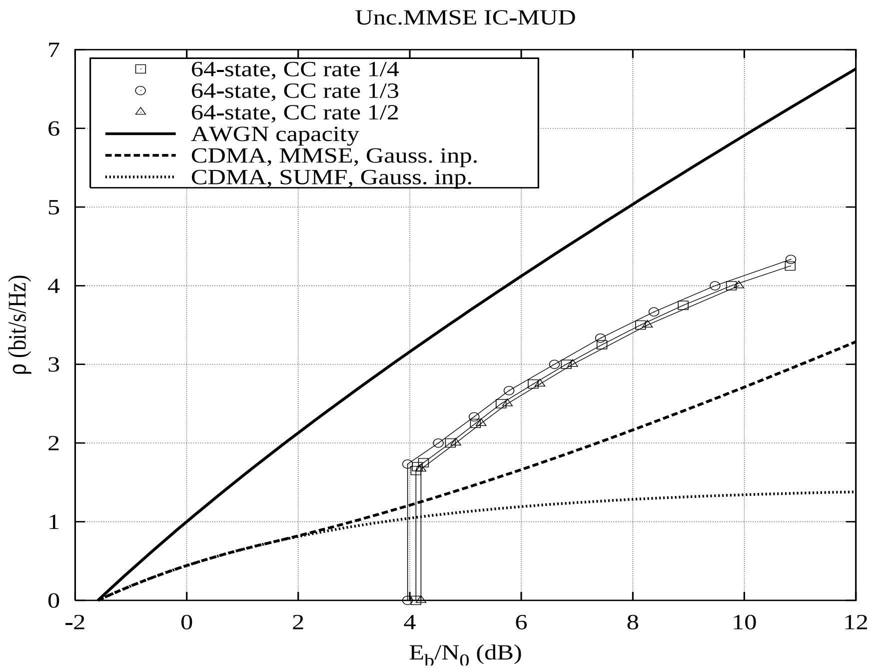

Figure 4: Spectral efficiency of 64-state codes with Unc.MMSE IC-MUD and optimal SNR distribution