ELSEVIER

Discrete Mathematics 239 (2001) 13–31

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Integer multiflows and metric packings beyond the cut condition

Karina Marcus[a,*,1], András Sebő[b]

[a] Corporate Communications Department, Institut Eurécom, 2229, route des Cretes - BP 193, 06904 Sophia Antipolis, France
[b] CNRS, Laboratoire Leibniz - Imag, 46, avenue Félix Viallet, 38031 Grenoble Cedex, France

Received 15 October 1997; revised 27 July 2000; accepted 14 August 2000

Abstract

Graphs, and more generally matroids, where the simplest possible necessary condition, the 'Cut Condition', is also sufficient for multiflow feasibility, have been characterized by Seymour. In this work we exhibit the 'next' necessary conditions — there are three of them — and characterize the subclass of matroids where these are also sufficient for multiflow feasibility, or for the existence of integer multiflows in the Eulerian case. Surprisingly, this subclass turns out to properly contain every matroid for which, together with all its minors, the metric packing problem — the 'polar' of the multiflow problem — has an integer solution for bipartite data (and a half integer solution in general). We also provide the excluded minor characterization of the corresponding subclass. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Metric packing; Cut condition; Integer multiflow

1. Introduction

Let M be a binary matroid defined on the finite set $E(M)$ and p a function assigning integer values to the elements of $E(M)$. The negative values of p are *demands* whereas the nonnegative values represent *capacities*. Define $F(p)=\{e\in E(M)\colon\ p(e)<0\}$. A *flow problem* is a pair (M,p), and it has a *solution* if there exists a *multiflow*, that is, a function $\Phi:\mathscr{C}_P(M)\to\mathbb{R}_+$ defined on the set $\mathscr{C}_p(M)$ of all circuits C of M with

* Corresponding author.
E-mail address: karina.marcus@eurecom.fr (K. Marcus).

[1] Most of this work was done when the author was at the Laboratoire Leibniz, supported by the brazilian Capes. Eurecom's research is partially supported by its industrial partners: Ascom, Cegetel, France Telecom, Hitachi, IBM France, Motorola, Swisscom, Texas Instruments, and Thomson CSF.

PII: S0012-365X(00)00372-1

$|C \cap F(p)| = 1$ such that

$$\sum_{C \in \mathscr{C}_p, C \ni e} \Phi(C) \begin{cases} \leqslant p(e) & \text{if } e \in E(M) - F(p), \\ = -\, p(e) & \text{if } e \in F(p). \end{cases}$$

In other words, Φ is a circuit-packing, where the circuits are restricted to $\mathscr{C}_p(M)$.

A function $m : E(M) \to \mathbb{R}_+$ is a *metric* if $m(e) \leqslant m(C - e)$ for all circuits C of M and all elements e of C. (We use the notation $m(X) = \sum_{e \in X} m(e)$ for subsets X of $E(M)$, and we replace $\{e\}$ by e.) It is apparent from this definition that the metrics of a matroid form a polyhedral cone. The extreme rays of this cone will be called *primitive metrics*. Metrics in matroids and some problems concerning them were introduced in [11]. For basics about cones see [7], in particular the cone generated by a set of vectors W will be denoted by cone(W).

To every binary matroid M we associate a set of metrics denoted by $\Delta(M)$, and $\Delta := \bigcup\{\Delta(M)$: M matroid$\}$ is called a *family of metrics*. For $A \subseteq \mathbb{R}_+$, a metric $m : E(M) \to A$ is called an *A-metric*; the family of A-metrics is denoted by Δ_A. In particular, $\Delta_{\mathbb{Z}^+}$ is the set of all integral metrics. A $\mathbb{Z}^+$-metric m is called *bipartite* if $m(C)$ is an even integer for all circuits C of M.

Let Δ be a family of metrics, and (M, p) be a flow problem. Consider the condition

$$mp \geqslant 0 \quad \text{for all } m \in \Delta(M). \tag{1}$$

This inequality is obviously necessary for the existence of multiflows, and it follows from linear programming (Farkas' lemma) that (1) with $\Delta = \Delta_{\mathbb{Z}^+}$ is also sufficient (see [11, (4.3)]; for graphs this is the celebrated 'Japanese theorem'). A basic question, well known for graphs, (for matroids see [11] Section 4) is the following: *is the restriction of* (1) *to smaller families of metrics already sufficient in some special but particularly interesting classes of graphs or matroids?* The sufficiency of (1) for $\Delta = \Delta_{\mathbb{Z}^+}$ can be reformulated in the following way.

A binary matroid M such that condition (1) is sufficient for the existence of a solution of (M, p) for arbitrary functions p, will be called *flowing with respect to* Δ. A flow problem (M, p) is *Eulerian* if $p(D)$ is even for all cocircuits D of M. If (1) is sufficient for the existence of an integer solution for all Eulerian problems (M, p), then M is called *cycling with respect to* Δ. Using Farkas' lemma again (like for $\Delta = \Delta_A$ in [10, Lemma 3.1]) the following holds.

Fact 1. *Let M be a matroid and Δ a class of metrics. Then M is flowing with respect to Δ if and only if $\Delta_{\mathbb{Z}_+}(M) \subseteq \mathrm{cone}(\Delta(M))$.*

In other words, if M is flowing with respect to Δ, then every metric can be written as a fractional linear combination of metrics in Δ. When can it be written as an integer combination?

Quite surprisingly, the existence of integer multiflows (in the Eulerian case) is correlated with the existence of such integer metric packings (for bipartite metrics m).

For the case of cut-metrics, Seymour [14] (see the sums of circuits property, for more explanations see Section 5), Karzanov [2] and Schrijver [8,9] have proved the existence of integer 'polars' of several well-known multiflow theorems. Karzanov [4] proved the existence of an integer packing of bip$(2,3)$-*metrics* and cuts for graphs with a demand-set adjacent to at most five vertices. For these problems, the cases where integer multiflow theorems hold are exactly the same as the cases when integer metric packing theorems are true. Contrary to what has been thought the same is not true in general! In this paper we characterize both properties which will show the difference between their domain of validity.

A binary matroid M, for which every metric is the nonnegative integer combination of metrics in Δ_A, is *packing with respect to* Δ_A. This means that Δ_A is a 'Hilbert basis' (see the definition of Hilbert basis in [7]). If M is packing with respect to its primitive metrics, we say simply that M is *packing*.

In Section 2 we give an overview of the multiflow problem in binary matroids and its relation to metrics; in Section 3 the K_5- and F_7-metrics are studied, and we prove that both are primitive and that condition (1) restricted to K_5- and F_7-metrics is sufficient for the existence of a multiflow in a certain class of matroids. Section 4 is about the matroid R_{10}, and a necessary and sufficient condition for the cyclingness of R_{10} is given. Finally, in Section 5 we show that $M(K_5)$, F_7 and R_{10} are packing, and characterize the class of packing matroids.

2. Multiflows

We shall denote by $\mathscr{C}(M)$ the set of cycles (that is, disjoint union of circuits) of the matroid M and by $\mathscr{C}^*(M)$ the set of cocycles. We refer to Welsh [15] for the basic concepts and facts of matroid theory.

The incidence vector χ_D of a cocycle D of M is called a *cut-metric*, and $\Delta_{(\mathrm{CC})}(M)$ denotes the set of all *cut-metrics* of the binary matroid M. We say that (M, p) satisfies the so-called *cut-condition* if and only if

$$mp \geqslant 0 \quad \text{for all } m \in \Delta_{(\mathrm{CC})}(M). \tag{CC}$$

The following result from Seymour [14] tells us that the metrics in $\Delta_{(\mathrm{CC})}$ are sufficient to describe the flowingness with respect to $\Delta_{\{0,1\}}$ and characterizes the related class of matroids.

Theorem 2. *For a binary matroid M the following are equivalent*:

(i) *M is cycling with respect to* $\Delta_{(\mathrm{CC})}$;
(ii) *M is flowing with respect to* $\Delta_{\{0,1\}}$;
(iii) *M has no* F_7, R_{10} *or* $M(K_5)$ *minor*.

F_7 is the Fano matroid on 7 elements, $M(K_5)$ is the graphic matroid of the complete graph on 5 nodes, and R_{10} is a special matroid on 10 elements used to characterize

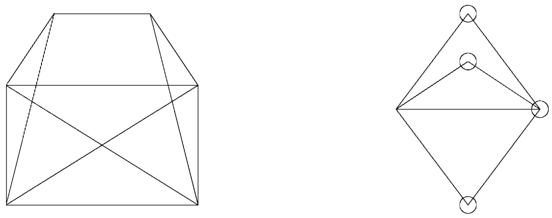

Fig. 1. H_6 and S_8.

regular matroids [13], that can be represented by the node-edge incidence matrix of the complete bipartite graph $K_{3,3}$, plus a column of 1.

Schwärzler and Sebő [10] have shown that extending the cut condition to a larger class of metrics, called CC3-metrics, a statement similar to Seymour's holds for a larger class of matroids. In the case where CC3 is replaced by the cut-condition or either of two conditions, which correspond to the only primitive metrics in CC3 for the matroids flowing with respect to $\Delta_{\{0,1,2\}}$, we will deduce the following sharper form in Section 3.

Theorem 3. *For a binary matroid M the following are equivalent*:

(i) *M is cycling with respect to* $\Delta_{(\mathrm{CC},\,F_7,K_5)}$;
(ii) *M is flowing with respect to* $\Delta_{\{0,1,2\}}$;
(iii) *M has no* AG(2,3), S_8, R_{10}, $M(H_6)$, $M(K_5)\oplus_2 F_7$, $M(K_5)\oplus_2 M(K_5)$, $F_7\oplus_2 F_7$ *minor*.

Here H_6 is the graphic matroid shown in Fig. 1(a), AG(2,3) is the representation of a projective plane and S_8 can be represented as the node-edge incidence matrix of the graph in Fig. 1(b), with a column with the circled elements. $M_1 \oplus M_2$ denotes the matroid resulting from the *2-sum* of binary matroids M_1 and M_2, where $E(M_1)\cap E(M_2)=f$. The cycles of $M_1\oplus M_2$ are $\mathscr{C}(M_1\oplus_2 M_2)=\{C_1\triangle C_2\colon C_1\in\mathscr{C}(M_1), C_2\in\mathscr{C}(M_2)\}$. The element f is called the *marker* of the 2-sum.

3. The two conditions

Let $\Delta_{(K_5)}(M)$ (resp. $\Delta_{(F_7)}(M)$) be the class of metrics $m\in\Delta_{\{0,1,2\}}$ such that, contracting the elements e with $m(e)=0$, the result is a $M(K_5)$ (resp. F_7), probably with some parallel elements, with the weights on each element of a parallel class defined below. A member of $\Delta_{(K_5)}$ ($\Delta_{(F_7)}$) will be called a K_5*-metric* (F_7*-metric*). In order to define the promised metric on $M(K_5)$, let $\{1,2,3,4,5\}$ be the vertex-set of K_5, and ij

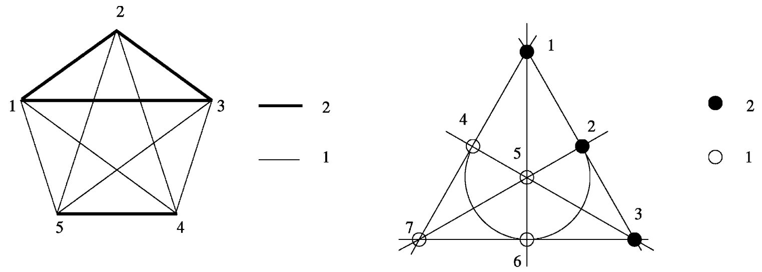

Fig. 2. K_5- and F_7-metrics.

be the edge between the vertices i and j. Define

$$m(ij) = \begin{cases} 2 & \text{if } ij \in \{12, 23, 13, 45\}, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, let C be a three-element circuit of $\mathscr{C}(F_7)$, and define

$$m(e) = \begin{cases} 2 & \text{if } e \in C, \\ 1 & \text{otherwise.} \end{cases}$$

We say that (M, p) satisfies the (CC, K_5, F_7) condition if

$$mp \geqslant 0, \quad \text{for all } m \in \Delta_{(\mathrm{CC},\, K_5,\, F_7)}(M).$$

Lemma 4. *The K_5- and F_7-metrics are primitive.*

Proof. It suffices to show that the F_7-metrics are extreme rays of the cone $\Delta_{\mathbb{Z}_+}(F_7)$ (for K_5 the proof works in the same way, see for example Karzanov [3]). If an F_7-metric m is not primitive, then m can be decomposed into a sum of primitive metrics, and the equalities $m(C-e) = m(e)$, $e \in C \in \mathscr{C}(F_7)$, satisfied by m, must be satisfied by any primitive metric in the decomposition. We check that the only solution to the system formed by these equalities is the original F_7-metric, and its positive multiples.

Without loss of generality, let m be the F_7-metric shown in Fig. 2. If x is in its decomposition, then following the numbering given at Fig. 2, x must satisfy the following equalities:

$$\left.\begin{aligned} x_1 = x_4 + x_7 = x_5 + x_6 \\ x_2 = x_5 + x_7 = x_4 + x_6 \end{aligned}\right\} \Rightarrow x_5 = x_4, \quad x_6 = x_7$$

and in the same way we obtain that $x_4 = x_7$, $x_5 = x_6$, and so $x_4 = x_5 = x_6 = x_7$ and $x_1 = x_2 = x_3 = 2x_4$; this corresponds to the original F_7-metric, proving that it is the only primitive metric in the decomposition. So m is an extreme ray of cone$(\Delta_{\mathbb{Z}_+}(F_7))$. □

Now, we prepare the proof of implication (iii) ⇒ (i) of Theorem 3. A twofold application of Seymour's 'Splitter Theorem' gives the following [14].

Proposition 5. *Every binary matroid with no* AG(2,3), S_8, R_{10} *or* $M(H_6)$ *minor may be obtained by* 1- *and* 2-*sums from matroids cycling with respect to* $\Delta_{(\mathrm{CC})}$, *and copies of* F_7 *and* $M(K_5)$.

And we can use it to prove the following result.

Proposition 6. *Any* 2-*sum* $M_1 \oplus_2 M_2$ *of a matroid* M_1 *cycling with respect to* $\Delta_{(\mathrm{CC},\,K_5,F_7)}(M_1)$ *and a matroid* M_2 *cycling with respect to* $\Delta_{(\mathrm{CC})}(M_2)$ *is cycling with respect to* $\Delta_{(\mathrm{CC},\,K_5,F_7)}(M_1 \oplus_2 M_2)$.

Proof. Let $E(M_1) \cap E(M_2) = \{f\}$ and $M = M_1 \oplus_2 M_2$. Choose $p : E(M) \to \mathbb{Z}$ such that (M, p) is Eulerian and (CC, K_5, F_7) is satisfied. We define functions $p_i : E(M_i) \to \mathbb{Z}$ $(i \in \{1,2\})$ in the following way:

$$p_i(e) = \begin{cases} p(e) & \text{if } e \in E(M_i) - f, \\ (-1)^{i-1} q & \text{if } e = f, \end{cases}$$

where $q = \min\{p(D - f)\colon f \in D \in \mathscr{C}^*(M_2)\}$. Let D_0 be a cocycle of M_2 with $p(D_0 - f) = q$.

Claim 1. p_i $(i \in \{1,2\})$ *is an Eulerian function.*

Proof. Let D_i be a cocycle of M_i. If $f \notin D_i$, then $p_i(D_i) = p(D_i) \equiv 0 \bmod 2$, because D_i is also a cocycle of M. If $f \in D_i$, then

$$\begin{aligned} p_i(D_i) &= p_i(D_i - f) + p_i(f) \\ &\equiv p(D_i - f) + p(D_0 - f) \equiv p(D_i \triangle D_0) \equiv 0 \bmod 2, \end{aligned}$$

since $D_i \triangle D_0$ is a cocycle of M. □

Claim 2. (M_2, p_2) *satisfies* (CC).

Proof. Let $D \in \mathscr{C}^*(M_2)$. If $f \notin D$, then D is also a cocycle of M and $p_2(D) = p(D) \geqslant 0$, because we assumed that (CC, K_5, F_7) and so, in particular, (CC) is satisfied by (M, p). If $f \in D$, then the definition of q implies that $p_2(D) = p_2(D - f) + p_2(f) = p(D - f) - p(D_0 - f) \geqslant 0$. □

Claim 3. (M_1, p_1) *satisfies* (CC, K_5, F_7).

Proof. We need to show that $pm_1 \geqslant 0$ for every choice of $m_1 \in \Delta_{(\mathrm{CC},\,K_5,F_7)}(M_1)$.

If m_1 is a CC-metric, then everything works as in Claim 2. Otherwise we associate to m_1 a metric $m \in \Delta_{(\mathrm{CC},\,K_5,F_7)}(M)$ defined as

$$m(e) = \begin{cases} m_1(e) & \text{if } e \in E(M_1) - f, \\ m_1(f) & \text{if } e \in D_0, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to see that if m_1 is a K_5- or F_7-metric on M_1, then m is a K_5- or F_7-metric on M. And so we have that

$$\begin{aligned} p_1 m_1 &= \sum_{e\in E(M_1)} p_1(e)m_1(e) = \sum_{e\in E(M_1-f)} p_1(e)m_1(e) + p_1(f)m_1(f) \\ &= \sum_{e\in E(M)-D_0} p(e)m(e) + p(D_0 - f)m_1(f) = \sum_{e\in E(M)} p(e)m(e) \\ &\geqslant 0, \end{aligned}$$

since (M, p) satisfies (CC, K_5, F_7). Thus, Claim 3 is proved. □

As M_1 (resp. M_2) was assumed to be cycling with respect to $\Delta_{(\mathrm{CC},\,K_5,F_7)}$ (resp. to $\Delta_{(\mathrm{CC})}$), the above claims guarantee the existence of integer flows ϕ_i in (M_i, p_i), $i\in\{1,2\}$. ϕ_i consists of a list of cycles of $\mathscr{C}_{p_i}(M_i)$. Suppose, without loss of generality, that precisely the first k_i cycles of each list contain the element f. It follows from the definition of a flow that $k_i \leqslant q = k_2$. After deleting the first $k_2 - k_1$ cycles from the second list ϕ_2, the union of the two lists contains exactly k_1 cycles of $\mathscr{C}(M_1)$ and k_1 cycles of $\mathscr{C}(M_2)$ passing through the element f. Build k_1 pairs (C_1, C_2), $C_i \in \mathscr{C}(M_i)$, of the cycles passing through f and replace each pair by $C_1 \triangle C_2$. It is easy to see that the list of cycles obtained in this way represents an integer flow of (M, p). □

Let us now prove our main theorem.

Theorem 3. *For a binary matroid M the following are equivalent*:

(i) *M is cycling with respect to $\Delta_{(\mathrm{CC},\,F_7,K_5)}$*;
(ii) *M is flowing with respect to $\Delta_{\{0,1,2\}}$*;
(iii) *M has no* AG(2,3), S_8, R_{10}, $M(H_6)$, $M(K_5)\oplus_2 F_7$, $M(K_5)\oplus_2 M(K_5)$, $F_7\oplus_2 F_7$ *minor*.

Proof. Condition (i) ⇒ (ii) trivial.

(ii) ⇒ (iii): There are several ways of proving this implication. Schwärzler and Sebő [10] checks it by showing multiflow problems that have no solution, but whose multiflow functions satisfy (1) for $\Delta_{\{0,1,2\}}$. We show that there are primitive metrics for these matroids that are not in $\Delta_{\{0,1,2\}}$, which is a shorter and easier way of proving the implication.

Let S_8, R_{10} and AG(2,3) be represented by the matrices in Fig. 3. We will prove the result for the R_{10} case, the other ones follow similarly. Let m_i denote the m value of the element corresponding to the ith column in the matrix.

Now let $m := (3,3,1,1,3,1,1,1,1,1)$. For this metric m we have the following equalities arising from the inequality $m(e) \leqslant m(C-e)$, where C is a circuit in R_{10}:

$$\begin{array}{lll} m_1 = m_3 + m_7 + m_9 & m_2 = m_3 + m_7 + m_8 & m_5 = m_6 + m_8 + m_9, \\ m_1 = m_4 + m_{10} + m_8 & m_2 = m_6 + m_7 + m_{10} & m_5 = m_4 + m_7 + m_8, \\ m_1 = m_3 + m_4 + m_6 & m_2 = m_4 + m_9 + m_{10} & m_5 = m_3 + m_9 + m_{10}. \end{array}$$

$$\begin{bmatrix} 1&0&0&0&1&1&1&1 \\ 0&1&0&0&1&0&0&1 \\ 0&0&1&0&0&1&0&1 \\ 0&0&0&1&0&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&0&0&0&1&1&1&0 \\ 1&0&0&1&1&0&1&1&0&1 \\ 0&1&0&1&0&1&1&0&1&1 \\ 0&0&1&0&1&1&0&1&1&1 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&1&0&1&1&1 \\ 0&0&1&1&1&0&1&0 \\ 0&1&1&1&0&0&0&1 \\ 1&1&1&1&1&1&1&1 \end{bmatrix}$$

Fig. 3. Matrix representation of S_8, R_{10} and AG(2,3).

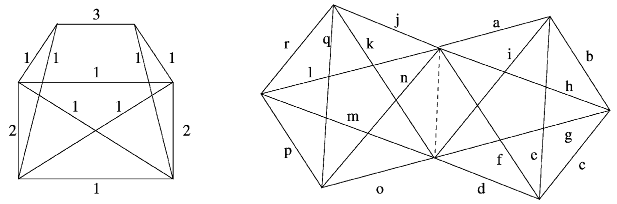

Fig. 4. H_6 and $M(K_5) \oplus_2 M(K_5)$.

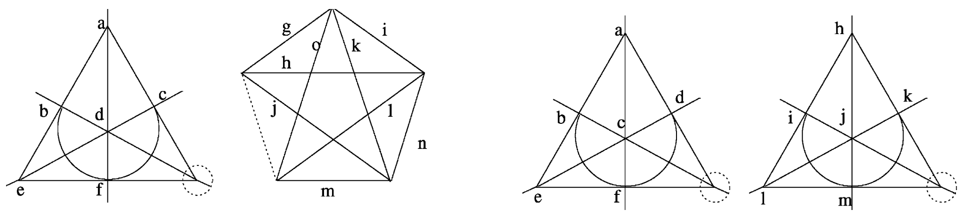

Fig. 5. $F_7 \oplus_2 M(K_5)$ and $F_7 \oplus_2 F_7$.

These equations are affinely independent and all solutions for this system are vectors of the form $(3a, 3a, a, a, 3a, a, a, a, a, a)$, $a \geqslant 0$, which is exactly the extreme ray of the cone of metrics $\Delta_{\mathbb{Z}_+}(R_{10})$ defined by m. Therefore m is primitive, but it is not a $(0,1,2)$-vector. Hence, by Fact 1, R_{10} is not flowing with respect to $\Delta_{\{0,1,2\}}$.

In the same way we can show that $m=(2,1,1,1,1,1,1,3)$ and $m=(1,1,1,4,1,1,1,1)$ define extreme rays of the cone of metrics $\Delta_{\mathbb{Z}_+}(S_8)$ and $\Delta_{\mathbb{Z}_+}(\mathrm{AG}(2,3))$, respectively, proving that they are not flowing with respect to $\Delta_{\{0,1,2\}}$.

A primitive metric for H_6 is represented in Fig. 4, and one can check that it is primitive in the same way as in the cases above.

Now let $M(K_5) \oplus_2 M(K_5)$ be as in Fig. 4, the marker is indicated in dashed line, and let $m : E(M(K_5) \oplus_2 M(K_5)) \to \mathbb{Z}_+$ be as follows:

$$m(x) := \begin{cases} 4 & \text{if } x \in \{j, l, o, r\}, \\ 2 & \text{if } x \in \{a, c, i, k, m, n, p, q\}, \\ 1 & \text{otherwise.} \end{cases}$$

And let $F_7 \oplus_2 M(K_5)$ and $F_7 \oplus_2 F_7$ be as in Fig. 5, where the markers are indicated by dashed lines, and $m_1 : E(F_7 \oplus_2 M(K_5)) \to \mathbb{Z}_+$ and $m_2 : E(F_7 \oplus_2 F_7) \to \mathbb{Z}_+$ be as follows:

$$m_1(x) := \begin{cases} 4 & \text{if } x \in \{g,h,i,m\}, \\ 2 & \text{if } x \in \{e,f,j,k,l,n,o\}, \\ 1 & \text{otherwise.} \end{cases} \qquad m_2(x) := \begin{cases} 4 & \text{if } x \in \{a,b,e\}, \\ 2 & \text{if } x \in \{c,d,f,l,m\}, \\ 1 & \text{otherwise.} \end{cases}$$

We can check in the same way as above that the metrics m, m_1 and m_2 are primitive, and since they are not $(0,1,2)$-vectors, using Fact 1, this implies that these matroids are not flowing with respect to $\Delta_{\{0,1,2\}}$.

(iii) $\Rightarrow$ (i) K_5 and F_7 are flowing with respect to $\Delta_{\{\mathrm{CC},\, K_5, F_7\}}$ (see [4,10]). These results with Proposition 6 give the desired implication. □

4. The R_{10} matroid

Now we consider the third excluded matroid in Theorem 2. We prove in this section that the matroid R_{10} is cycling with respect to a well-defined set of metrics, and use this property to characterize the packing matroids in Section 5.

A metric $m : E(M) \to \{0,1,3\}$ is an R_{10}-*metric*, if after contracting the elements e such that $m(e) = 0$, a matroid R_{10} is obtained, possibly with parallel elements, and m is as follows. There is a circuit C of size 4 and $\{a,b,c\} \subseteq C$ such that

$$m(e) = \begin{cases} 3 & \text{if } e \in \{a,b,c\}, \\ 1 & \text{otherwise.} \end{cases}$$

In this case we denote m by $m_{(a,b,c)}$. The proof in the last section showing that R_{10} is not cycling with respect to $\Delta_{(0,1,2)}$ can be easily adapted to show that any $m_{(a,b,c)}$ is primitive.

We say that (M,p) satisfies the (CC, R_{10}) condition if

$$mp \geqslant 0, \quad \text{for all } m \in \Delta_{(\mathrm{CC},\, R_{10})}(M).$$

We consider the matroid R_{10} as represented by the node-edge incidence matrix of the graph $K_{3,3}$ depicted in Fig. 6, and a column of 1, that is called the element t. We use the numbering given by Fig. 6 throughout this section.

Using this representation, the cycles that contain t are the t-joins in $K_{3,3}$, i.e., the set of edges T such that $d_T(v) \equiv 1 \bmod 2$ (the degree of v in T is odd); the circuits of

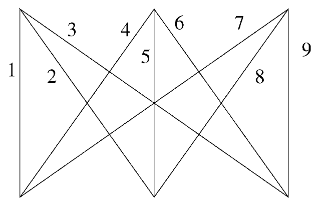

Fig. 6. $K_{3,3}$.

R_{10} that do not contain t are exactly the circuits of $K_{3,3}$. We may also check that the cocircuits of R_{10} that contain t are the sets $C+t$, where C is a t-cut, i.e., $C=\delta(X)$[2], for an $X \subseteq V(K_{3,3})$, with $|X \cap V(K_{3,3})| = |X| \equiv 1 \bmod 2$; the cocircuits in R_{10} that do not contain t are the cocircuits of $K_{3,3}$ that are not t-cuts.

We will use all these remarks to prove the following.

Theorem 7. *The matroid R_{10} is cycling with respect to $\Delta_{(\mathrm{CC},\, R_{10})}(R_{10})$.*

Proof. Let (R_{10}, p) be an Eulerian multiflow problem such that $pm \geqslant 0$, for all $m \in \Delta_{(\mathrm{CC},\, R_{10})}$. We will prove by induction on $\sum_{e \in E(R_{10})} |p(e)|$ that there is a solution for (R_{10}, p). We suppose that $p(e) \neq 0$, for all $e \in E(R_{10})$, otherwise there is an integer multiflow, since every minor of R_{10} is cycling with respect to $\Delta_{(\mathrm{CC})}$ (Theorem 2).

If $|F(p)| \leqslant 2$, then there is an integer solution to (R_{10}, p), since R_{10} is cycling for every p such that $|F(p)| \leqslant 2$ [14, 14.7].

If $|F(p)| \geqslant 6$, then $F(p)$ contains a cocircuit, and the cut condition is always violated.

For the other cases, we proceed as follows. We suppose that $t \in F(p)$, without loss of generality, and we search for all circuits C such that $C \cap F(p) = \{t\}$. Given such a circuit C, we define $p_C : E(R_{10}) \to \mathbb{Z}$ as

$$p_C(e) = \begin{cases} p(e)+1 & \text{if } e = t, \\ p(e)-1 & \text{if } e \in C - t, \\ p(e) & \text{otherwise.} \end{cases}$$

If there is such a circuit C, and if (R_{10}, p_C) satisfies (CC, R_{10}), then we apply induction on (R_{10}, p_C), obtaining the result. We will show that such circuit C exists.

A cocircuit D *forbids* a circuit C if $p_C(D) < 0$. First, we give some results concerning the cocircuits that cannot forbid a circuit. All these results have as hypotheses that $p(e) \neq 0$, $e \in E(R_{10})$ and $p(t) < 0$. Their proofs are quite easy, so we prove only the first lemma, the others following analogously.

Lemma 8. *If $D \in \mathscr{C}^*(R_{10})$, $|D| = 6$* and *$t \notin D$, then $p(D) > 0$.*

Proof. If $|D| = 6$ and $t \notin D$, then there exist $u_1, u_2 \in V(K_{3,3})$, $u_1 \neq u_2$, such that $D = \delta(u_1) \cup \delta(u_2)$. This means that $D = D_1 \triangle D_2$, where $D_i = \delta(u_i) \cup \{t\}$, $D_i \in \mathscr{C}^*(R_{10})$, $i = 1, 2$. Therefore $p(D) = p(D_1) + p(D_2) - 2p(t) > 0$. $\square$

Lemma 9. *Let $D \in \mathscr{C}^*(R_{10})$ be such that $|D| = 6$* and *$t \in D$. If there is some $x \in F(p) \backslash D$, $x \neq t$, then $p(D) > 0$.*

Lemma 10. *Let $D \in \mathscr{C}^*(R_{10})$ be such that $|D| = 4$ and $t \notin D$. If there is some $x \in F(p) \backslash D$, $x \neq t$, such that x is adjacent to all the edges in D, then $p(D) > 0$.*

[2] $\delta(X)$ is the set of edges in G that have exactly one extremity in X.

Lemma 11. *Let $D \in \mathscr{C}^*(R_{10})$ be such that $|D| = 4$ and $t \notin D$. If there are two edges $x, y \in F(p) \backslash D$, $x \neq y \neq t$, and $a \in D$ such that $a \cap x \cap y = \{u\}$, $u \in V(K_{3,3})$, then $p(D) > 0$.*

Lemma 12. *Let $D_1, D_2 \in \mathscr{C}^*(R_{10})$ be such that $|D_1| = |D_2| = 4$, $t \notin D_1 \cup D_2$ and $D_1 \cap D_2 \neq \emptyset$. If there exists a node u such that $\delta(u) \subseteq D_1 \cup D_2$, then $p(D_1) + p(D_2) > 0$.*

Now we consider the R_{10}-metrics. Actually, these metrics are simple to handle, as the next lemma shows.

Lemma 13. *Let $m_{(a,b,c)} \in \Delta_{(R_{10})}(R_{10})$ and $C \in \mathscr{C}(R_{10})$, $|C| = 4$, be such that $C \cap F(p) = \{t\}$ and $p_C \chi_D \geqslant 0$, for all cocircuits D. Then we have the following inequalities*:

$$m_{(a,b,c)} p_C \geqslant \begin{cases} 8 & \text{if } |C \cap \{a,b,c\}| = 3, \\ 6 & \text{if } |C \cap \{a,b,c\}| = 2, \\ 4 & \text{if } |C \cap \{a,b,c\}| = 1, \\ 2 & \text{if } |C \cap \{a,b,c\}| = 0. \end{cases}$$

Proof. We present here the proof of some cases, the others are dealt similarly, using a convenient decomposition of $m_{(a,b,c)} p$.

We suppose that $C = \{t, e_1, e_5, e_9\}$. We first consider $m_{(1,5,9)}$. The following equality holds:

$$m_{(1,5,9)} p = p(t, e_1, e_2, e_3) + p(t, e_4, e_5, e_6) + p(t, e_7, e_8, e_9) + 2p(e_1, e_5, e_9) - 2p(t).$$

This implies $m_{(1,5,9)} p \geqslant 8$, since we supposed that $p(e) \neq 0$, for all $e \in E(R_{10})$.

Now we consider the case where $\{a, b\} = \{e_1, e_5\}$, and, suppose $c = e_2$. The following equality results:

$$m_{(1,2,5)} p = p(t, e_1, e_2, e_3) + p(e_2, e_4, e_6, e_8) + p(e_2, e_5, e_7, e_9) + 2p(e_1, e_5).$$

As $p_C(e_2, e_5, e_7, e_9) \geqslant 0$, this implies that $m_{(1,2,5)} p \geqslant 6$.

If $|C \cap \{a,b,c\}| = 1$, suppose that $\{a,b,c\} = \{e_1, e_2, e_4\}$. In this case we have the following equality:

$$m_{(1,2,4)} p = p(t, e_1, e_2, e_4, e_5, e_9) + p(e_2, e_3, e_4, e_7) + p(e_2, e_4, e_6, e_8) + 2p(e_1),$$

since by hypotheses $p_C(t, e_1, e_2, e_4, e_5, e_9) \geqslant 0$, and $p(e_1) > 0$, therefore $m_{(1,2,4)} p \geqslant 4$.

At last we consider $|C \cap \{a,b,c\}| = 0$. Suppose that $\{a,b,c\} = \{e_2, e_3, e_6\}$ and we obtain the following equality:

$$m_{(2,3,6)} p = p(t, e_1, e_2, e_3) + p(t, e_3, e_6, e_9) + p(t, e_2, e_3, e_5, e_6, e_7) + p(e_2, e_4, e_6, e_8) - 2p(t),$$

therefore $m_{(2,3,6)} p \geqslant 2$. □

Corollary 14. *Let $C \in \mathscr{C}(R_{10})$ be such that $|C| = 4$, and $C \cap F(p) = \{t\}$. If there is no cocircuit that forbids C, then $p_C m \geqslant 0$, for all $m \in \Delta_{(\mathrm{CC},\, R_{10})}(R_{10})$.*

We treat now the cases $3 \leqslant F(p) \leqslant 5$, using the previous lemmas.

(a) $F(p) = \{t, a, b\}$, where a, b are not adjacent in $K_{3,3}$.

Without loss of generality, let $\{a, b\} = \{e_1, e_5\}$. In this case, a circuit C such that $C \cap F(p) = \{t\}$ must contain e_2 or e_3. The possible circuits are $C_1 = \{t, e_2, e_4, e_9\}$, $C_2 = \{t, e_3, e_4, e_8\}$, $C_3 = \{t, e_2, e_6, e_7\}$ and $C_4 = \{t, e_3, e_6, e_7, e_8, e_9\}$. The cocircuits that may forbid C_1 are $D_1 = \{e_1, e_2, e_6, e_9\}$, $D_2 = \{e_3, e_4, e_5, e_9\}$, $D_3 = \{e_1, e_4, e_8, e_9\}$, $D_4 = \{e_2, e_5, e_7, e_9\}$, and there is only one cocircuit of size 6 that does not satisfy the conditions of Lemma 8 or 9, namely $D_5 = \{t, e_1, e_2, e_4, e_5, e_9\}$; the cocircuits that may forbid C_2 are $D_6 = \{e_1, e_3, e_5, e_8\}$, D_2 and D_3, and some cocircuits of size 6, all of them satisfying the hypotheses of Lemma 8 or 9; the cocircuits that may forbid C_3 are D_1, D_4 and $D_7 = \{e_1, e_5, e_6, e_7\}$, and the above cocircuits of size 4 may forbid C_4 as well.

So now we must check that not all circuits are forbidden by some cocircuit. First, we do not have any of $p(D_6) = p(D_1) = 0$, $p(D_2) = p(D_7) = 0$, $p(D_4) = p(D_6) = 0$, $p(D_1) = p(D_2) = 0$, $p(D_3) = p(D_7) = 0$, or $p(D_3) = p(D_4) = 0$. Indeed, all these cases satisfy the condition of Lemma 12. Next, it holds that

$$p(D_2) + p(D_4) = p(t, e_1, e_3, e_5, e_7, e_9) + p(t, e_1, e_2, e_4, e_5, e_9) \\ -2p(t) - 2p(e_1) > 0,$$

$$p(D_1) + p(D_3) = p(t, e_1, e_2, e_4, e_5, e_9) + p(t, e_1, e_5, e_6, e_8, e_9) - 2p(t) \\ - 2p(e_5) > 0.$$

The last case is where $p(D_5) = p(D_6) = p(D_7) = 0$. However,

$$0 \leqslant m(t, e_1, e_5) = 3p(t, e_1, e_5) + p(e_2, e_3, e_4, e_6, e_7, e_8, e_9) \\ = p(D_5) + p(D_6) + p(D_7) + 2p(t).$$

Therefore one in $p(D_5), p(D_6), p(D_7)$ must be positive. So we showed that there must be a circuit that is not forbidden by a cocircuit.

Now Corollary 14 implies that if (R_{10}, p_{C_i}) $i \in \{1, 2, 3\}$ satisfies the cut condition, then $p_{C_i} m \geqslant 0$, for all $m \in \Delta_{(R_{10})}(R_{10})$. Thus, we may apply induction on (R_{10}, p_{C_i}) and get the result.

For the other cases we proceed in the same way: We first find all the circuits containing t that are candidates for composing the multiflow, then we look at each circuit and consider the cocircuits that could forbid it. Using the previous lemmas one easily concludes that it is not possible that all circuits are forbidden, and so we may apply induction. The details may be found in [5]. □

5. Packing matroids

The 'packing' property of matroids in the special case when all the packed metrics are cuts is nothing else but the dual of the 'sums of cuts property'. M has *the sums* *of circuits property* (see [12]) if the following are equivalent for all $p : E(M) \to \mathbb{Z}_+$

(i) There is a function $\alpha : \mathscr{C}(M) \to \mathbb{R}_+$ such that $\sum \alpha(C)\chi_C = p$.
(ii) For every cocircuit D and $f \in D$, $p(f) \leqslant p(D - \{f\})$.

In [14] Seymour characterized matroids that have the sums of circuits property — they are the duals of those flowing with respect to $\Delta_{(\mathrm{CC})}$—, and conjectured the following result, proved by Fu and Goddyn [1].

Theorem 15. *If M is a binary matroid and has no F_7^*, R_{10}, $M^*(K_5)$ or $M(P_{10})$ minor, and p satisfies* (ii) *and is Eulerian, then there is an integral α satisfying* (i).

$M(P_{10})$ is the cycle-matroid of the Petersen graph. Dualizing this result, the following holds.

Corollary 16. *If M is a binary matroid and has no F_7, R_{10}, $M(K_5)$ or $M^*(P_{10})$ minor, then M is packing with respect to $\Delta_{(\mathrm{CC})}(M)$.*

Compare this with Theorem 2: The only difference between the class of matroids cycling and those packing with respect to $\Delta_{(\mathrm{CC})}$ is that the latter must also not contain $M(P_{10})$ as minor.

We state now some positive facts about K_5 and F_7. The statement for K_5 can be checked in a similar way to the proof below for F_7. However, the proof of the first is omitted, since one can simply refer to a theorem of Karzanov [4].

Lemma 17. *In the matroid $M(K_5)$ every bipartite metric can be expressed as a positive integer sum of metrics in $\Delta_{(\mathrm{CC},\, K_5)}(M(K_5))$.*

We say that a metric m_2 can be *subtracted* from m_1 if $m_1 - m_2$ is a metric. If both m_1 and m_2 are bipartite, then obviously $m_1 - m_2$ is also bipartite.

Lemma 18. *In F_7 every bipartite metric may be expressed as a positive integer sum of metrics in $\Delta_{(\mathrm{CC},\, F_7)}(F_7)$.*

Proof. Let m be a bipartite metric on F_7, and suppose that for every bipartite metric $m' < m$ the statement is true. By Theorem 3 and Fact 1 we know that m can be expressed as $m = v_1\chi_{D_1} + \cdots + v_n\chi_{D_n} + \lambda_1 m_1 + \cdots + \lambda_k m_k$, where D_i is a cocircuit, m_j is a F_7-metric, and $v_i, \lambda_j > 0$ $(i = 1, \ldots, n)$, $(j = 1, \ldots, k)$.

Claim 1. *If D_i is a cocircuit, then χ_{D_i} can be subtracted from m.*

Proof. Indeed, χ_{D_i} can be subtracted from m if and only if

$$m(C - e) - m(e) \geqslant \chi_{D_i}(C - e) - \chi_{D_i}(e), \quad \text{for all } e \in C \in \mathscr{C}(F_7).$$

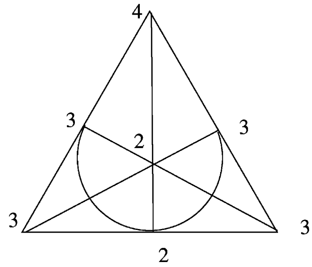

Fig. 7. A sum of two F_7-metrics.

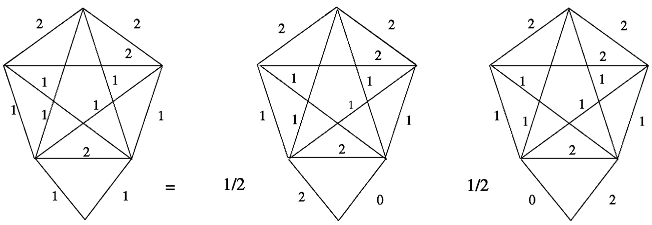

Fig. 8. A half-integer metric-packing.

Since $\lambda_i > 0$, the equality $m(C-e) - m(e) = 0$ implies that $\chi_{D_i}(C-e) - \chi_{D_i}(e) = 0$, therefore we only have to check the above inequality when $m(C-e) - m(e) > 0$. Since m is bipartite, $m(C-e) - m(e) \geqslant 2$ holds. Now, if $\chi_{D_i}(C) \geqslant 4$, then $D_i = C$, and $\chi_{D_i}(C-e) - \chi_{D_i}(e) = 2$; if $4 > \chi_{D_i}(C) \geqslant 2$, obviously $\chi_{D_i}(C-e) - \chi_{D_i}(e) \leqslant 2$. So we have the desired inequality for every circuit C. □

Claim 2. *If m_i and m_j are different F_7-metrics, then $m_i + m_j$ can be written as a sum of cut-metrics.*

Proof. Fig. 7 shows the unique sum of two F_7-metrics in F_7, up to isomorphism. It can be decomposed into cut-metrics: $\chi_{\{1,2,4,5\}} + \chi_{\{1,3,5,7\}} + \chi_{\{1,2,6,7\}} + \chi_{\{1,3,4,6\}} + \chi_{\{2,3,4,7\}}$ (with the numbering of Fig. 2). □

Now if $n \geqslant 1$ it follows from Claim 1 that $m - m_1$ is also a metric, and then by the minimal choice of m: $m - m_1$ is a positive integer sum of metrics in $\Delta_{(\mathrm{CC},\, F_7)}(F_7)$. Consequently so is m.

Now by Claim 2, if $k \geqslant 2$, there is another decomposition where $n \geqslant 1$, and then we have already proved the statement in the previous paragraph. If $n = 0$, $k = 1$, m is an integer multiple of an F_7 metric. □

Fig. 8 gives an example of a bipartite metric on the 2-sum of two packing matroids, but its unique decomposition into primitive metrics is not integer. (The uniqueness of the decomposition easily follows from the fact that K_5-metrics are primitive.)

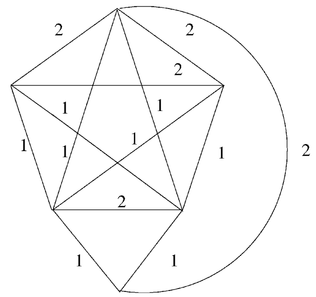

Fig. 9. A graph which is packing and contains a non-packing graph as minor.

A matroid M is *half-packing* with respect to $\Delta_A(M)$ if for every bipartite metric m in $\Delta(M)$ there is a half-integer decomposition in metrics in $\Delta_A(M)$. In [6], the following characterization of half-packing matroids was given.

Theorem 19. *If a binary matroid M has no* $M^*(P_{10})$, *then M is cycling with respect to* $\Delta_{\{0,1,2\}}(M)$ *if and only if M is half-packing with respect to* $\Delta_{(\mathrm{CC},\,K_5,F_7)}(M)$.

Moreover, for graphs this turns into the following:

Corollary 20. *For a graph G the following are equivalent*:

(i) *G is cycling with respect to* $\Delta_{(\mathrm{CC},\,K_5)}$;
(ii) *G is flowing with respect to* $\Delta_{\{0,1,2\}}$;
(iii) *G is half-packing with respect to* $\Delta_{(\mathrm{CC},\,K_5)}$;
(iv) *G has no* H_6, $K_5 \oplus_2 K_5$ *as minor*.

The 2-sum of $M(K_5)$ or F_7 or R_{10} with a matroid consisting on two circuits C_1, C_2 such that $C_1 \cap C_2 = \{f\}$, where f is the marker of the 2-sum, and $|C_1| = 3$, $|C_2| = 2$, will be denoted by $\bar{K}_5$, $\bar{F}_7$, $\bar{R}_{10}$, respectively (see $\bar{K}_5$ in Fig. 8). $\bar{K}_5$, $\bar{F}_7$, $\bar{R}_{10}$ are not packing; the metric that shows the 'non-packingness' of $\bar{K}_5$ is depicted in Fig. 8; for $\bar{F}_7$, a F_7 metric is extended with value 1 on the new elements; for $\bar{R}_{10}$, a R_{10} metric is extended with value 2 on one new element, and 1 on the other.

As far as packingness is concerned, we are indebted to Monique Laurent to have warned us about the fact that it is not necessarily closed under minors. Indeed, a graph G which is packing and contains the nonpacking graph of Fig. 8 can be found, see Fig. 9. To see that G is packing, consider a bipartite metric m on G, and consider G as $K_5 + \{a,b,c\}$, where a,b,c are the three edges incident to the degree 3 node. If $m(e) = 0$ for some $e \in A(K_5)$, then e can be contracted, and the obtained graph is packing; if $m(e) = 0$, for some $e \in \{a,b,c\}$, then by contracting e we obtain a K_5, which is packing. Finally, if $m(e) > 0$, for all $e \in A(G)$, then the restriction of m to K_5 is integer-decomposable in cuts and at most one K_5-metric, and it is not very difficult (case checking) to verify that this decomposition can be extended to an integer decomposition in G. One needs only to notice that there are (1,2)-metrics in $\Delta(G)$ that are not in $\Delta_{K_5}(G)$.

Hence in Theorem 23(i) below the property has to be required for all minors! That is, in what follows, we are willing to characterize matroids which are packing as well as all their minors.

We first show that R_{10} is packing.

Lemma 21. *In the matroid R_{10} every bipartite metric can be expressed as a positive integer sum of metrics in $\Delta_{(\mathrm{CC},\, R_{10})}(R_{10})$.*

Proof. Let m be a bipartite metric on R_{10}. We want to write it as an integer sum of cuts and R_{10}-metrics. By Theorem 7 and Fact 1 m can be expressed as $m = v_1\chi_{C_1} + \cdots + v_n\chi_{C_n} + \lambda_1 m_1 + \cdots + \lambda_k m_k$, where C_i is a cut, m_i is a R_{10}-metric, and $v_i, \lambda_i > 0$.

Claim 1. *Let D be a cocircuit on R_{10}. If $(m - \chi_D)(C - e) - (m - \chi_D)(e) < 0$, for a circuit C, $|C| = 6$, and $e \in C$, then there exists a circuit C' of cardinality 4, and an element $f \in C'$, such that $(m - \chi_D)(C' - f) - (m - \chi_D)(f) < 0$.*

Proof. Let $l = m - \chi_D$. One can easily see that there are two circuits $C_1, C_2 \in \mathscr{C}(R_{10})$, $|C_1| = |C_2| = 4$, such that $C_1 \triangle C_2 = C$. Suppose that $C_1 \cap C_2 = \{g\}$, and without loss of generality, let $e \in C_1$. Then the following is true.

$$0 > l(C - e) - l(e) = l(C_1 - e) - l(e) + l(C_2 - g) - l(g).$$

Therefore either $l(C_1 - e) - l(e) < 0$ or $l(C_2 - g) - l(g) < 0$ must hold. □

Claim 2. *If C_i is a cocircuit, then χ_{C_i} can be subtracted from m.*

If χ_{C_i} cannot be subtracted from m, then by Claim 1, there is a circuit K, $|K| = 4$, and $e \in K$, such that

$$(m - \chi_{C_i})(K - e) - (m - \chi_{C_i})(e) < 0. \tag{2}$$

The equality $m(K - e) - m(e) = 0$ implies that $\chi_{C_i}(K - e) - \chi_{C_i}(e) = 0$, so we may suppose that $m(K - e) - m(e) \geqslant 2$. And as $\chi_{C_i}(K - e) - \chi_{C_i}(e) \leqslant 2$ holds, equality (2) above is not true. □

Claim 3. *If m_i and m_j are different R_{10}-metrics, then $m_i + m_j$ can be written as a sum of cut-metrics.*

Proof. We present the three possible cases for a sum of two R_{10}-metrics, $m_{(a,b,c)}$ and $m_{(e,f,g)}$, where $\{a,b,c\} \neq \{e,f,g\}$. We refer to the representation of R_{10} used in Theorem 7.

(I) $\{a,b,c\} \cap \{e,f,g\} = \emptyset$. Without loss of generality, let $m_{(a,b,c)} = m_{(e_2,e_6,e_7)}$ and $m_{(e,f,g)} = m_{(t,e_1,e_5)}$. Then

$$m_{(e_2,e_6,e_7)} + m_{(t,e_1,e_5)} = \chi_{\{t,e_2,e_3,e_5,e_6,e_7\}} + \chi_{\{e_1,e_5,e_6,e_7\}} + \chi_{\{e_1,e_2,e_4,e_5,e_7,e_8\}} + \chi_{\{t,e_3,e_6,e_9\}} + \chi_{\{t,e_1,e_2,e_4,e_5,e_9\}} + \chi_{\{t,e_1,e_2,e_6,e_7,e_8\}}.$$

(II) $|\{a,b,c\} \cap \{e,f,g\}| = 1$. Without loss of generality, let $m_{(a,b,c)} = m_{(t,e_1,e_5)}$ and $m_{(e,f,g)} = m_{(e_2,e_4,e_5)}$. Then

$$m_{(t,e_1,e_5)} + m_{(e_2,e_4,e_5)} = \chi_{\{t,e_4,e_5,e_6\}} + \chi_{\{t,e_1,e_2,e_4,e_5,e_9\}} + \chi_{\{t,e_2,e_5,e_8\}} + \chi_{\{e_1,e_2,e_4,e_5,e_7,e_8\}} + \chi_{\{e_1,e_5,e_6,e_7\}} + \chi_{\{t,e_1,e_2,e_3\}} + \chi_{\{e_3,e_4,e_5,e_9\}}.$$

(III) $|\{a,b,c\} \cap \{e,f,g\}| = 2$. Without loss of generality, let $m_{(a,b,c)} = m_{(t,e_1,e_5)}$ and $m_{(e,f,g)} = m_{(e_1,e_2,e_5)}$. Then

$$m_{(t,e_1,e_5)} + m_{(e_1,e_2,e_5)} = 2(\chi_{\{e_1,e_5,e_6,e_7\}}) + \chi_{\{t,e_2,e_5,e_8\}} + \chi_{\{t,e_1,e_2,e_3\}} + 2(\chi_{\{t,e_1,e_2,e_4,e_5,e_9\}}) + \chi_{\{e_1,e_3,e_5,e_8\}}.$$

The three previous claims prove Lemma 21. □

And now we prove the following constructive lemma.

Lemma 22. *The matroid M, resulting from the 2-sum of a matroid M_1 that is packing with respect to $\Delta_{(\mathrm{CC},\,K_5,F_7,R_{10})}$ with a matroid M_2, that is a circuit, is metric packing with respect to $\Delta_{(\mathrm{CC},\,K_5,F_7,R_{10})}$.*

Proof. If M_1 does not contain $M(K_5)$ and F_7 as a minor, the result is trivial. So suppose M_1 contains one of $M(K_5)$ or F_7 as minor. Given a bipartite metric m on the matroid M, we will find an integral decomposition for m as a sum of cocircuits, F_7- or K_5-metrics. Let $M_1 \cap M_2 = \{f\}$.

We define two metrics $m_1 : E(M_1) \to \mathbb{Z}_+$ and $m_2 : E(M_2) \to \mathbb{Z}_+$ such that

$$m_i(e) = \begin{cases} m(e) & \text{if } e \in E(M_i) - f, \\ q & \text{if } e = f, \end{cases}$$

where $q = \min\{m(C - f) \colon C \in \mathscr{C}(M_1) \cup \mathscr{C}(M_2)\}$.

It is not difficult to see that each m_i, $i = 1, 2$, is a bipartite metric, and so there are cocircuits $C_1, \ldots, C_r \in \mathscr{C}^*(M_2)$ such that $\sum_{j=1}^{r} \chi_{C_j} = m_2$. We assume that the first k_2 cocircuits contain f; and there are cocircuits $D_1, \ldots, D_s \in \mathscr{C}^*(M_1)$, and K_5- or F_7-metrics $l_1, \ldots, l_t$ such that $\sum_{j=1}^{r} \chi_{D_j} + \sum_{j=1}^{t} l_j = m_1$. We suppose that the first k_1 cocircuits contain f, and that the first k_3 F_7- or K_5-metrics l_i are such that $l_i(f) = 1$. Notice that $k_2 = k_1 + k_3 + 2(t - k_3)$.

To each l_j, $1 \leqslant j \leqslant k_3$, and to each D_i, $1 \leqslant i \leqslant k_1$, associate a cocircuit $C_k \in \mathscr{C}^*(M_2)$. In each of them replace f by C_k, and the result is clearly a cocircuit, a K_5- or an F_7-metric.

Now associate to each l_i, $k_3 + 1 \leqslant i \leqslant t$, two cocircuits C_j, C_k, $1 \leqslant j < k \leqslant k_1$ (i.e., $f \in C_j \cap C_k$), and consider B_1 and B_2 defined as follows. Let $B_1 = C_j \triangle C_k$, B_1 is a cocircuit in M_2, and so in M. Let $B_2 = C_j \cap C_k$.

If $|B_2| \equiv 0 \bmod 2$, then B_2 is a cocycle in M_2, since M_2 is a circuit. Define l_i' on M as

$$l_i'(e) = \begin{cases} l_i(e) & \text{if } e \in E(M_1) - f, \\ 2 & \text{if } e \in B_2 - f. \end{cases}$$

Clearly l_i' is an F_7- or K_5-metric. Replace l_i, C_j, C_k with l_i', B_1.

If $|B_2| \equiv 1 \bmod 2$, consider l_i' on M_1 defined as

$$l_i'(e) = \begin{cases} l_i(e) & \text{if } e \in E(M_1) - f, \\ 0 & \text{if } e = f. \end{cases}$$

As it is, l_i' can be decomposed into a family of cocircuits $\mathscr{S}$. Replace l_i, C_j, C_k with $\mathscr{S}$, B_1 and twice $B_2 - f$, if $B_2 - f \neq \emptyset$ (in this case $B_2 - f$ is a cocycle in M_2 and in M). Proceeding this way we get an integer packing of cocircuits and F_7- or K_5-metrics for m.

For the case concerning the matroid R_{10}, the proof is similar, we just replace the coefficients 2 with 3. □

And the complete characterization of packing matroids follows.

Theorem 23. *For a binary matroid M the following statements are equivalent*:

(i) *M and all its minors are packing*,
(ii) *M is packing with respect to $\Delta_{(\mathrm{CC}, F_7, K_5, R_{10})}$*,
(iii) *M has no $M^*(P_{10})$, $\bar{F}_7$, $\bar{K}_5$, $\bar{R}_{10}$ minor.*

Proof. If M does contain one of the excluded minors, then M is not packing. Indeed, $M(P_{10})$ does not have the integer sum of circuits property, and the example in Fig. 8 shows that $\bar{K}_5$ ($\bar{F}_7$ and $\bar{R}_{10}$, analogously) is not packing. So (i) implies (iii).

Now let M be a matroid that does not contain any of the excluded minors. If M does not contain any $M(K_5)$, F_7 or R_{10} minor, then, by Corollary 16, M is packing with respect to $\Delta_{(\mathrm{CC})}$. If it does contain any, then each connected component of M is the result of several 2-sums of only one $M(K_5)$, F_7 or R_{10}, and matroids that are circuits, otherwise M would contain a $\bar{F}_7$, $\bar{K}_5$, or $\bar{R}_{10}$ minor. Using now Lemma 22, we conclude that M is packing. So (iii) implies (ii) and (i). □

Acknowledgements

The authors are indebted to Monique Laurent for various comments, in particular for pointing out a flaw in a previous version (see the remark above Lemma 21). We thank also the anonymous referee who contributed with valuable suggestions.

References

[1] X. Fu, L. Goddyn, Matroids with the circuit cover property, preprint, 1995.
[2] A.V. Karzanov, Metrics and undirected cuts, Math. Programming 32 (1985) 183–198.

[3] A.V. Karzanov, Half-integer five-terminus flows, Discrete Appl. Math. 18 (1987) 263–278.
[4] A.V. Karzanov, Sums of cuts and bipartite metrics, European J. Combin. 11 (1990) 473–484.
[5] K. Marcus, Multiflows, metrics and *h*-perfect graphs: the odd cycles in the combinatorial optimization, Ph.D. Thesis, Université Joseph Fourier, Grenoble, France, 1996 (in French).
[6] K. Marcus, A. Sebő, On integer multiflows and metric packings in matroids, in: R. Euler, Y. Yannakakis (Eds.), Combinatorics in Computer Science, Lecture Notes in Computer Science, Springer, Berlin, 1995.
[7] A. Schrijver, Theory of Linear and Integer Programming, Wiley, Chichester, 1986.
[8] A. Schrijver, Distances and cuts in planar graphs, J. Combin. Theory, Ser. B 46 (1989) 46–57.
[9] A. Schrijver, Short proofs on multicommodity flows and cuts, J. Combin. Theory, Ser. B 53 (1991) 32–39.
[10] W. Schwärzler, A. Sebő, A generalized cut-condition for multiflows in matroids, Discrete Math. 113 (1993) 207–221.
[11] A. Sebő, The Cographic Multiflow Problem: An Epilogue, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 1, 1990, pp. 203–234.
[12] P.D. Seymour, Sums of circuits, Graph Theory and Related Topics, Academic Press, New York, 1979.
[13] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory 28 (1980) 305–359.
[14] P.D. Seymour, Matroids and multicommodity flows, European J. Combin. 2 (1981) 257–290.
[15] D.J.A. Welsh, Matroid Theory, Academic Press, London, 1976.