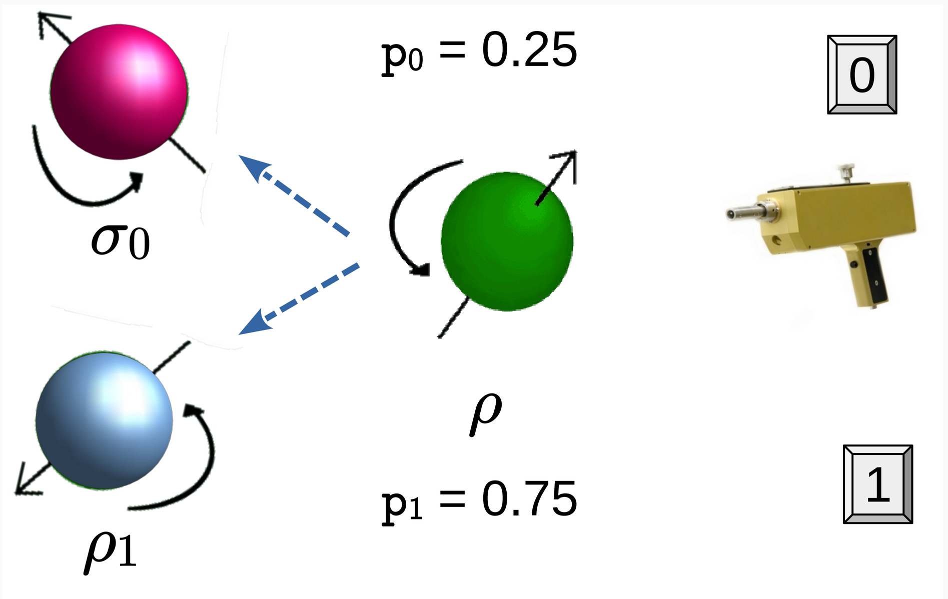
How Informative are Network Quantum Measurements?

Arun Padakandla

EURECOM, Sophia-Antipolis, France.

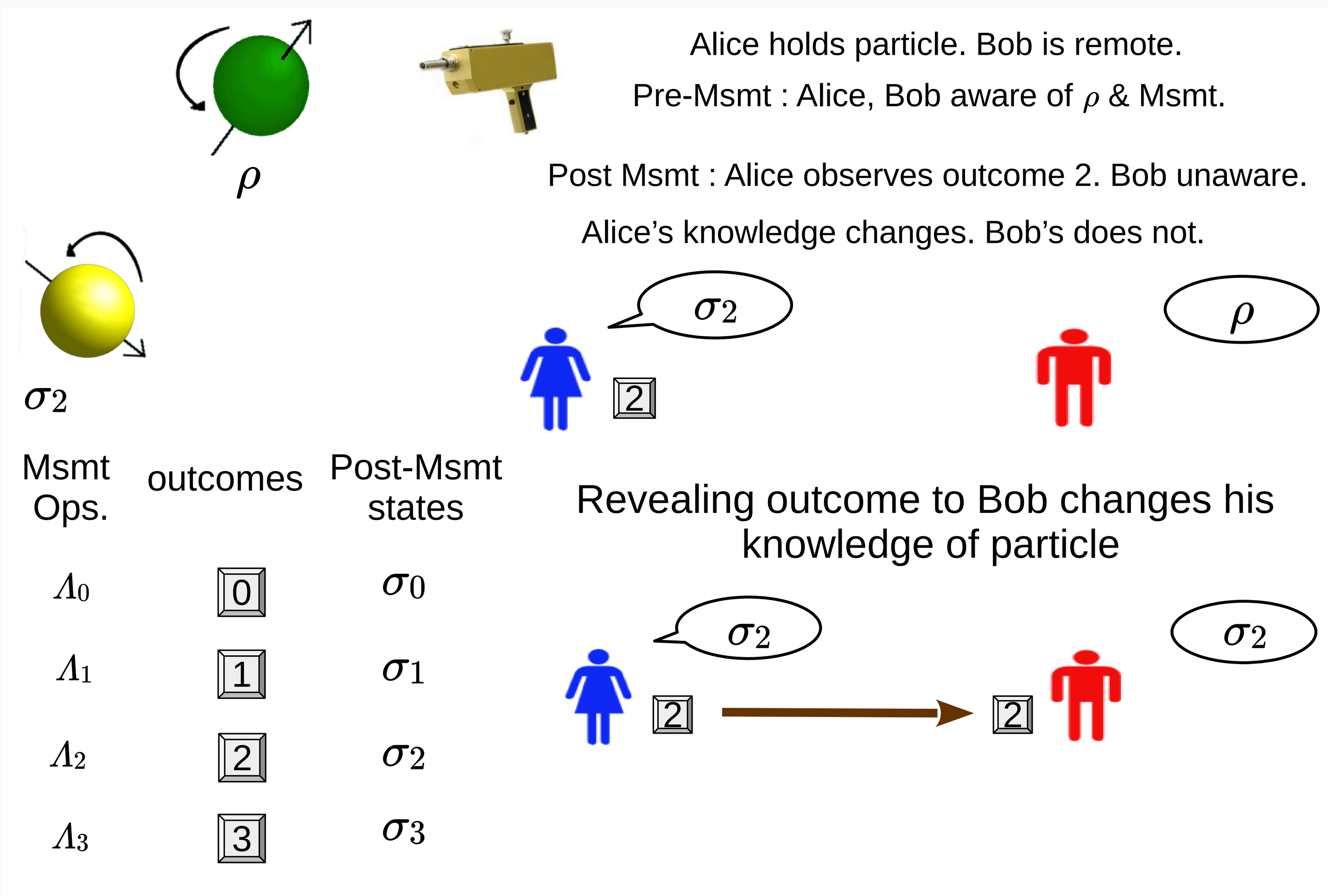
A Quantum Measurement (Msmt) is our **only** window into the quantum world. Any interaction with a quantum state is via quantum measurement. The outcome of a quantum measurement is **random**. ***How much information does the outcome of a quantum measurement contain?*** Can we ***quantify*** the information content in the outcome of a quantum measurement? How do we even formulate this question? How do we *quantify the information content in a random outcome*?

1. Quantum Measurement with post-Msmt States

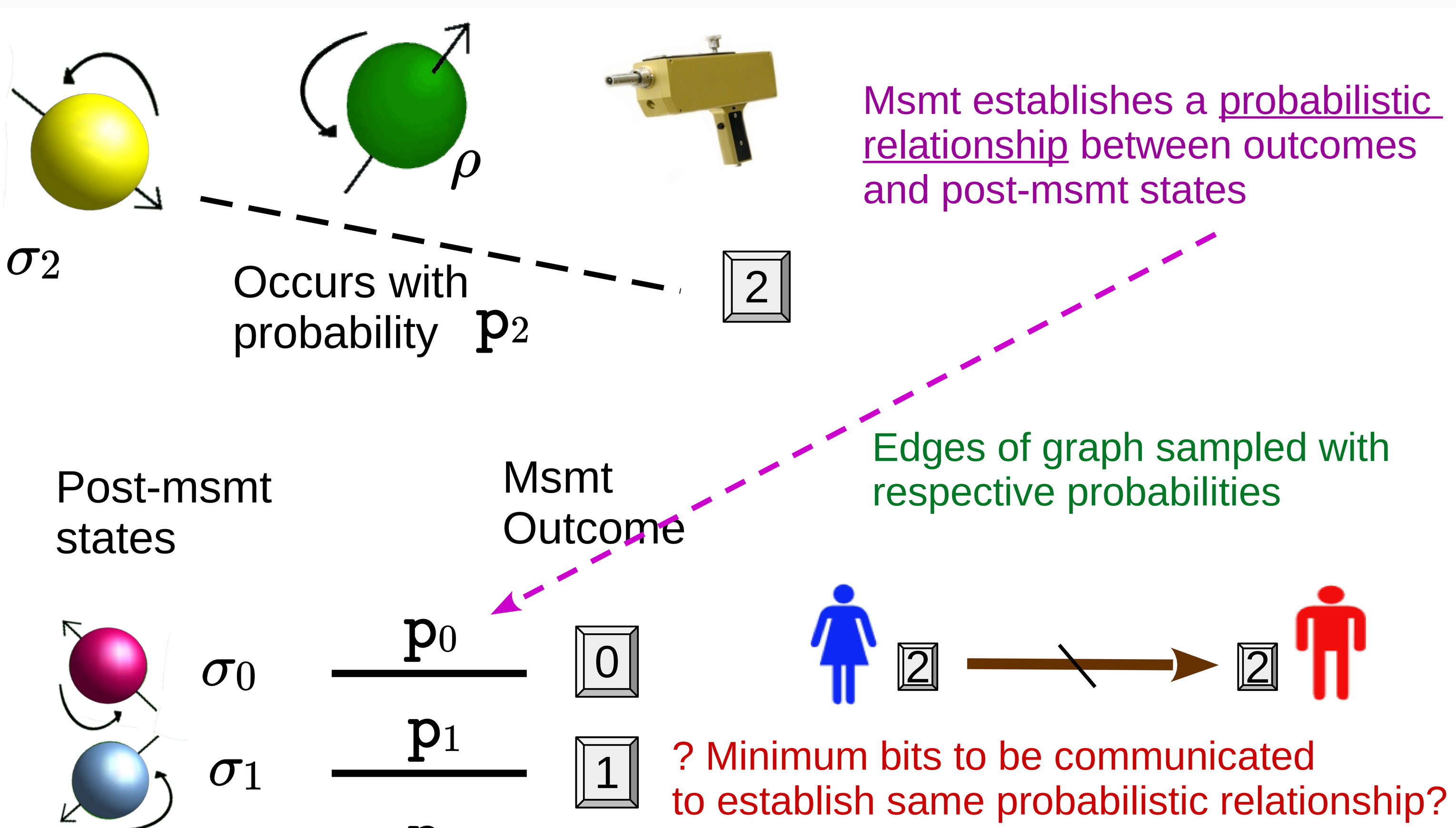


Even a perfectly well prepared & controlled quantum states yield **random** outcomes to precise Msmts. *Does it carry information?* If yes, how much?

Formulation : What does the outcome tell us?

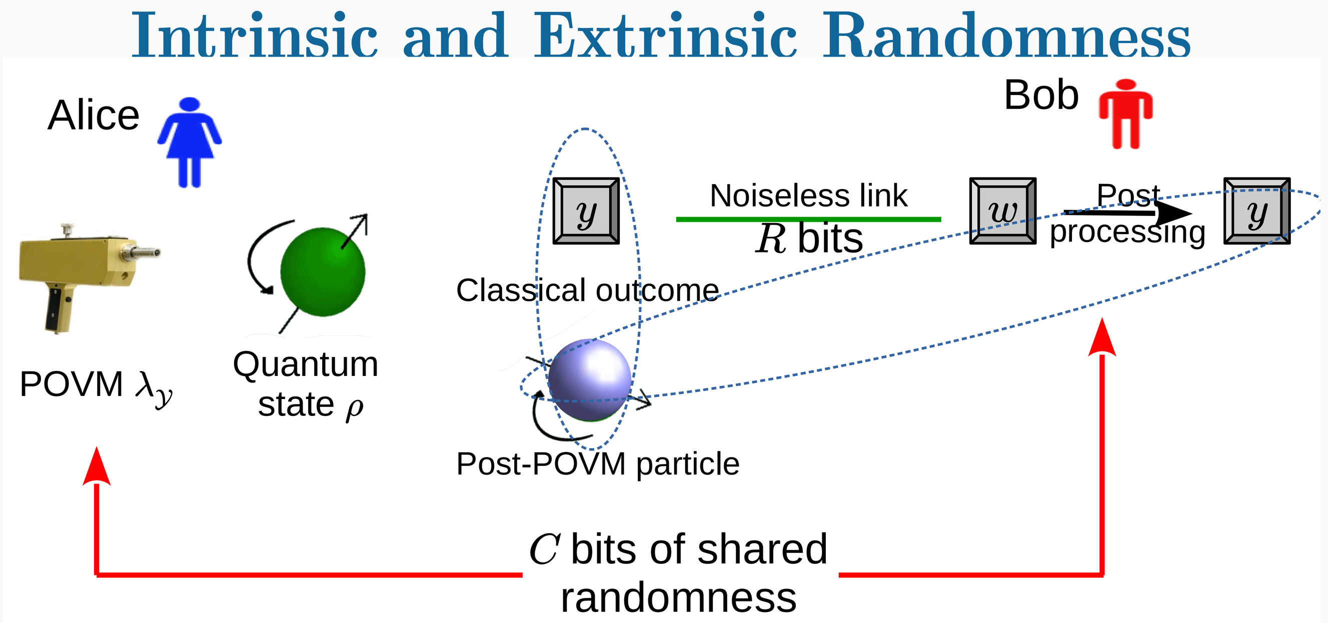


Formulation : A Probabilistic Relationship



- Amount of information obtained by a quantum measurement Massar & Popescu Phys. Rev. A 61, May 2000
- A. Winter and S. Massar, "Compression of quantum-measurement operations," Phys. Rev. A, vol. 64, Jun 2001.
- A. Winter, ""Extrinsic"and "Intrinsic" Data ….. Positive Operator Valued Measures,"
- Communications in mathematical physics, vol. 244, 2004.
- M. M. Wilde, P. Hayden, F. Buscemi, and M.-H. Hsieh, "The information-theoretic costs of simulating quantum measurements," Jrnl of Physics A: Mathematical and Theoretical, 2012.

Intrinsic and Extrinsic Randomness



What are the **minimum values for** R **and** C that can ensure the components in the two blue ellipses are **statistically indistinguishable**?

An Information-Theoretic Formulation

Given a quantum state φ_ρ and a POVM $\lambda_y = \{\lambda_y : y \in \mathcal{Y}\}$, the post Msmt state is

$$\Upsilon_\lambda \triangleq \mathrm{tr}_A \circ \mathcal{E}^{\lambda}(\varphi_\rho) = (i_R \otimes \mathrm{tr}_A) \circ (i_R \otimes \mathcal{E}^{\lambda})(\varphi_\rho) = \sum_{y \in \mathcal{Y}} \sqrt{\rho}\lambda_y\sqrt{\rho} \otimes |y\rangle\langle y| \quad (1)$$

Defn. 1 *An $(n, K, M, \theta, \Delta)$ (POVM) simulation protocol consists of (i) a bank of K POVMs $\theta_k : k \in [K]$, wherein each POVM $\theta_k = \{\theta_{k,m} : m \in [M]\}$ has at most M outcomes, (ii) a (decoder) POVM $\Delta = \{\Delta_{y^n} : y^n \in \mathcal{Y}^n\}$ wherein each Δ_{y^n} is an operator on $\mathcal{H}_{KM}$. We let θ denote the POVM $\{\theta_{k,m} \otimes |k\rangle\langle k| : (k, m) \in [K] \times [M]\}$. The post-measurement state of the $(n, K, M, \theta, \Delta)$ simulation protocol is*[1]

$$\Upsilon_{\theta,\Delta} \triangleq \mathbf{\mathfrak{E}}_{sim}(\varphi_{\rho_K}) \triangleq (\mathrm{tr}_K \otimes \mathrm{tr}_A \circ \mathcal{E}^{\theta} \otimes \mathrm{tr}_{KM} \circ \mathcal{E}^{\Delta})(\varphi_{\rho_K}). \quad (2)$$

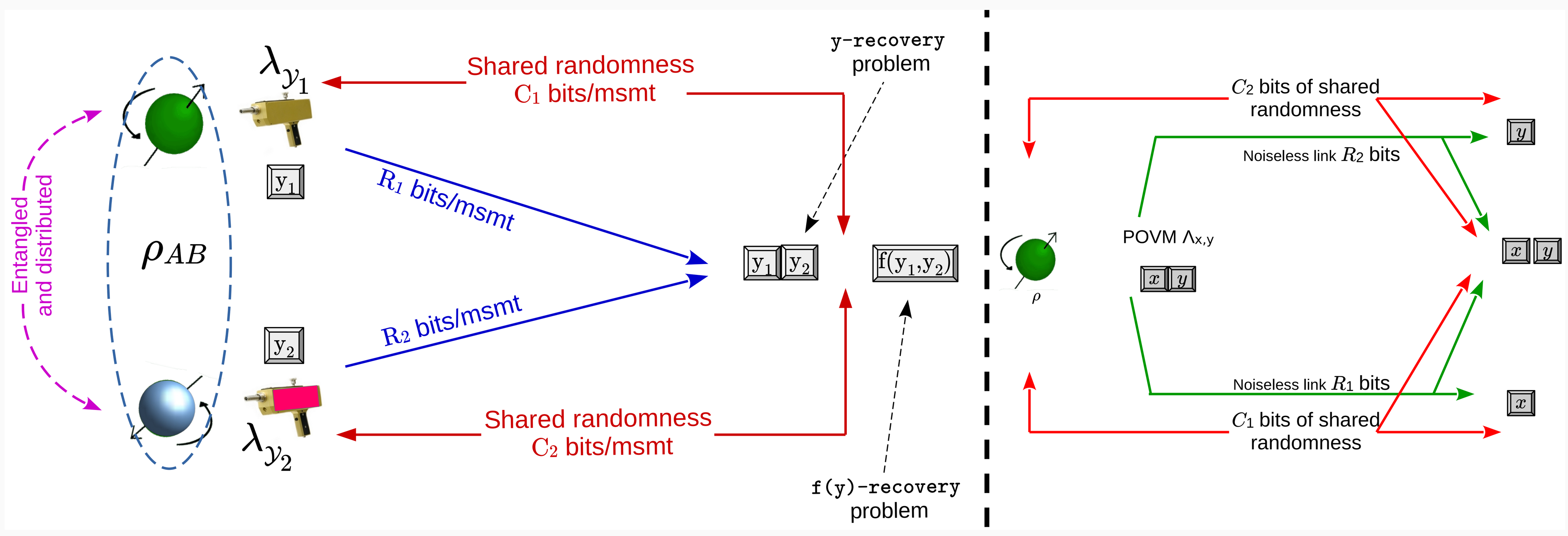
Defn. 2 *An $(n, K, M, \theta, \Delta)$ POVM simulation simulates action of $\lambda_{\mathcal{Y}}$ on ρ within $\eta > 0$ if $\|\Upsilon_\lambda - \Upsilon_{\theta,\Delta}\|_1 \le \eta$. The action of $\lambda_{\mathcal{Y}}$ on ρ can be simulated with communication cost (R, C) if for every $\eta > 0$, there exists an $N_\eta \in \mathbb{N}$ such that for all $n \ge N_\eta$ there exists an $(n, K, M, \theta, \Delta)$ POVM simulation protocol that simulates the action of $\lambda_{\mathcal{Y}}$ on ρ within $\eta > 0$ and $\frac{\log K}{n} \le C + \eta$ and $\frac{\log M}{n} \le R + \eta$. We define*

$$\mathscr{C}(\rho, \lambda_{\mathcal{Y}}) \triangleq \{(R, C) : \text{ Action of } \lambda_{\mathcal{Y}} \text{ on } \rho \text{ can be simulated with communication cost } (R, C)\} \quad (3)$$

Theorem

The action of $\lambda_{\mathcal{Y}}$ on ρ can be simulated with communication cost (R, C) iff there exists $(\mathcal{W}, \mu_{\mathcal{W}}, p_{Y|W}) \in \mathcal{C}(\rho, \lambda_{\mathcal{Y}})$ for which $(R, C) \in \mathscr{A}(\mathcal{W}, \mu_{\mathcal{W}}, p_{Y|W})$.

Distributed Separable and Jointly Measurable POVMs



The two problems addressed and solved in this work are depicted on the left and right panels above. Left Panel : How much information is contained in the outcome of a separable measurement on a pair of distributed entangled particles? Right Panel : How much information is contained in the outcome of jointly measurable POVMs operated on a single particle? ***Our Contributions are*** :

1. Developed a new simplified measurement simulation protocol based on **Likelihood POVMs**
2. Developed new techniques for analyzing Performance Of simulation POVMs
3. Established Likelihood POVMs can ***optimally simulate*** POVMs in diverse network scenarios including the single terminal and the above two scenarios depicted.

Inner Bound for Simulating Distributed Separable POVMs

Defn. 3 *For $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and POVMs $\lambda_{\mathcal{Y}_j} = \{\lambda_{y_j} \in \mathcal{P}(\mathcal{H}_j) : y_j \in \mathcal{Y}_j\} : j \in [2]$, let $\mathcal{C}(\rho_{AB}, \lambda_{\underline{\mathcal{Y}}})$ be a collection of quintuples $(\mathcal{W}_1, \mathcal{W}_2, \mu_{\mathcal{W}_1}, \mu_{\mathcal{W}_2}, p_{\underline{Y}|\underline{W}})$ wherein, for $j \in [2]$ (i) $\mathcal{W}_j$ is a finite set, (ii) $\mu_{\mathcal{W}_j} = \{\mu_{w_j} \in \mathcal{P}(\mathcal{H}_j) : w_j \in \mathcal{W}_j\}$ is a POVM, and (iii) $(p_{\underline{Y}|\underline{W}}(\underline{y}|\underline{w}) : (\underline{w}, \underline{y}) \in \underline{\mathcal{W}} \times \underline{\mathcal{Y}})$ is a stochastic matrix such that*

$$\mathrm{tr}_{\underline{W}}(\sigma^{R\underline{W}\underline{Y}}) = (i_R \otimes \mathrm{tr}_{AB} \circ \mathcal{E}^{\lambda_{y_1} \otimes \lambda_{y_2}})(\varphi_{\rho_{AB}}), \text{ where } \sigma^{R\underline{W}\underline{Y}} \triangleq (i_R \otimes \mathrm{tr}_{AB} \circ \mathcal{E}^{\mu_{\mathcal{W}_1} \otimes \mu_{\mathcal{W}_2}} \otimes \mathcal{E}_p^{\underline{Y}|\underline{W}})(\varphi_{\rho_{AB}}) \quad (4)$$

where, as stated earlier, $\underline{W}, \underline{Y}, \underline{w}, \underline{y}$ abbreviate W_1, W_2, Y_1, Y_2, (w_1, w_2) and (y_1, y_2) respectively. For $(\underline{\mathcal{W}}, \mu_{\underline{\mathcal{W}}}, p_{\underline{Y}|\underline{W}}) \in \mathcal{C}(\rho_{AB}, \lambda_{\underline{\mathcal{Y}}})$, we let $\sigma_1^{RBW_1} \triangleq (i_R \otimes \mathrm{tr}_A \circ \mathcal{E}^{\mu_{\mathcal{W}_1} \otimes I_B})(\varphi_{\rho_{AB}})$, $\sigma_2^{RAW_2} \triangleq (i_R \otimes \mathrm{tr}_B \circ \mathcal{E}^{I_A \otimes \mu_{\mathcal{W}_2}})(\varphi_{\rho_{AB}})$ and $\mathscr{A}(\underline{\mathcal{W}}, \mu_{\underline{\mathcal{W}}}, p_{\underline{Y}|\underline{W}})$ be the set of all quadruples $(\underline{R}, \underline{C})$ satisfying

$$\begin{aligned} R_1 &> I(W_1; RB)_{\sigma_1} - I(W_1; W_2)_\sigma & R_1 + R_2 &> I(W_1; RB)_{\sigma_1} + I(W_2; RA)_{\sigma_2} - I(W_1; W_2)_\sigma \\ R_2 &> I(W_2; RA)_{\sigma_2} - I(W_1; W_2)_\sigma & R_1 + R_2 + C_1 &> I(W_1; R\underline{Y})_\sigma + I(W_2; RA)_{\sigma_2} - I(W_1; W_2)_\sigma \\ R_1 + C_1 &> I(W_1; R\underline{Y})_\sigma - I(W_1; W_2)_\sigma & R_1 + R_2 + C_2 &> I(W_2; R\underline{Y})_\sigma + I(W_1; RB)_{\sigma_1} - I(W_1; W_2)_\sigma \\ R_2 + C_2 &> I(W_2; R\underline{Y})_\sigma - I(W_1; W_2)_\sigma & R_1 + R_2 + C_1 + C_2 &> I(\underline{W}; R\underline{Y})_\sigma - I(W_1; W_2)_\sigma. \end{aligned}$$

Theorem

The action of $\lambda_{\underline{\mathcal{Y}}}$ on ρ_{AB} can be simulated with communication cost $(\underline{R}, \underline{C})$ if there exists $(\underline{\mathcal{W}}, \mu_{\underline{\mathcal{W}}}, p_{\underline{Y}|\underline{W}}) \in \mathcal{C}(\rho_{AB}, \lambda_{\underline{\mathcal{Y}}})$ for which $(\underline{R}, \underline{C}) \in \mathscr{A}(\underline{\mathcal{W}}, \mu_{\underline{\mathcal{W}}}, p_{\underline{Y}|\underline{W}})$.

Inner Bound for Multiple Jointly Measurable POVMs

Suppose $\rho \in \mathcal{D}(\mathcal{H})$, $\lambda_{\mathcal{Y}_1}, \lambda_{\mathcal{Y}_2}, \lambda_{\mathcal{Y}_3}$ are POVMs such that there exists *commuting* POVMs $\mu_{\mathcal{W}_1}, \mu_{\mathcal{W}_2}, \mu_{\mathcal{W}_3}$ and a conditional PMF $p_{\underline{Y}|\underline{W}}$ such that certain Markovity conditions hold, then we provide sufficient conditions analogous to the above theorem under which the simultaneous action of jointly measurable POVMs $\lambda_{\mathcal{Y}_1}, \lambda_{\mathcal{Y}_2}, \lambda_{\mathcal{Y}_3}$ can be simultaneously at the decoder.