Coloring of Cyclic Graphs

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Motivation for functional compression

- An example: Consider a student database with information including the rental records, and health, etc, of individuals.
- The Ministry of Science wants to offer housing aid to a particular group of students, which does not require any other information than the rental contract, and the payslips of the students, due to privacy and redundancy constraints.
- This scenario avoids compressing and transmitting large volumes of distributed data and is tailored to the specifics of the function.

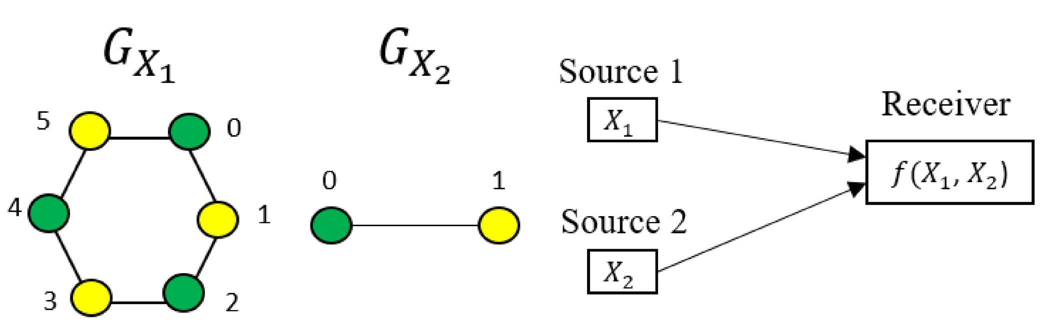
Functional compression (NP-hard)

- Goal: to compute a function $f(X_1, X_2)$ of the distributed sources X_1 and X_2 with a joint distribution $p(x_1, x_2)$.
- Source X_1 builds a characteristic or confusion graph G_{X_1} for distinguishing the outcomes of $f(X_1, X_2)$.
- G_{X_1} is represented by $G_{X_1} = G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{X}_1$, and for $x_1^1, x_1^2 \in \mathcal{V}$ that are distinct vertices, $\exists$ an edge $(x_1^1, x_1^2) \in \mathcal{E}$ iff $\exists$ a $x_2^1 \in \mathcal{X}_2$ such that
$$p(x_1^1, x_2^1) \cdot p(x_1^2, x_2^1) > 0 \text{ and } f(x_1^1, x_2^1) \neq f(x_1^2, x_2^1)\,.$$

OR powers of characteristic graphs

- For $n > 1$, the n-th OR power of characteristic graph $G_X = G(\mathcal{V}, \mathcal{E})$ is represented as $G^n_{\mathbf{X}} = (\mathcal{V}^n, \mathcal{E}^n)$ where $\mathcal{V}^n = \mathcal{X}^n$, and for distinct vertices $\mathbf{x}_1^n = x_{11}, \dots, x_{1n} \in \mathcal{V}^n$, $\mathbf{x}_2^n = x_{21}, \dots, x_{2n} \in \mathcal{V}^n$.
- It holds that $(\mathbf{x}_1^n, \mathbf{x}_2^n) \in \mathcal{E}^n$, when $\exists$ at least one $q \in [n]$ such that $(x_{1q}^1, x_{1q}^2) \in \mathcal{E}$, which is determined for characteristic graphs.

Distributed functional compression



- Comparison of distributed sources: Let $f(X_1, X_2) = (X_1 + X_2) \mod 2$, two sources X_1 and X_2 and one receiver.
- Source one X_1 is uniform over the alphabet $\mathcal{X}_1 = \{0, 1, 2, 3, 4, 5\}$, and X_2 is uniform over $\mathcal{X}_2 = \{0, 1\}$.
- At each source, even outcomes do not need to be distinguished from each other and are assigned the color G, while odd outcomes are assigned Y.
- To decode f, the receiver needs the color pairs (Y, G) or (G, Y), which correspond to outcome of 1.
- Conversely, the pairs (Y, Y) and (G, G) indicate an outcome of 0.

OR power graph degree

- Derivation of the degrees in the n-th OR power of a cycle graph C_i^n for $n \geq 2$:
$$deg(\mathbf{x}^n) = 2 + \sum_{j=1}^{n-1} 2(V^j)\,, \quad \forall \mathbf{x}^n \in \mathcal{V}^n\,.$$

Main contributions

- Evaluation of the exact degree of a vertex of n-th OR power for both odd and even cycles.
- Characterization of the exact value of the chromatic number, denoted by $\chi_{C_{2k}}$, for even cycles C_{2k} and their OR powers, for $k \in \mathbb{Z}^+$.
- An achievable coloring scheme for odd cycles C_{2k+1} for $k \in \mathbb{Z}^+$.
- Computation of the largest eigenvalue, λ_1, of the adjacency matrix A_f^n for the n-th OR power of cycle graphs. Which helps with bounding the chromatic number of a graph.
- A polynomial time valid coloring of a characteristic graph, which is in the form of a cycle, exploiting the structure of the characteristic graph and its OR powers.

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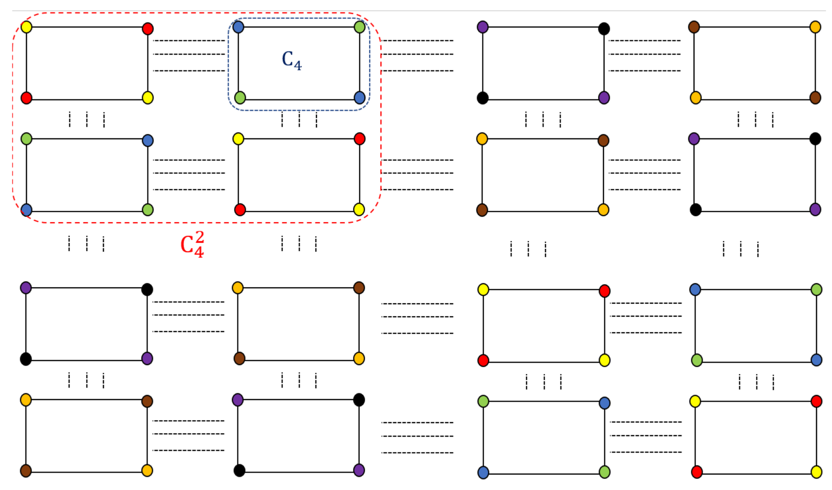
Coloring Even and odd Cycles Power Graphs

Even cycles

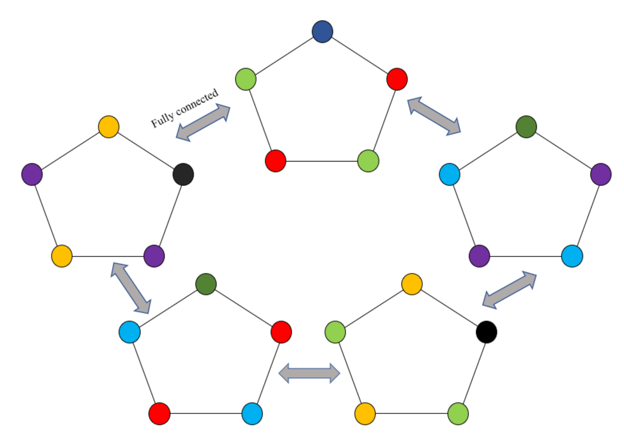
1. The graph, denoted by C_{2l}, $l \in \mathbb{Z}^+$, is colored in an alternating fashion.
2. Even vertices are assigned one color, while odd vertices are represented with a different color.

$$\chi(C_{2l}^n) = 2^n\,, \quad l \in \mathbb{Z}^+\,,\ n \geq 1\,.$$



Odd cycles

1. Odd cycles, denoted as C_{2l+1}, $l \in \mathbb{Z}^+$, have an odd number of vertices, e.g., C_3 is a cycle with 3 vertices, requiring 3 distinct colors.
2. For coloring the second power, C_3^2, sub-graphs $\{C_3(1), C_3(2), C_3(3)\}$, requires 3 different colors each (complete graph C_3).
3. For C_5, it needs 3 colors to cover all nodes.
4. A coloring scheme was devised for the coloring of C_5 powers and other odd cyclic graphs based on the color set of sub-graphs.
5. Coloring set of sub-graphs in C_5^2:
$$C_5^2(1) = \{c_1, c_2, c_3\}, \quad C_5^2(2) = \{c_4, c_5, c_6\},$$
$$C_5^2(3) = \{c_7, c_8, c_1\}, \quad C_5^2(4) = \{c_2, c_3, c_4\},$$
$$C_5^2(5) = \{c_5, c_6, c_7\}\,.$$



Chromatic number of odd cycles: Our scheme for optimal coloring
$$\chi_{C_i^{n+1}} = 2\chi_{C_i^n} + \left\lceil \frac{\chi_{C_i^n}}{2} \right\rceil\,, \quad n \geq 1\,.$$

Bounds on Chromatic Number

1. Bounds on χ_G, i.e., the chromatic number of graph G, where the lower bound is based on the work of Hoffman and the upper bound is derived from Wilf:
$$1 - \frac{\lambda_1(G)}{\lambda_V(G)} \leq \chi_G \leq \lfloor \lambda_1(G) \rfloor + 1\,.$$

> The eigenvalues of the all-ones matrix J_V with size $V \times V$ are 0 with an algebraic multiplicity $V - 1$ and V with multiplicity 1.

2. The adjacency matrix of the n-th OR power, A_f^n, of a cycle $C_i = G(\mathcal{V}, \mathcal{E})$ is presented as follows:
$$A_f^n = \begin{bmatrix} A_f^{n-1} & J_{V^{n-1}} & 0 & \dots & 0 & J_{V^{n-1}} \\ J_{V^{n-1}} & A_f^{n-1} & J_{V^{n-1}} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{V^{n-1}} & 0 & 0 & \dots & J_{V^{n-1}} & A_f^{n-1} \end{bmatrix},$$
3. The largest eigenvalue $\lambda_1(C_i^n)$ of the n-th OR power of a cycle graph C_i^n is determined as follows:
$$\lambda_1(C_i) = 2\,, \quad \lambda_1(C_i^n) = 2 + \sum_{j=0}^{n-1}(2V^j)\,, \quad n \geq 2\,.$$
4. The A_f^n for the n-th OR power of cyclic graphs, where $n \geq 2$, has the same distinct eigenvalues of A_f^{n-1} as well as two new distinct eigenvalues.
5. The following bound holds for $\chi_{C_i^n}$:
$$1 - \frac{2 + \sum_{j=0}^{n-1}(2V^j)}{-\sqrt{(V^n/2)[(V^n+1)/2]}} \leq \chi_{C_i^n} \leq \sum_{j=0}^{n-1}(2V^j) + 3\,.$$