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Training sequence for a particular channel

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Abstract

In this report, we look for an invertible training sequence whose inverse is of minimal energy. In usual sense, the product of an invertible sequence by its inverse yields an infinite Kronecker sequence. Here, "invertible" is taken in the sense that the product yields a finite sequence of fixed odd length, having value 1 in the middle and null otherwise. We look for an exact inverse, trading this constraint against the fact that the energy is not minimal. A recursive optimization method similar to Greedy's algorithm is proposed. Simulations are performed on various kinds of sequences. Simulations show that, like for the GSM, cyclic training sequences provide better results.

Contents

1 Notations

The notations are taken from Troulis' Phd thesis.

We estimate the channel by the means of a training binary ± 1 sequence u of length L_u. It has a pseudo-inverse c of length L_c such that the product (the channel response) is a pseudo Kronecker sequence h, 1 in the middle and made up of $L_{h1} = L_h - 1$ zeros on each side. L_h is the channel length and L_{h1} the channel memory. Thus h has total length $2L_h - 1 = 2L_{h1} + 1$.

2 Kinds of sequences tested

The kinds of sequences tested are the followings:

- totally random sequences (random case for short) : all bits are random.
- random sequences with bounded running digital sum (RDS).
- cyclic (for short) : the pattern of the sequence is BXB where B (the beginning) and X are random. The cyclic case is inspired by the GSM sequence.
- padded with 0 (zero-sequences for short) : the central part $u(L_{h1} : L_{h1} + L_c - 1)$ is random. The pattern of the total sequence is

$$u = [zeros(L_{h1}, 1)\ u\ zeros(L_{h1}, 1)]$$

  in MATLAB notations.
- Legendre sequences : a Legendre sequence is the characteristic sequence for quadratic residues modulo p where p is some prime. That is we consider the integers from 1 to p and associate +1 to those which are equal to some square modulo p and -1 to the other ones.
- cyclic Legendre sequences : these are Legendre sequences at the end of which we repeat the beginning.

The sequences do not necessarily meet the balance constraint.

3 Optimization method

One has to find out $\min(E = \sum c_i^2)$ under the constraint $Uc = h$. The c which minimizes the energy is of the form $c = U^t\lambda$ with $\lambda = (UU^t)^{-1}h$. Then the partial derivative of E with respect to u_i is $\partial^i E = -2\lambda^t \partial^i U c$.

From the theorem of bounded variations

$$\Delta E \approx \Delta u_i \; \partial^i E. \qquad (1)$$

The energy E is decreasing iff Δu_i and $\partial^i E$ have opposite signs. Then, since $|\Delta u_i = 2|$, the variation is maximal when $|\partial^i E|$ is maximal.

Now we have to distinguish the cyclic and the non-cyclic case. In the non-cyclic case,

$$(\partial^i E)_{i=1\ldots L_u} = -2\lambda \;.*\, \text{flip}(c) \qquad (2)$$

in MATLAB notations. Then the problem reduces to find out the biggest element of $\lambda \;.*\, \text{flip}(c)$ with index i such that u_i and $(\lambda \;.*\, \text{flip}(c))_i$ are of opposite signs.

In the cyclic case, we have prefered to use a matrix formula. We remark that

$$(\partial^i Uc)_{i=1\ldots L_c} = \text{toeplitz}(c(L_c : -1 : 1), c(L_c)c(1 : L_{h1} - 1)). \qquad (3)$$

We replace u_i with its opposite and go again.

4 Other possible optimization methods

We just give a few ideas we have investigated or not. Results are not detailed.

4.0.1 Soft decoding

We start from a random binary sequence u. We compute the sequence c of minimal energy among the solutions of $Uc = h$ (the pseudo-inverse of u).

From c we get back a sequence u_r, the inverse of c, which is real valued and then determine the binary sequence u_b at minimal distance of u_r. This problem amounts to the one of soft decoding.

Suppose that the length of the sequences u is of n. We look for the vector u_b which minimizes

$$\|u_r - u_b\| = u_r^2 + n - 2u_r.u_b.$$

Clearly the solution is the sequence $u_b = (u_{b,i})_{i=1\ldots n}$ such that $u_{b,i}$ is the sign of $u_{r,i}$. Then we compute a new c and so on.

Calculations have been performed but the results were not so good as for the method retained.

4.0.2 Various remarks

An other possible optimization method would be simulated annealing which is worth for any global optimization problem with several variables. This idea has not been furthered.

Looking for λ, we must solve $UU^t\lambda = h$. When u is cyclic, UU^t is Toeplitz and symmetric. This allows to use Levinson's algorithm to solve the system and speeds up the calculations.

5 Initial values, numerical parameters

For every kind of sequence except of Legendre sequences, one takes

- $L_c = 100$.
- $L_{h1} = 41$. We have $L_c \approx (2 + 20\%)L_{h1}$ which is an empirical formula.
- $L_u = 2L_{h1} + L_c = 182$.
- Number of iterations : 20.
- Number of simulations per kind of sequences : 10.

Legendre sequences are a particular case because they are deterministic and that their length must be a prime. We take L_u a prime so that the formulas $L_c \approx (2 + 20\%)L_{h1}$ and $L_u = 2 * L_{h1} + L_c$ are met. We consider the primes 179, 283 and 379, $L_c = 99, 149, 199$, $L_{h1} = 40, 67, 90$ and make 20,20 and 30 iterations respectively.

For cyclic Legendre sequences, L_c is chosen to be 100, 150 and 200 respectively, L_{h1} to be 41, 67 and 90 respectively so that the prime considered is the nearest to L_u.

For random sequences with bounded running digital sum, the bound is chosen to be $K = 20$.

Moreover, three very long random sequences with $L_c = 600$, $L_{h1} = 250$, $L_u = 1100$ and with bounded running digital have been tested for $K = 20, 15, 10$.

6 Results and comments

The energy is lower bounded by $1/L_c$. We compute the relative difference between the energy of the best inverse sequence found and the lower bound i.e. $(\sum c_i^2 - 1)L_c$. We give here the best result provided by the simulations.

- Random case : 21% reached after 17 iterations.

$$
\begin{array}{llllllllllll}
u = [-1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1]
\end{array}
$$

- Random case, bounded RDS : 19% reached after 14 iterations.

$$
\begin{array}{lllllllllllllll}
u = [1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1]
\end{array}
$$

- Zero case : 23% reached after 8 iterations.

$$
\begin{array}{ccccccccccccc}
u=[0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0] & & & & & & & & & & & &
\end{array}
$$

- Cyclic case : 11% reached after 15 iterations.

$$
\begin{array}{ccccccccccc}
u=[1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & & & & &]
\end{array}
$$

- Legendre sequences :
  - $p = 179$. Best relative error : 25% reached after 13 iterations.

$$
\begin{array}{llllllllllll}
u = [1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1] &
\end{array}
$$

– $p = 283$. Best relative error : 34% reached after 17 iterations.

$$
\begin{array}{llllllllllll}
u = [1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & & & & &]
\end{array}
$$

– $p = 379$. Best relative error : 35% reached after 26 iterations.

$$
\begin{array}{rrrrrrrrrrrr}
u = [1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1] & & & & &
\end{array}
$$

- very long random sequence with running sum bounded by 20 : best relative error 33 % attained after 57 iterations.

$$
u = [\begin{array}{rrrrrrrrrrrrrrr}
-1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \ldots
\end{array}
$$

...1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	−1	1
1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	−1
−1	1	−1	1	1	1	−1	−1	1	1	1	−1	−1	1	−1
−1	1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1
1	1	1	−1	−1	1	1	1	−1	−1	−1	1	1	−1	−1
1	1	−1	−1	1	1	1	1	−1	−1	−1	1	−1	−1	−1
1	1	−1	1	1	1	1	1	−1	1	−1	1	−1	1	−1
1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	−1	1	1
1	1	−1	1	−1	1	−1	−1	−1	1	−1	1	−1	−1	1
−1	−1	1	1	1	−1	1	1	−1	−1	−1	1	1	1	1
1	1	1	1	−1	1	1	1	−1	1	1	1	1	1	−1
1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	−1	−1	−1
1	−1	1	−1	1	−1	1	1	1	1	−1	1	−1	−1	1
−1	−1	−1	−1	−1	1	1	1	1	−1	1	1	−1	1	1
1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	−1	−1
1	1	1	−1	1	−1	−1	1	1	1	−1	−1	−1	−1	−1
1	1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	−1	−1
1	−1	1	−1	−1	−1	−1	1	−1	1	1	−1	1	−1	−1
−1	1	−1	−1	1	1	−1	1	1	1	1	−1	−1	1	1
−1	1	1	−1	−1	−1	1	1	−1	1	1	−1	1	−1	1
−1	1	−1	1	−1	1	−1	−1	1	−1	1	1	1	1	−1
−1	1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	1	−1
−1	−1	1	1	1	−1	−1	1	1	1	1	1	−1	1	−1
−1	−1	−1	1	1	1	1	1	1	1	−1	1	−1	−1	1
1	−1	1	−1	−1	1	−1	1	−1	−1	−1	−1	−1	1	−1
1	−1	−1	1	1	1	−1	−1	1	1	−1	−1	−1	−1	−1
1	1	−1	1	1	1	−1	1	−1	−1	1	−1	1	−1	1
1	1	1	−1	−1	−1	−1	−1	−1	−1	1	−1	−1	−1	1
1	1	−1	1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1
−1	1	−1	−1	1	−1	1	1	1	−1	1	−1	1	−1	1
1	−1	1	1	−1	1	−1	−1	−1	1	−1	1	1	−1	−1
−1	−1	1	−1	−1	−1	1	1	−1	1	−1	1	−1	−1	1
1	−1	1	1	−1	1	−1	−1	−1	1	−1	−1	−1	1	1
−1	−1	1	−1	−1	−1	1	−1	1	−1	−1	−1	−1	1	−1
1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	−1	−1
1	1	1	1	−1]										

- very long random sequence with running sum bounded by 15 : best relative error 29% attained after 72 iterations.

$$
\begin{array}{llllllllllllll}
u = [-1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \ldots
\end{array}
$$

...1	−1	−1	−1	1	1	−1	1	1	−1	−1	−1	1	1
1	−1	1	−1	−1	−1	1	−1	−1	−1	1	1	−1	−1
−1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1
1	−1	1	−1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1
−1	1	1	−1	1	−1	1	−1	−1	−1	−1	−1	1	−1
−1	−1	1	−1	−1	1	−1	1	1	1	1	1	1	−1
1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1
−1	−1	−1	1	1	−1	−1	1	−1	1	−1	−1	−1	1
1	1	−1	1	−1	1	1	1	−1	−1	1	−1	−1	1
1	1	−1	−1	−1	1	1	−1	1	−1	−1	1	−1	−1
1	1	−1	−1	−1	1	−1	1	−1	−1	1	1	1	1
1	−1	−1	1	−1	−1	1	1	−1	1	1	−1	−1	−1
1	1	−1	1	1	1	−1	1	−1	−1	1	−1	1	1
1	1	1	−1	1	1	1	−1	−1	1	1	−1	1	−1
−1	−1	1	1	−1	−1	1	−1	1	−1	−1	−1	1	1
1	−1	1	1	−1	1	1	−1	−1	−1	1	1	−1	1
−1	1	1	−1	−1	−1	1	1	−1	1	1	1	−1	−1
1	−1	1	−1	1	−1	1	−1	−1	1	−1	−1	1	−1
−1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	−1
−1	1	−1	−1	1	−1	−1	−1	−1	−1	1	1	−1	−1
1	−1	1	1	1	1	1	1	−1	1	−1	1	1	1
−1	1	1	−1	1	1	1	1	1	1	1	−1	−1	1
1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1
1	1	1	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1
1	−1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	−1
−1	1	1	−1	1	−1	−1	1	−1	−1	−1	1	−1	−1
1	−1	1	−1	1	−1	−1	−1	1	−1	−1	1	1	1
−1	1	1	1	−1	−1	−1	1	1	1	1	−1	−1	−1
−1	1	1	−1	1	1	−1	−1	1	−1	1	−1	1	−1
−1	−1	1	−1	1	−1	1	−1	−1	1	1	−1	1	1
−1	−1	−1	1	1	1	−1	−1	−1	−1	1	−1	1	1
−1	1	−1	−1	−1	1	1	−1	−1	1	1	−1	−1	−1
1	1	1	−1	−1	1	1	1	1	1	−1	1	−1	1
1	−1	1	1	1	1	1	1	1	−1	1	1	−1	1
−1	1	−1	1	−1	−1	1	1	−1	1	1	1	1	1
−1	1	1	−1	−1	1	−1	−1	1	1	1	1	1	−1
1	1	−1	−1	1	1	−1	1	−1	−1	1	−1	1	1
−1	1	−1	−1	−1	−1	1	1	1	−1	1	−1	1	1
−1	1	−1	−1	−1	1	−1	1	−1	−1	1	1	−1	−1
1	−1	1	1	1	−1	−1	1	1	1	1	1	−1	1
1	−1	−1	−1	1	−1	−1	1	1	−1	1	−1	−1	−1
−1	1	1	1	1	−1	−1	−1	−1	−1	1	−1	1	1
−1	−1	1	1	1	1	−1	−1]						

- very long random sequence with running sum bounded by 10 : best relative error 51 % attained after 55 iterations.

$$
\begin{array}{lcccccccccccccc}
u = [-1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1\ldots
\end{array}
$$

. . . −1	1	1	1	1	1	1	−1	1	1	−1	1	−1	−1	−1
−1	−1	−1	−1	−1	−1	1	1	1	1	−1	1	−1	−1	−1
−1	−1	1	−1	−1	−1	1	−1	−1	−1	−1	1	1	1	1
−1	−1	−1	1	1	−1	1	1	1	1	−1	−1	−1	1	−1
1	1	−1	−1	−1	1	1	−1	−1	1	−1	−1	−1	−1	−1
1	1	−1	−1	1	−1	1	1	1	−1	−1	1	−1	−1	1
1	−1	1	−1	−1	1	−1	−1	1	1	1	1	1	−1	1
1	−1	−1	−1	−1	−1	1	−1	1	−1	1	1	1	1	−1
−1	1	−1	1	−1	1	1	−1	1	−1	−1	1	1	−1	−1
1	1	1	−1	1	1	1	1	−1	1	1	−1	1	1	−1
1	1	1	−1	1	1	1	−1	1	1	1	−1	−1	1	−1
−1	1	−1	−1	−1	−1	1	1	1	1	−1	1	1	1	−1
1	−1	−1	−1	1	1	−1	1	−1	1	1	1	−1	−1	−1
−1	1	1	1	1	−1	−1	1	−1	−1	−1	1	−1	1	−1
−1	1	−1	−1	1	−1	1	1	−1	−1	1	−1	1	−1	−1
−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	−1	−1
1	1	1	−1	1	−1	−1	−1	1	1	1	1	−1	1	−1
1	1	1	1	1	1	−1	1	1	1	−1	−1	1	1	−1
−1	1	1	1	−1	1	1	−1	1	−1	−1	−1	1	1	1
1	1	1	1	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	1
−1	−1	1	1	1	1	−1	1	1	−1	1	1	−1	1	−1
1	1	−1	−1	−1	−1	1	−1	−1	1	1	−1	−1	−1	−1
1	−1	1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	1
1	1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	−1	1
1	1	−1	−1	1	−1	1	−1	1	1	−1	−1	1	−1	1
−1	−1	−1	−1	1	1	−1	1	1	−1	−1	−1	1	1	1
−1	−1	−1	−1	−1	−1	1	1	−1	1	−1	−1	−1	1	1
−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1	1
1	−1	−1	1	−1	1	1	−1	1	1	1	−1	1	1	1
−1	1	1	−1	1	−1	1	−1	1	−1	−1	1	−1	−1	−1
1	1	1	1	−1	1	1	−1	−1	1	−1	−1	1	1	1
1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1	−1
1	1	−1	−1	−1	−1	−1	−1	1	1	1	−1	1	−1	1
1	−1	1	−1	−1	−1	1	−1	1	−1	−1	1	1	−1	−1
1	−1	1	1	1	−1	−1	1	1	1	1	1	−1	1	1
−1	−1	−1	1	−1	−1	1	1	−1	1	−1	−1	−1	−1	1
1	1	1	−1	−1	−1	−1	−1	1	−1	1	1	−1	−1	1
1	1	1	−1	−1]										

It seems that it remains a gap that can not be filled between the best norm attained and the lower bound.

Results provided by random sequences or sequences padded with zeros are equally good. It is surprising and very positive that the constraint on the RDS does not deteriorate the performances of random sequences. The results obtained with cyclic sequences are particularly good and encouraged us to go further with longer sequences.

6.1 Further study of the cyclic case

The new values for the parameters are $Lc = 200$, $Lh = 90$. The best relative error attained is 15% after 24 iterations.

$$
\begin{array}{llllllllllll}
u = [-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 \\
-1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1] & & & &
\end{array}
$$