2D DoA-based Positioning with Phase Jump Corrections and An Approximate Maximum Likelihood Estimator

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***Abstract*—Direction of arrival (DoA) estimation is crucial to improve communications systems' performance, leading to much more accurate results in localization, one of the most vital applications in the Internet of Things (IoT). Unlike the range-based ones, the direction-based positioning algorithms estimate the unknown position by the measured angles whose values must be predefined in an interval of 2π-length. Noisy measurements with values near the edges of this interval can lead to drastic estimation errors, making the convergence of iterative procedures much more challenging. In this paper, we propose a Maximum Likelihood (ML) estimator, which applies iterative procedures for position estimation. Our procedure is based on the atan2 function, which has the 2π-long codomain to map the DoA. Moreover, a novel mechanism to make the estimation near the edges much more robust, phase jump corrections are proposed to rectify the final estimates. In addition, a new approximate ML estimator, where the effects of approximately normal distributed DoA estimation errors are limited to first-order perturbations, is also introduced. Outputs of this approximate estimator help to enhance the accuracy of the true ML estimator. Simulation results show significant performance improvements.**

***Index Terms* - direction-based, 2D positioning, DoA, Direction of Arrival, localization, Maximum Likelihood, approximate ML.**

I. Introduction

So far, we have several main traditional positioning techniques in 2D schemes: Time of Arrival (ToA), Time Difference of Arrival (TDoA), Received Signal Strength (RSS), and Direction of Arrival (DoA) (in some documents, it is also called Angle of Arrival - AOA) [1]. ToA-based [2]–[4] and TDoA-based [5], [6] positioning require highly accurate clock synchronization among all BSs and mobile device. RSS-based technique [7], [8], on the other hand, is very sensitive to the log normal fadings to it provides rough estimates for localization. DoA-based systems do not require such a synchronization. Instead, the resolution of DoA measurements is limited by the Signal-to-Noise Ratio (SNR), the number of sensors in the array, and the separation between these sensors. DoA estimation schemes are usually thought of as computationally expensive. However, recent developments propose computationally simple DoA estimation schemes that enable small antenna arrays with a reduced number of elements [9].

Direction-based localization computes the coordinate of the mobile device based on the direction of incident waves to base stations. The numerical expression of this direction is the trigonometric angle between the x-direction and the wave (Fig. 1). To avoid confusion in measuring angle, all the angles' value must be defined in an interval whose length is 2π. At the bound of the interval, the DoA is very sensitive to noise. On condition that the DoA's set of definition is $(-\pi;\pi]$, when the true value of an angle is $\widehat{aOb} = \pi - \varepsilon_a$, a small noise of $\varepsilon_a + \varepsilon_b$ (ε_a and ε_b are small positive values) can make the angle's measured value $\widehat{aOb'} = -\pi + \varepsilon_b$ (Fig. 2).

Related papers [10]–[13] about 2D DoA-based localization use arctan function to define DoA; meanwhile the codomain of that function is $[-\pi/2;\ \pi/2]$ (we assume that arctan $(-\infty)$ = $-\pi/2$ and arctan $(+\infty)$ = $\pi/2$). This codomain does not cover all the possible values of an angle. In the paper [13], a ML estimator is proposed to optimize the positioning. However, the sensitivity to noise of a value near the bound of the set of definition is not well considered. In addition, the localization is studied when all the DoA measurements are considered to be distributed with one common variance.

Our contributions in this paper are:

- Expressing location in terms of the atan2 function; introducing and optimizing associated phase wrapping correction terms for ML estimator in **section II**. In the definition of DoA, the atan2 function is utilized instead of arctan function. Furthermore, phase jump corrections are added in estimating the estimated DoA to avoid possible huge computing errors caused by small mistakes in practical measurements. Evaluations on the effect of the phase jump corrections are carefully analyzed.
- Propose an approximate ML estimator for DoA-based localization (**section III**).
- In **section IV**, we analyse the true ML estimator with the new DoA definition and the additional correction proposed in section II.

In the **last two sections**, we illustrate the simulations and then compare our results to the related results, in order to

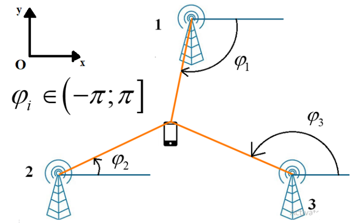


Fig. 1. DoA positioning in noiseless scenario

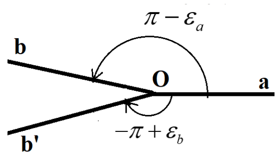


Fig. 2. Sensitivity to noise of an angle's measured value

prove the superiority of our proposed algorithm.

Notation:

mod(x,a) denotes x modulo a; **diag**$(a_1,\ a_2,\ \dots,\ a_n)$ is the diagonal matrix whose diagonal elements are $a_1,\ a_2,\ \dots,\ a_n$ respectively; $[a;b)$ is the interval of real numbers from a to b which includes a but excludes b, **atan2** means 2-argument which is defined as: φ = atan2(y, x) $\iff$ x + jy = $\mathrm{r}e^{\mathrm{j}\varphi}$ with r = $\sqrt{\mathrm{x}^2+\mathrm{y}^2}$, $\varphi \in (-\pi;\pi]$ and j is the imaginary unit. The standard arctangent function **arctan** has values in $[-\frac{\pi}{2},\frac{\pi}{2}]$. Let

$$\operatorname{sign}(x) = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0\,. \end{cases} \tag{1}$$

Then for $(x,y) \neq (0,0)$, we have

$$\operatorname{atan2}(y,x) = \arctan\left(\frac{y}{x}\right) - (\operatorname{sign}(x)-1)\operatorname{sign}(y)\frac{\pi}{2}\,. \tag{2}$$

II. DoA-based localization by Least Squares algorithm

A. Definition of DoA

We define φ_i to be the **trigonometric angle** between the x axis and the signal ray received at the i-th base station (Fig. 1). Let (x,y) be the coordinates of the mobile device and (x_i,y_i) be the coordinates of the i-th base station. The relation between φ_i and all the related coordinates can be stated as follows:

$$\tan\varphi_i = \frac{y-y_i}{x-x_i} \tag{3}$$

In [10]–[13], the arctan function (or tan^{-1} for other notation) is used to compute φ_i from the related coordinates

$$\varphi_i = \arctan\frac{y-y_i}{x-x_i} \tag{4}$$

However, the equation (4) can perform in low accuracy when the value of φ_i is larger than $\pi/2$ or smaller than $-\pi/2$. Consequently, we propose the definition for the true DoA of the signal to the i-th base station:

$$\varphi_i = \operatorname{atan2}(y-y_i,\ x-x_i) \tag{5}$$

According to the definition of atan2 function, the true value of DoA $\varphi_i \in (-\pi;\pi]$ so its measured value $\hat{\varphi}_i$ must be in that range, too. Nonetheless, practical measurements always contain an additive noise. When the true value of the angle is close to $-\pi$ or π, the measured value is very sensitive to noise (a small change in noise can cause a big difference in measured value) (Fig. 2). To avoid this unexpected difference, a phase jump correction of $k_i 2\pi$ is added, where the value of k_i can be -1, 0 or 1, depending on the difference between the true value and the measured value caused by the small noise.

In practical measurements, the measured value of i-th DoA is:

$$\hat{\varphi}_i = \varphi_i + n_i + k_i 2\pi \tag{6}$$

where n_i is assumed to be a Gaussian distributed angle estimation with zero-mean and variance σ_i^2.

Assuming that all n_i are independent. Therefore we have covariance matrix of the noise vector $\boldsymbol{n}$:

$$\boldsymbol{C} = E\{\boldsymbol{n}\boldsymbol{n}^T\} = \operatorname{diag}\{\sigma_1^2,\sigma_2^2,\dots,\sigma_N^2\} \tag{7}$$

where $\boldsymbol{n} = [n_1\ \ n_2\ \ \dots\ \ n_N]^T$ and N is the number of base stations.

We name the action of adding a phase jump correction of $k_i 2\pi$ as **k-correction**. We have the definition of k_i

$$k_i = \begin{cases} 1 & , \varphi_i + n_i \leq -\pi \\ -1 & , \varphi_i + n_i > \pi \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

B. Effect of the phase jump corrections

We evaluate the effect of the phase jump corrections on localization. A phase jump correction is significant when it is non-zero. Let $\rho(\varphi_i)$ be the probability of that event, at the argument φ_i . We have

$$\begin{aligned} \rho(\varphi_i) &= p(n_i < -\pi - \varphi_i) + p(n_i \geq \pi - \varphi_i) \\ &= \Phi\left(\frac{-\pi-\varphi_i}{\sigma_i}\right) + 1 - \Phi\left(\frac{\pi-\varphi_i}{\sigma_i}\right) \end{aligned} \tag{9}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}\,dt$

Since φ_i is in $(-\pi,\pi]$, the probability p_i that the phase jump correction is non-zero is

$$p_i = \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\varphi_i)\,d\varphi_i \tag{10}$$

In [14], it is approximated that $\Phi(x) \approx \dfrac{e^{2\sqrt{\frac{2}{\pi}}x}}{1+e^{2\sqrt{\frac{2}{\pi}}x}}$ when x is very large.

Thus, when σ_i is small enough

$$p_i \approx \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{2\sqrt{\frac{2}{\pi}}\frac{-\pi-\varphi_i}{\sigma_i}}}{1+e^{2\sqrt{\frac{2}{\pi}}\frac{-\pi-\varphi_i}{\sigma_i}}} + 1 - \frac{e^{2\sqrt{\frac{2}{\pi}}\frac{\pi-\varphi_i}{\sigma_i}}}{1+e^{2\sqrt{\frac{2}{\pi}}\frac{\pi-\varphi_i}{\sigma_i}}}\right)d\varphi_i \tag{11}$$

$$p_i \approx \sigma_i\frac{\ln 2 - \ln\left(e^{\frac{4\sqrt{2\pi}}{\sigma_i}}+1\right)}{2\sqrt{2\pi}} + 2 \tag{12}$$

Since σ_i is small, we approximate

$$\ln\left(e^{\frac{4\sqrt{2\pi}}{\sigma_i}}+1\right) \approx \ln\left(e^{\frac{4\sqrt{2\pi}}{\sigma_i}}\right) = \frac{4\sqrt{2\pi}}{\sigma_i} \tag{13}$$

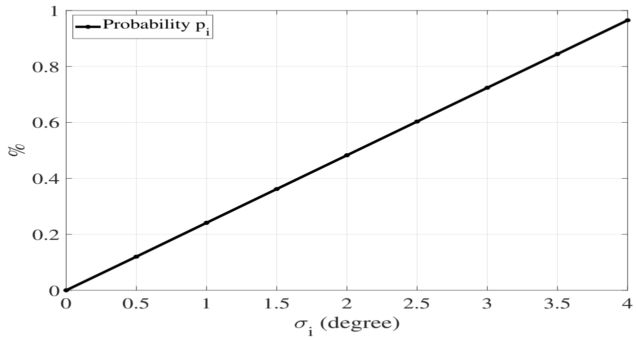


Fig. 3. Probability (%) that a phase jump correction is non-zero, in terms of σ_i, when σ_i is small.

As a result, (11) and (12) give an approximation:

$$p_i \approx \sigma_i \frac{\ln 2 - \frac{4\sqrt{2\pi}}{\sigma_i}}{2\sqrt{2\pi}} + 2 = \sigma_i \frac{\ln 2}{2\sqrt{2\pi}} \tag{14}$$

Therefore, when σ_i is small enough, p_i is proportional to σ_i with the coefficient of $\frac{\ln 2}{2\sqrt{2\pi}}$ (Fig. 3). Since the network has multiple base stations, the probability that **at least** one phase jump correction is non-zero is much higher in practical localization.

C. Estimating position by Least Squares method

From equation (3), we have

$$\tan\varphi_i = \frac{\sin\varphi_i}{\cos\varphi_i} = \frac{y - y_i}{x - x_i} \tag{15}$$

$$x\sin\varphi_i - y\cos\varphi_i = x_i\sin\varphi_i - y_i\cos\varphi_i \tag{16}$$

In this method, as n_i is very small, we approximate that $\sin n_i \approx 0$ and $\cos n_i \approx 1$. Thus

$$\sin\varphi_i = \sin(\hat{\varphi}_i - n_i - k_i 2\pi) = \sin(\hat{\varphi}_i - n_i) \approx \sin\hat{\varphi}_i \tag{17}$$

$$\cos\varphi_i = \cos(\hat{\varphi}_i - n_i - k_i 2\pi) = \cos(\hat{\varphi}_i - n_i) \approx \cos\hat{\varphi}_i \tag{18}$$

Hence, from (16), it is approximated that

$$x\sin\hat{\varphi}_i - y\cos\hat{\varphi}_i = x_i\sin\hat{\varphi}_i - y_i\cos\hat{\varphi}_i \tag{19}$$

In matrix approach, we define

$$\hat{\boldsymbol{A}} = \begin{bmatrix} \sin\hat{\varphi}_1 & -\cos\hat{\varphi}_1 \\ \sin\hat{\varphi}_2 & -\cos\hat{\varphi}_2 \\ \cdots & \cdots \\ \sin\hat{\varphi}_N & -\cos\hat{\varphi}_N \end{bmatrix}; \; \hat{\boldsymbol{b}} = \begin{bmatrix} x_1\sin\hat{\varphi}_1 - y_1\cos\hat{\varphi}_1 \\ x_2\sin\hat{\varphi}_2 - y_2\cos\hat{\varphi}_2 \\ \cdots \\ x_N\sin\hat{\varphi}_N - y_N\cos\hat{\varphi}_N \end{bmatrix}$$

$\boldsymbol{x} = \begin{bmatrix} x & y \end{bmatrix}^T$ is the coordinate vector of the mobile device. We have the equation of approximation

$$\hat{\boldsymbol{A}}\boldsymbol{x} = \hat{\boldsymbol{b}} \tag{20}$$

We have

$$\hat{\boldsymbol{x}} = \min_{\boldsymbol{x}} \|\hat{\boldsymbol{A}}\boldsymbol{x} - \hat{\boldsymbol{b}}\|^2 \tag{21}$$

leading to the estimate of $\boldsymbol{x}$ being calculated by Least Squares estimation of $\boldsymbol{x}$

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{A}}^{\dagger}\hat{\boldsymbol{b}} \tag{22}$$

where $\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ is the Moore-Penrose pseudo inverse of matrix $\boldsymbol{A}$.

III. Optimizing position estimation by an approximate Maximum Likelihood estimator

Unlike Least Squares method, in this approximate estimator, we assume that the effect of approximately Gaussian DoA estimation errors n_i can be limited to first-order perturbations, so we have the approximations: $\sin n_i \approx n_i$ and $\cos n_i \approx 1$. As a result, we have:

$$\sin\varphi_i = \sin(\hat{\varphi}_i - n_i - k_i 2\pi) = \sin(\hat{\varphi}_i - n_i) \approx \sin\hat{\varphi}_i - n_i\cos\hat{\varphi}_i \tag{23}$$

$$\cos\varphi_i = \cos(\hat{\varphi}_i - n_i - k_i 2\pi) = \cos(\hat{\varphi}_i - n_i) \approx \cos\hat{\varphi}_i + n_i\sin\hat{\varphi}_i \tag{24}$$

Therefore, from (16), we approximate

$$(x - x_i)(\sin\hat{\varphi}_i - n_i\cos\hat{\varphi}_i) - (y - y_i)(\cos\hat{\varphi}_i + n_i\sin\hat{\varphi}_i) = 0 \tag{25}$$

$$n_i = \frac{-(x - x_i)\sin\hat{\varphi}_i + (y - y_i)\cos\hat{\varphi}_i}{-(x - x_i)\cos\hat{\varphi}_i - (y - y_i)\sin\hat{\varphi}_i} \tag{26}$$

The Cost Function of this approximate ML estimator is expressed as

$$L = \sum_{i=1}^{N} \frac{n_i^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left(\frac{-(x - x_i)\sin\hat{\varphi}_i + (y - y_i)\cos\hat{\varphi}_i}{-(x - x_i)\cos\hat{\varphi}_i - (y - y_i)\sin\hat{\varphi}_i} \right)^2 \tag{27}$$

Our task is to find an estimate $\hat{\boldsymbol{x}}$ that minimizes the Cost Function, or

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \sum_{i=1}^{N} \left(\frac{1}{\sigma_i} . \frac{-(x - x_i)\sin\hat{\varphi}_i + (y - y_i)\cos\hat{\varphi}_i}{-(x - x_i)\cos\hat{\varphi}_i - (y - y_i)\sin\hat{\varphi}_i} \right)^2 \tag{28}$$

Finding $\hat{\boldsymbol{x}} = [\hat{x} \;\; \hat{y}]^T$ is really a challenging task. A solution is to use a local search, which is an iterative algorithm requiring an initial position estimate. We consider Gauss Newton algorithm [15] for $\hat{\boldsymbol{x}}$. At the iteration $(u+1)$:

$$\hat{\boldsymbol{x}}^{(u+1)} = \hat{\boldsymbol{x}}^{(u)} - (\boldsymbol{G}_n^T(\hat{\boldsymbol{x}}^{(u)})\boldsymbol{G}_n(\hat{\boldsymbol{x}}^{(u)}))^{-1}\boldsymbol{G}_n^T(\hat{\boldsymbol{x}}^{(u)})\boldsymbol{n}^T \tag{29}$$

where $\hat{\boldsymbol{x}}^{(u)}$ is the estimated coordinate vector of the mobile at the u-th iteration. $\boldsymbol{G}_n(\boldsymbol{x})$ is the following Jacobian matrix:

$$\boldsymbol{G}_n(\boldsymbol{x}) = \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{x}^T} = \frac{\partial \left(\frac{1}{\sigma_i} \frac{-(x - x_i)\sin\hat{\varphi}_i + (y - y_i)\cos\hat{\varphi}_i}{-(x - x_i)\cos\hat{\varphi}_i - (y - y_i)\sin\hat{\varphi}_i} \right)}{\partial \boldsymbol{x}^T} \tag{30}$$

The procedure is expected to terminate when $\|\hat{\boldsymbol{x}}^{(u+1)} - \hat{\boldsymbol{x}}^{(u)}\|_2 < \varepsilon_1$, for the stopping criterion ε_1 sufficiently small. Then, the final position of the procedure is considered to be the coordinates of the mobile device in the xy plane. However, the iterative procedures do not always converge. In [16], we demonstrate that there are **three** possible outcomes for an iterative procedure: **Convergence**, **Divergence** and **Oscillation**.

If a procedure is diverging or oscillating, we will take its initialization as the estimated mobile position. As for a converging procedure, the final position is selected as estimate.

The Algorithm 1 illustrates the Gauss-Newton iterative procedure of the approximate ML estimator.

Algorithm 1: Proposed Approximate ML Estimator

Take the measured Direction of Arrival $\hat{\varphi}_i$.
Assign $u = 1$ and ε_1 sufficiently small.
Assign the coordinate vector computed by (22) as the first estimated coordinate vector $\hat{\boldsymbol{x}}^{(1)}$ of the mobile device.
repeat
  Compute the following estimated coordinate vector $\hat{\boldsymbol{x}}^{(u+1)}$ of the mobile device by (29).
  $u = u+1$;
until $\|\hat{\boldsymbol{x}}^{(u+1)} - \hat{\boldsymbol{x}}^{(u)}\|_2 < \varepsilon$ *or* $u > 1000$ *or* $\|\hat{\boldsymbol{x}}^{(u+1)}\|_2 = \pm\infty$;
if $u > 1000$ *or* $\|\hat{\boldsymbol{x}}^{(u+1)}\|_2 = \pm\infty$ **then**
  $\hat{\boldsymbol{x}}^{(1)}$ is the estimated position of the mobile device;
else
  $\hat{\boldsymbol{x}}^{(u)}$ is the estimated position of the mobile device;

IV. Optimizing position estimation by the true Maximum Likelihood estimator

In the Least Squares method and approximate ML, the existence of the k-correction does not matter, and there is no difference in the final estimates if we use arctan or atan2 function in DoA definition. However, in the true ML estimator, it is important to use the DoA definition with atan2 function in equation (5) and the **k-correction** plays a crucial role, too. This true ML estimator is expected to give more accurate position estimations than the approximate ML.

In vector form, we denote

$$\hat{\boldsymbol{\varphi}} = \begin{bmatrix} \hat{\varphi}_1 & \hat{\varphi}_2 & \dots & \hat{\varphi}_N \end{bmatrix}^{\mathrm{T}} \tag{31}$$

$$\boldsymbol{f}(\hat{\boldsymbol{x}}, \boldsymbol{k}) = \begin{bmatrix} \hat{\varphi}_1(\hat{\boldsymbol{x}}) + k_1 2\pi \\ \hat{\varphi}_2(\hat{\boldsymbol{x}}) + k_2 2\pi \\ \cdots \\ \hat{\varphi}_N(\hat{\boldsymbol{x}}) + k_N 2\pi \end{bmatrix} \tag{32}$$

where $\boldsymbol{k} = [k_1 \ \ k_2 \ \ \cdots \ \ k_N]^T$, $\hat{\varphi}_i(\hat{\boldsymbol{x}})$ is the estimated DoA depending on $\hat{\boldsymbol{x}} = [\hat{x} \ \ \hat{y}]$ and computed by:

$$\hat{\varphi}_i(\hat{\boldsymbol{x}}) = \mathrm{atan2}(\hat{y} - y_i, \, \hat{x} - x_i) \tag{33}$$

Treating the phase shift vector $\boldsymbol{k}$ as unknown parameters and ignoring their dependence on the noise, the measurement vector $\hat{\boldsymbol{\varphi}}$ is Gaussian with mean vector of $\boldsymbol{f}$ and covariance matrix $\boldsymbol{C}$, we have the probability density function (pdf) [17]:

$$\mathrm{p}(\hat{\boldsymbol{\varphi}}|\boldsymbol{x}, \boldsymbol{k}) = \frac{(2\pi)^{-\frac{N}{2}}}{|\boldsymbol{C}|^{\frac{1}{2}}} \exp\left[\frac{-1}{2}(\hat{\boldsymbol{\varphi}} - \boldsymbol{f})^T \boldsymbol{C}^{-1} (\hat{\boldsymbol{\varphi}} - \boldsymbol{f})\right] \tag{34}$$

Maximizing the pdf in (34) is equivalent to

$$\hat{\boldsymbol{x}}, \widehat{\boldsymbol{k}} = \arg\min_{\boldsymbol{x}, \boldsymbol{k}} (\hat{\boldsymbol{\varphi}} - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}))^T \boldsymbol{C}^{-1} (\hat{\boldsymbol{\varphi}} - \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k})) \tag{35}$$

which we shall perform alternatingly.

Like the section III, the Gauss Newton algorithm [15] is applied for $\hat{\boldsymbol{x}}$. At the iteration $(u+1)$:

$$\hat{\boldsymbol{x}}^{(u+1)} = \hat{\boldsymbol{x}}^{(u)} + (\boldsymbol{G}^T \boldsymbol{C}^{-1} \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{C}^{-1} (\hat{\boldsymbol{\varphi}} - \boldsymbol{f}(\hat{\boldsymbol{x}}^{(u)}, \boldsymbol{k}^{(u+1)})) \tag{36}$$

where $\boldsymbol{G}$ is the Jacobian matrix.

$$\boldsymbol{G} = \boldsymbol{G}(\hat{\boldsymbol{x}}^{(u)}, \boldsymbol{k}^{(u+1)}), \ \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{k}) = \frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k})}{\partial \boldsymbol{x}^T}. \tag{37}$$

At this point, it is important to determine the value of k_i. As we do not know the additive noise in each DoA measurement, k_i cannot be determined by equation (8). From (6), we have

$$|n_i| = |\hat{\varphi}_i - \varphi_i - k_i 2\pi| \tag{38}$$

We assume n_i small enough, $|n_i| < \pi$. Thus $\widehat{k_i}$ is estimated by

$$\hat{k}_i^{(u+1)} = \arg\min_{k_i \in \{0; \pm 1\}} |\hat{\varphi}_i(\hat{\boldsymbol{x}}^{(u)}) - \hat{\varphi}_i - k_i 2\pi| \tag{39}$$

where $\hat{\boldsymbol{x}}^{(u)} = [\hat{x}^{(u)} \ \ \hat{y}^{(u)}]$ is the estimated coordinate vector of the mobile device at the u-th iteration. The rests of the true ML estimator are very similar to the approximate ML estimator shown in section III.

In a nutshell, the Algorithm 2 is proposed for the Gauss-Newton iterative procedure of ML estimator.

Algorithm 2: Proposed True ML Estimator with the simultaneous estimation of $\widehat{\boldsymbol{k}}$

Take the measured Direction of Arrival $\hat{\varphi}_i$.
Assign $u = 1$ and ε_2 sufficiently small.
Assign a vector as the first estimated coordinate vector $\hat{\boldsymbol{x}}^{(1)}$ of the mobile device. This vector can be the estimate of Least Squares in equation (22) **or** the the result of approximate ML by Algorithm 1 in section III.
repeat
  Compute the estimated DoA $\hat{\varphi}_i(\hat{\boldsymbol{x}}^{(u)})$ by (33).
  if $|\hat{\varphi}_i(\hat{\boldsymbol{x}}^{(u)}) - \hat{\varphi}_i| \geq \pi$ **then**
    $\widehat{k_i} = \mathrm{sign}(\hat{\varphi}_i(\hat{\boldsymbol{x}}^{(u)}) - \hat{\varphi}_i)$
  else
    $\widehat{k_i} = 0$;
  Compute the following estimated coordinate vector $\hat{\boldsymbol{x}}^{(u+1)}$ of the mobile device by (36).
  $u = u+1$;
until $\|\hat{\boldsymbol{x}}^{(u+1)} - \hat{\boldsymbol{x}}^{(u)}\|_2 < \varepsilon_2$ *or* $u > 1000$ *or* $\|\hat{\boldsymbol{x}}^{(u+1)}\|_2 = \pm\infty$;
if $u > 1000$ *or* $\|\hat{\boldsymbol{x}}^{(u+1)}\|_2 = \pm\infty$ **then**
  $\hat{\boldsymbol{x}}^{(1)}$ is the estimated position of the mobile device;
else
  $\hat{\boldsymbol{x}}^{(u)}$ is the estimated position of the mobile device;

V. Simulation Results

A. Analysis of Cramer-Rao Bound (CRB)

To evaluate the quality of the algorithm based on DoA, we need to calculate the CRB, via the Fisher Information Matrix (FIM):

$$\boldsymbol{I}(\boldsymbol{x}) = \boldsymbol{G}^T(\boldsymbol{x}) \boldsymbol{C}^{-1} \boldsymbol{G}(\boldsymbol{x}). \tag{40}$$

The CRB is the trace of the inverse of FIM:

$$\mathrm{CRB} = tr(\boldsymbol{I}^{-1}) = [\boldsymbol{I}^{-1}]_{1,1} + [\boldsymbol{I}^{-1}]_{2,2} \tag{41}$$

B. Simulation Setup

To compare the quality among the algorithms and CRB, we use Root Mean Square Position Error (RMSE) which is defined by

$$\mathrm{RMSE} = \sqrt{\mathrm{E}\left(\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|^2\right)} \tag{42}$$

where $\boldsymbol{x}$ is the true position of the mobile device and $\hat{\boldsymbol{x}}$ is its estimate. In the xy plane, RMSE averaging is over 1000 mobile positions picked randomly in a square of 1000m x 1000m centered in the BS circle. The network of 8 Base stations (numbered from 1 to 8) forms the circumscribed circle of this

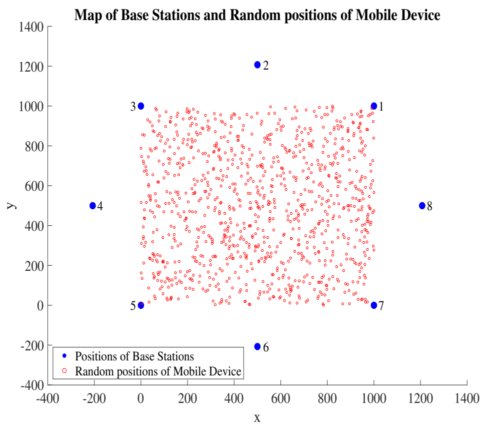


Fig. 4. Map of base stations' network, random points of the mobile device.

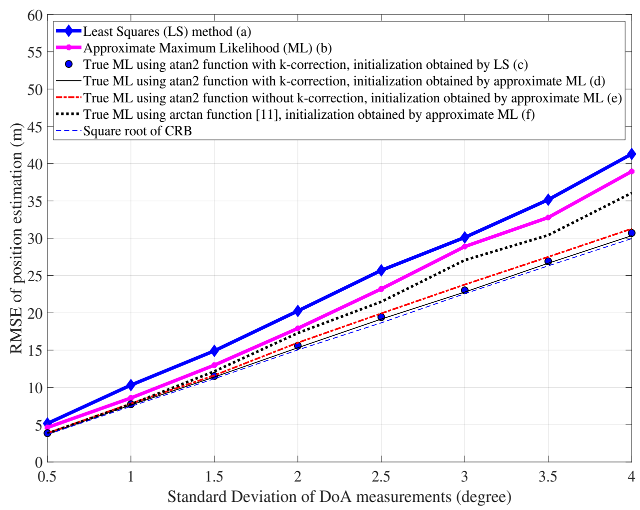


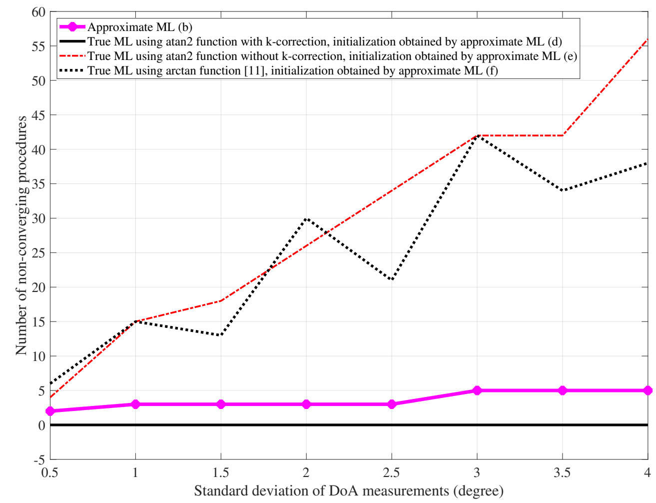
Fig. 5. Comparisons of RMSEs among the algorithms when $N = 8$ BSs, the standard deviation of DOA measurements varies from 0.5° to 4°.

square. All the related points are shown in Fig. 4. Stopping criteria for the approximate ML and true ML estimators are $\varepsilon_1 = \varepsilon_2 = 0.01$.

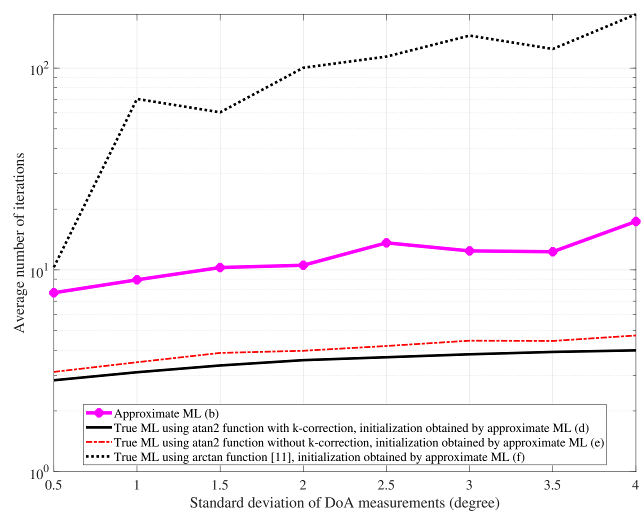
C. Results

Instead of comparing the MSEs to the CRB, we compare their square roots: The Root Mean Square Error (RMSE) = $\sqrt{\text{MSE}}$ and square root of CRB ($\sqrt{\text{CRB}}$). In each scheme, a common standard deviation is assumed for all the DoA measurements. More comprehensively, Fig. 5 compares the RMSEs of the 6 positioning algorithms:

(a) Least Squares method shown in section II-C.
(b) Approximate ML estimator shown in section III, with the initialization obtained by Least Squares method.
(c) True ML estimator (section IV), **with** k-correction; the DoA definition using atan2 function. The initialization is obtained by Least Squares method.
(d) True ML estimator (section IV), **with** k-correction; the DoA definition using atan2 function. The initialization is obtained by Approximate ML.



(a) Number of non-converging procedures out of 1000 testing ones



(b) Average number of iterations

Fig. 6. Comparisons among the iterative procedures of the algorithms when $N = 8$ BSs, the standard deviation of DOA measurements varies from 0.5° to 4°.

(e) True ML estimator (section IV), **without** k-correction; the DoA definition using atan2 function. The initialization is obtained by Approximate ML.
(f) True ML estimator (section IV), with the DoA definition using arctan function [13]. The initialization is obtained by the Approximate ML.

The $\sqrt{\text{CRB}}$ is also added to validate their performances.

Section III introduces 3 possible outcomes of an iterative procedure. Table I compares their results on RMSE and number of iterations. We compare the algorithms in terms of the accuracy (evaluated by the RMSEs) and the time delay (evaluated by the average number of iterations). When a procedure is **diverging** or **oscillating**, the initial position is taken as its estimate, which increases the RMSE and thus makes the location less exact. Moreover, an **oscillating** procedure raises remarkably the number of iterations, which makes the localization processes slower. As a result, the figures, which compare the RMSEs, the number of **non-converging** procedures and the average number of iterations among the algorithms, are demonstrated.

TABLE I: Possible outcomes of an iterative procedure.

Possible outcomes	**Estimated position of mobile**	**RMSE**	**Number of iterations**
Convergence	Final position of the procedure	Low	Low (fewer than 10)
Divergence	First position of the procedure	High	Low (fewer than 10)
Oscillation	First position of the procedure	High	High (1000)

Fig. 5 compares the RMSEs of the 6 algorithms above in the scenario of 8 base stations and the standard deviation of DoA measurements varies from $0.5°$ to $4°$. RMSE stands for the accuracy in estimation. Furthermore, Fig. 6 gives us an overview on how efficient the iterative procedures are in approximate ML and true ML estimators. More specifically, Fig 6a illustrates the number of non-converging procedures out of 1000 testing procedures. **Non-converging** procedures are the combination of **diverging** procedures and **oscillating** procedures defined in section III. Our proposed algorithm for true ML estimator has zero non-converging procedure, or in other words, all the procedures converge to local minima. Fig. 6b presents the average number of iterations. Our proposed algorithm has the fewest average number of iterations, which reduces the time delay for localization processes.

From the results above, it is obvious that

- The true ML estimator gives the best accurate position estimation, compared to approximate ML estimator and Least Squares method.
- In the true ML estimator, the initialization obtained by approximate ML gives a little smaller RMSE than that has initialization obtained by Least Squares. On the other hand, this RMSE is still larger than the $\sqrt{\text{CRB}}$, which assures the unbiased property of the estimator.
- The positioning algorithm with the k-correction and the definition of DoA using atan2 function has the best performance, in both accuracy and time delay, compared to the algorithms, of which the DoA is defined by arctan function, or by atan2 function but no k-correction for the iterative procedures. With this proposed algorithm, the RMSE is lowest but still higher than the $\sqrt{\text{CRB}}$, and the average number of iterations is noticeably smaller than the 2 other algorithms. This is the most important contribution of our paper.

In essence, the true ML likelihood estimator, of which the DoA is defined with atan2 function and an addition k-correction is used, is the best estimator. The initialization by an approximate ML can enhance the performance of the true ML.

VI. Conclusions

This paper thoroughly analyzes a ML estimator with the DoA-based positioning algorithms using atan2 function and the k-correction in 2D schemes. Moreover, an approximate ML estimator is also proposed. The simulations demonstrate the superior properties of our proposed algorithm: maintaining the unbiased property with the most accurate results and the shortest time delay. The approximate ML gives a better initialization for the true ML, which can augment the accuracy.

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