

Cramér–Rao Bounds for Blind Multichannel Estimation

Elisabeth de Carvalho and John Cioffi
Stanford University
Stanford, CA 94305-9515, USA
{carvalho,cioffi}@dsl.stanford.edu

Dirk Slock
Institut Eurecom, B.P. 193
06904 Sophia Antipolis Cedex, FRANCE
slock@eurecom.fr

Abstract—Certain blind channel estimation techniques allow the identification of the channel up to a scale or phase factor. This results in singularity of the Fisher Information Matrix (FIM). The Cramér–Rao Bound, which is the inverse of the FIM, is then not defined. To regularize the estimation problem, one can impose constraints on the parameters. In general, many sets of constraints are possible but are not always relevant. We propose a constrained CRB, the pseudo-inverse of the FIM, which gives, for a minimum number of constraints, the lowest bound on the mean squared estimation error.

I. INTRODUCTION

The Cramér–Rao Bound (CRB) is a powerful tool in estimation theory as it gives a performance lower bound for parameter estimation problems. It is computed as the inverse of the Fisher Information Matrix (FIM). When the parameters cannot all be identified, the FIM is singular, and the classical CRB results cannot be applied directly.

The main motivation for this work is the study of the performance of certain blind channel estimation problems where the parameters can indeed be identified only up to a scale or phase factor. Blind estimation is usually done under certain parameter constraints to regularize the problem. The performance of blind methods is not correctly evaluated in general or remains somewhat vague. The performance is often compared to “CRBs” that do not correspond to the regularization technique used in the estimation. A constraint often used is to consider one coefficient of the channel as known (which is sufficient to render the estimation problem regular): the resulting performance and its bound depend on the choice of this coefficient and appear arbitrary. One of the contributions of this work will be to give a less arbitrary bound and the corresponding set of constraints.

In this paper, we study the CRBs for estimation under parameter constraints in the case where the unconstrained problem leads to nonidentifiability, *i.e.* the FIM is singular. We furthermore outline the correspondence between the number and characteristics of FIM singularities and the number and characteristics of independent constraints needed in order to regularize the estimation problem and to be able to define the constrained CRB. Furthermore, assuming that we can choose the set of constraints, we propose a particular CRB for the case of an unidentifiable unconstrained estimation problem: this CRB is the Moore–Penrose pseudo-inverse of the FIM. It corresponds to a minimum number of independent constraints and gives the lowest bound on the mean square estimation error, *i.e.*

$\text{tr}(\text{CRB})$.

We apply these results to two classes of blind FIR multichannel estimation problems corresponding to two different models for the input symbols. The deterministic model, which exploits no statistical information on the input symbols, takes the input symbols to be deterministic quantities whereas in the Gaussian model we consider them to be uncorrelated Gaussian random variables to exploit their second-order statistics. The deterministic model leads to the class of methods that are directly based on the structure of the received signal; the Gaussian approach includes methods based on the second-order moments of the data, like certain prediction approaches [1] or the covariance matching method [2]. The deterministic methods can identify the channel up to a scale factor only and the Gaussian methods up to a phase factor, resulting in singularities of the FIM.

Throughout the paper, we distinguish between the real and complex parameter cases since they lead to different FIMs, with different singularities, and require different regularization constraints. The blind deterministic CRB is computed under the commonly used norm constraint which imposes the norm of the channel to be constant. This constraint is sufficient to regularize the problem when the channel is real, but not when it is complex in which case an additional constraint is required to adjust the phase of the channel. This constraint is chosen so that the resulting constrained CRB is the Moore–Penrose pseudo-inverse of the FIM and corresponds to a minimal constrained CRB. When the channel is real the Gaussian FIM is regular. When it is complex however, the FIM is singular: a constraint on the phase is necessary as in the deterministic case and the constrained CRB is again the pseudo-inverse of the FIM.

II. CRBS FOR REAL AND COMPLEX PARAMETERS

We review here CRBs for the case of regular FIMs.

A. CRBs for Real Parameters

Let θ be a deterministic real parameter vector and $f(\mathbf{Y}|\theta)$ the probability density function of the vector of real observations \mathbf{Y} . The FIM for θ is:

$$\mathcal{J}_{\theta\theta} = \text{E}_{\mathbf{Y}|\theta} \left(\frac{\partial \ln f(\mathbf{Y}|\theta)}{\partial \theta} \right) \left(\frac{\partial \ln f(\mathbf{Y}|\theta)}{\partial \theta} \right)^T. \quad (1)$$

Let $\hat{\theta}$ be an unbiased estimate of θ and $\tilde{\theta} = \hat{\theta} - \theta$ the estimation error. Hence $E\tilde{\theta} = 0$ and $C_{\tilde{\theta}\tilde{\theta}} = E\tilde{\theta}\tilde{\theta}^T$ is the error covariance matrix. When $\mathcal{J}_{\theta\theta}$ is nonsingular and under certain regularity conditions [3], $\mathcal{J}_{\theta\theta}^{-1}$ is the Cramér–Rao Bound and:

$$C_{\tilde{\theta}\tilde{\theta}} \geq CRB = \mathcal{J}_{\theta\theta}^{-1}. \quad (2)$$

B. CRB for Complex Parameters, Complex CRB.

When θ is a complex deterministic parameter, the previous results can be applied to $\theta_R = [\text{Re}(\theta)^T \text{Im}(\theta)^T]^T$ and $\mathbf{Y}_R = [\text{Re}(\mathbf{Y})^T \text{Im}(\mathbf{Y})^T]^T$, the associated real parameters and real observations.

It is however possible to define the FIM for θ_R w.r.t. complex FIM-like matrices. Let $J_{\varphi\psi}$ be defined as:

$$J_{\varphi\psi} = E_{\mathbf{Y}|\theta} \left(\frac{\partial \ln f(\mathbf{Y}|\theta)}{\partial \varphi^*} \right) \left(\frac{\partial \ln f(\mathbf{Y}|\theta)}{\partial \psi^*} \right)^H \quad (3)$$

Derivation w.r.t. the complex vector $\theta = \alpha + j\beta$ is defined as: $\frac{\partial}{\partial \theta} = \frac{1}{2} \left(\frac{\partial}{\partial \alpha} - j \frac{\partial}{\partial \beta} \right)$. The parameterization in $(\text{Re}(\theta), \text{Im}(\theta))$ is equivalent to a parameterization in terms of (θ, θ^*) via:

$$\theta_R = \begin{bmatrix} \text{Re}(\theta) \\ \text{Im}(\theta) \end{bmatrix} = \mathcal{M} \begin{bmatrix} \theta \\ \theta^* \end{bmatrix}, \quad \mathcal{M} = \frac{1}{2} \begin{bmatrix} I & I \\ -jI & jI \end{bmatrix} \quad (4)$$

where \mathcal{M} is non-singular. Knowing that $J_{\theta\theta} = J_{\theta^*\theta^*}^*$ and $J_{\theta\theta^*} = J_{\theta^*\theta}^*$, equation (4) implies:

$$\mathcal{J}_{\theta_R\theta_R} = \mathcal{M} \begin{bmatrix} J_{\theta\theta} & J_{\theta\theta^*} \\ J_{\theta^*\theta^*}^* & J_{\theta\theta}^* \end{bmatrix} \mathcal{M}^H. \quad (5)$$

When $J_{\theta\theta^*} = 0$, $\mathcal{J}_{\theta_R\theta_R}$ is completely determined by $J_{\theta\theta}$. In that case, $J_{\theta\theta}$ can be considered as the *complex FIM* and the corresponding *complex CRB* is such that:

$$C_{\tilde{\theta}\tilde{\theta}} = E\tilde{\theta}\tilde{\theta}^H \geq CRB = J_{\theta\theta}^{-1}. \quad (6)$$

If $J_{\theta\theta^*} \neq 0$, $J_{\theta\theta}^{-1}$ is also a lower bound on $C_{\tilde{\theta}\tilde{\theta}}$, but not as tight as the (real) CRB, $CRB_R = \mathcal{J}_{\theta_R\theta_R}^{-1}$.

C. Correspondence between Identifiability and FIM Regularity for a Gaussian Data Distribution

We consider identifiability as defined in [4], [5]: θ is said to be identifiable if

$$\forall \mathbf{Y}, \quad f(\mathbf{Y}|\theta) = f(\mathbf{Y}|\theta') \Rightarrow \theta = \theta'. \quad (7)$$

When the observations $\mathbf{Y} \sim \mathcal{N}(m_Y(\theta), C_{Y_Y}(\theta))$ have a normal distribution, identifiability is based on the mean and covariance: θ is said to be identifiable if:

$$m_Y(\theta) = m_Y(\theta') \text{ and } C_{Y_Y}(\theta) = C_{Y_Y}(\theta') \Rightarrow \theta = \theta'. \quad (8)$$

We have local identifiability at θ if identifiability holds for θ' being restricted to some open neighborhood of θ . In general, under some mild conditions on the FIM, we have equivalence between local identifiability and FIM regularity [6].

III. CRBS FOR ESTIMATION WITH CONSTRAINTS

In this section, we consider real parameters (hence θ stands for θ_R if θ is complex). When the estimation is (locally) unidentifiable, the FIM is singular and the classical CRB result (2) cannot be applied.

In order to characterize the non regular estimation performance, we define CRBs for estimation under a certain set of equality constraints: this set of constraints should allow to adjust the parameters that cannot be identified and in this way to regularize the estimation problem.

CRBs for parameter estimation under constraints were derived in [7] in the case where the unconstrained estimation problem is regular. A simpler derivation of these results was presented in [8]. The main ingredient of this simpler derivation was used in [9] to give an alternative expression for the CRB in the case where the unconstrained problem is unidentifiable. We shall succinctly restate these results, which appeared already in [10] for the case of linear constraints.

Consider a k -fold constraint of the form: $\mathcal{K}_\theta = 0$ where $\mathcal{K}_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is continuously differentiable and $k < n$, n being the number of parameters in the vector θ . A constrained parameter estimator $\hat{\theta}$ is called unbiased if it satisfies the constraints ($\mathcal{K}_{\hat{\theta}} = 0$) and if the parameter estimation bias is zero for all parameter values that satisfy the constraints [8]. The constrained CRB depends on the constraints only through the tangents to the constraint set at the true value of θ :

$$\mathcal{M}_\theta = \left\{ Z \in \mathbb{R}^n ; Z^T \frac{\partial \mathcal{K}_{\theta^o}^T}{\partial \theta} = 0 \right\}, \quad (9)$$

where θ^o denotes the true value of θ . We introduce a full column rank matrix \mathcal{V}_θ such that $\text{range}\{\mathcal{V}_\theta\} = \mathcal{M}_\theta$.

Theorem 1 (Constrained CRB) Assume the constrained estimator $\hat{\theta}$ to be unbiased ($\hat{\theta}$ and θ satisfy the constraints $\mathcal{K}_\theta = 0$), then

$$C_{\tilde{\theta}\tilde{\theta}} \geq CRB_C = \mathcal{V}_\theta (\mathcal{V}_\theta^T \mathcal{J}_{\theta\theta} \mathcal{V}_\theta)^{-1} \mathcal{V}_\theta^T. \quad (10)$$

A necessary and sufficient condition for the boundedness of CRB_C is the nonsingularity of $\mathcal{V}_\theta^T \mathcal{J}_{\theta\theta} \mathcal{V}_\theta$.

CRB_C is independent of the choice of \mathcal{V}_θ and in particular

$$CRB_C = P_{\mathcal{V}_\theta} (P_{\mathcal{V}_\theta} \mathcal{J}_{\theta\theta} P_{\mathcal{V}_\theta})^+ P_{\mathcal{V}_\theta} = (P_{\mathcal{V}_\theta} \mathcal{J}_{\theta\theta} P_{\mathcal{V}_\theta})^+. \quad (11)$$

There is a direct correspondence between the number of FIM singularities and the number of constraints necessary to have a finite constrained CRB, which is also the number of constraints necessary to have local identifiability.

Theorem 2: For the constrained CRB to be defined, it is necessary and sufficient to fulfill the following conditions.

- (i) The number of independent constraints should be at least equal to $n - r$ ($r = \text{rank}(\mathcal{J}_{\theta\theta})$).
- (ii) At least $n - r$ independent columns of $\frac{\partial \mathcal{K}_\theta^T}{\partial \theta}$ should not be orthogonal to the null space of $\mathcal{J}_{\theta\theta}$.

A constraint of the form $\mathcal{K}_\theta = 0$ has a local effect only (through its tangent) and can be linearized locally.

Theorem 3: The constrained CRB (10) can also be interpreted as the CRB under the linear constraint:

$$\theta^T \frac{\partial \mathcal{K}_{\theta^\circ}^T}{\partial \theta} = \theta^\circ{}^T \frac{\partial \mathcal{K}_{\theta^\circ}^T}{\partial \theta} \quad (12)$$

which means that the components of θ in span $\left\{ \frac{\partial \mathcal{K}_{\theta^\circ}^T}{\partial \theta} \right\}$ are known (here we emphasized that $\frac{\partial \mathcal{K}_{\theta^\circ}^T}{\partial \theta}$ is evaluated at $\theta = \theta^\circ$). This linearization of the constraints is very convenient, as the derivation of the CRBs or estimation performance using this linearization gets very simple [11].

A. Minimal constrained CRB

We assume here that $\mathcal{J}_{\theta\theta}$ is singular. When $\text{range}\{\mathcal{V}_\theta\} = \text{range}\{\mathcal{J}_{\theta\theta}\}$ and since \mathcal{V}_θ has full column rank, $\mathcal{V}_\theta^T \mathcal{J}_{\theta\theta} \mathcal{V}_\theta$ is regular (minimal number of independent constraints in this case) and the constrained CRB is:

$$CRB_C = \mathcal{J}_{\theta\theta}^+ . \quad (13)$$

This is a particular constrained CRB: we prove in [12] that, among all sets of a minimal number of independent constraints, $CRB_C = \mathcal{J}_{\theta\theta}^+$ yields the lowest value for $\text{tr}\{CRB_C\}$. This means that if we want to introduce a priori information in the form of independent constraints, enough to regularize the estimation problem, but not more (minimal number), then all the constraints should concentrate on the unidentifiable part of the parameters only ($\text{range}\left\{ \frac{\partial \mathcal{K}_{\theta^\circ}^T}{\partial \theta} \right\} = \text{null}\{\mathcal{J}_{\theta\theta}\}$) to minimize $\text{tr}\{CRB_C\}$.

Consider also the case of the estimation of θ_1 with θ_2 being a nuisance parameter. The overall parameter vector is $\theta = [\theta_1^T \ \theta_2^T]^T$. Assume that $\mathcal{J}_{\theta\theta}$ is singular but $\mathcal{J}_{\theta_2\theta_2}$ is regular. To regularize the estimation problem, we consider the introduction of (independent) constraints on θ_1 only: $\mathcal{K}_{\theta_1} = 0$. Assume that $\text{range}\{\mathcal{V}_{\theta_1}\} = \text{range}\{\mathcal{J}_{\theta_1\theta_1}(\theta)\}$, with $\mathcal{J}_{\theta_1\theta_1}(\theta) = \mathcal{J}_{\theta_1\theta_1} - \mathcal{J}_{\theta_1\theta_2} \mathcal{J}_{\theta_2\theta_2}^{-1} \mathcal{J}_{\theta_2\theta_1}$ ($\mathcal{J}_{\theta_1\theta_1}(\theta)$ would be the unconstrained CRB for θ_1 if $\mathcal{J}_{\theta\theta}$ were regular). Then it can be proven that the constrained CRB for θ_1 separately is [12]:

$$CRB_{C,\theta_1} = \mathcal{J}_{\theta_1\theta_1}^+(\theta) \quad (14)$$

Such constraints give the minimal constrained CRB for θ_1 over all sets of a minimal number of independent constraints on θ_1 .

IV. CRBS FOR BLIND FIR MULTICHANNEL ESTIMATION

These results are now applied to blind FIR multichannel estimation. Two models are presented here: the deterministic model and the Gaussian model. We first present the multichannel model, which is fundamental in blind channel estimation (from second-order statistics).

A. The Multichannel Model

Consider a sequence of symbols $a(k)$ received through m channels of length N with coefficients $\mathbf{h}(i)$:

$$\mathbf{y}(k) = \sum_{i=0}^{N-1} \mathbf{h}(i)a(k-i) + \mathbf{v}(k), \quad (15)$$

$\mathbf{v}(k)$ is an additive independent white Gaussian noise and $r_{\mathbf{v}\mathbf{v}}(k-i) = \text{E} \mathbf{v}(k)\mathbf{v}(i)^H = \sigma_v^2 I_m \delta_{ki}$. Assume we receive M samples, concatenated in the vector $\mathbf{Y}_M(k)$:

$$\mathbf{Y}_M(k) = \mathcal{T}_M(h) A_M(k) + \mathbf{V}_M(k) \quad (16)$$

$\mathbf{Y}_M(k) = [\mathbf{y}^T(k) \cdots \mathbf{y}^T(k-M+1)]^T$, similarly for $\mathbf{V}_M(k)$, and $A_M(k) = [a(k) \cdots a(k-M-N+2)]^T$. $(\cdot)^T$ denotes transpose and $(\cdot)^H$ hermitian transpose. The channel transfer function is $\mathbf{H}(z) = \sum_{i=0}^{N-1} \mathbf{h}(i)z^{-i} = [\mathbf{H}_1^T(z) \cdots \mathbf{H}_m^T(z)]^T$. $\mathcal{T}_M(h)$ is a block Toeplitz matrix filled out with the channel coefficients grouped in $h = [\mathbf{h}^T(0) \cdots \mathbf{h}^T(N-1)]^T$. We shall simplify the notation in (16) with $k = M-1$ to:

$$\mathbf{Y} = \mathcal{T}(h) A + \mathbf{V} . \quad (17)$$

A channel will be said to be irreducible if its subchannels $\mathbf{H}_i(z)$ have no zeros in common, and reducible otherwise. A reducible channel can be decomposed as $\mathbf{H}(z) = \mathbf{H}_I(z)\mathbf{H}_c(z)$ where $\mathbf{H}_I(z)$ of length N_I is irreducible and $\mathbf{H}_c(z)$ of length $N_c = N - N_I + 1$ is a monochannel.

B. Deterministic Model

The deterministic model considers the joint estimation of the unknown input symbols A and the channel coefficients h . The parameter vector is $\theta = [A^T \ h^T]^T$. When the channel is complex, as \mathbf{Y} is circular, we can work with the complex probability density function of the Gaussian random variable $\mathbf{Y} \sim \mathcal{N}(\mathcal{T}(h)A, \sigma_v^2 I)$. As $J_{\theta\theta^*} = 0$, the complex FIM $J_{\theta\theta}$ is equivalent to the real one $\mathcal{J}_{\theta_R\theta_R}$ and is equal to [10]:

$$J_{\theta\theta} = \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}^H(h) \\ \mathcal{A}^H \end{bmatrix} \begin{bmatrix} \mathcal{T}(h) & A \end{bmatrix} \quad (18)$$

When the channel is real, the FIM is the same as in (18). This equality of the expressions will allow us to treat the complex and real cases simultaneously.

B.1 Singularities of the FIMs

Identifiability of (A, h) occurs from $m_Y(\theta) = \mathbf{X} = \mathcal{T}(h)A$, the signal part of \mathbf{Y} . (h, A) can at best be estimated up to a scale factor: indeed $\mathcal{T}(h)A = \mathcal{T}(h/\alpha)\alpha A$. Blind identifiability in the deterministic model (*i.e.* identifiability up to a scale factor) requires the channel to be irreducible and the burst length and the number of input excitation modes to satisfy certain minimum requirements [5]. It can be proven that a channel is blindly identifiable up to a scale factor if and only if the complex FIM $J_{\theta\theta}$ has exactly one singularity [5]. It can also

be verified that the complex FIM $J_{\theta\theta}$ has the same number of singularities as $J_{hh}(\theta) \triangleq \frac{1}{\sigma_v^2} \mathcal{A}^H P_{T(h)}^\perp \mathcal{A}$. If $J_{hh}(\theta)$ were regular, its inverse would be the CRB for h (with A considered as nuisance parameters). The unique null vector of $J_{hh}(\theta)$ is h .

In the complex channel case, the real FIM $\mathcal{J}_{\theta_R\theta_R}(\theta_R)$ has 2 singularities spanned by:

$$h_{S_1} = \begin{bmatrix} \text{Re}(h) \\ \text{Im}(h) \end{bmatrix} = h_R \quad \text{and} \quad h_{S_2} = \begin{bmatrix} -\text{Im}(h) \\ \text{Re}(h) \end{bmatrix}. \quad (19)$$

The first null vector can be interpreted as corresponding to the ambiguity in the norm of the channel and the second one to the ambiguity in the phase factor.

B.2 Regularized Blind CRB

Blind methods commonly consider the quadratic constraint on the norm of the channel $h^H h = 1$ (see [9]). This constraint does not render the problem identifiable: it leaves a sign ambiguity when h is real and a continuous phase ambiguity when h is complex. In the former case, the computation of mean squared error (MSE) assumes the right sign (the right sign could be taken as the sign giving the smallest error). In the complex case however, which phase factor should be chosen? A frequent choice consists in imposing one element of h to be real and positive; the resulting CRB depends on the choice of this element however.

The blind regularized CRB proposed here is computed under the following constraints:

- (1) A quadratic constraint:

$$h^H h = h^{oH} h^o \quad (20)$$

which allows to adjust the norm of the channel.

- (2) In the complex case, an additional constraint is necessary to adjust the phase factor:

$$h_{S_2}^{oT} h_R = h_{S_2}^{oT} h_R^o = 0. \quad (21)$$

where h^o denotes the true value of the channel, h denotes the variable. In both real and complex cases, these constraints leave a sign ambiguity which does not lead to FIM singularity, as we have local identifiability in that case. For MSE evaluation, the ambiguity can be resolved by requiring $h^{oT} h > 0$.

Result 1: Under constraint (20) (and (21) for the complex case) the CRB for h is the Moore-Penrose pseudo-inverse of $J_{hh}(\theta)$:

$$CRB_{C,h} = J_{hh}^+(\theta) = \sigma_v^2 \left(\mathcal{A}^H P_{T(h)}^\perp \mathcal{A} \right)^+. \quad (22)$$

This result is an application of equation (14): note that in the complex case, the real FIM should be used (and not the complex one) to apply the result in (14).

The choice of the first constraint is not only motivated by its common use. As mentioned in section III-A, this constraint (possibly combined with the linear constraint on the phase) leads to the minimal constrained CRB.

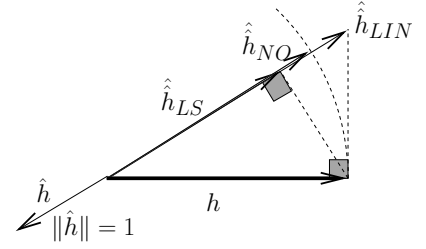


Fig. 1. Deterministic case: asymptotically equivalent constraints.

B.3 Some Equivalent Constraints

Another choice for the constraints leading to the same range for $\frac{\partial \mathcal{K}_{h_R}^T}{\partial h_R}$ is the linear constraint: $h^{oH} h = h^{oH} h^o$. This constraint, which leaves no sign ambiguity, corresponds to forcing the components of h in the nullspace of J_{hh} to their true values.

Often, h is estimated under a unit norm constraint $\|\hat{h}\| = 1$, and the scale factor is adjusted in different ways. The following adjustments lead to the same minimal CRB.

- The norm of the channel is adjusted so that $\|\hat{h}\| = \|h^o\|$ and the phase using the phase constraint (21). We denote the resulting estimate \hat{h}_{NO} .

- The scale factor is adjusted in the least-square sense [13] through the criterion $\min_\alpha \|h^o - \alpha \hat{h}\|^2$ to get \hat{h}_{LS} . To be more precise, in this case the trace of the corresponding constrained CRB is $\text{tr}\{CRB_C\}$ of equation (22).

Another way to adjust the scale factor consists of adjusting α by the following linear constraint $h^{oH} \hat{h}_{LIN} = h^{oH} \alpha \hat{h} = h^{oH} h^o$, leading to the following channel estimate: $\hat{h}_{LIN} = \frac{\hat{h} h^{oH}}{h^{oH} \hat{h}} h^o$. When the estimation of h is consistent, then, asymptotically, the CRB for this constrained channel estimate is the same $CRB_{C,h}$ of (22). In figure 1, we show \hat{h}_{NO} , \hat{h}_{LS} , \hat{h}_{LIN} for a real channel of length $N = 1$ and with 2 subchannels.

B.4 Reducible Channel Case

In this case, $\mathbf{H}(z) = \mathbf{H}_I(z) \mathbf{H}_c(z)$ where $\mathbf{H}_c(z)$ is a monic (first coefficient equal to 1) polynomial in z^{-1} . h can be decomposed as $h = T_I h_c$ where T_I is block Toeplitz with $[h_I^T \ 0_{1 \times (N_c - 1)m}]^T$ as first column. Then, we can prove as previously that the CRB for estimating a reducible h under the constraint $T_I^{oH} h = T_I^{oH} h^o$ is

$$CRB_{C,h} = J_{hh}^+(\theta) = \sigma_v^2 (\mathcal{A}^H P_{T(h)}^\perp \mathcal{A})^+. \quad (23)$$

C. Gaussian Model

In the Gaussian model, the estimation parameter is $\theta = [h^T \ \sigma_v^2]^T$. When the input constellation is complex, the FIM computation is based on the complex probability density function of \mathbf{Y} : $\mathbf{Y} \sim \mathcal{N}(m_Y, C_{YY})$, with $C_{YY} = \sigma_a^2 \mathcal{T}(h) \mathcal{T}^H(h) + \sigma_v^2 I$, $m_Y = 0$. Let $h_R = [\text{Re}(h^T) \ \text{Im}(h^T)]^T$ and $\bar{\theta}_R = [h_R^T \ \sigma_v^2]^T$, the real parameter vector. As $J_{\theta\theta^*}$ is non zero,

we cannot consider the complex CRB anymore: the real FIM $\mathcal{J}_{\bar{\theta}_R \bar{\theta}_R}$ is determined via (5) thanks to the quantities:

$$J_{\theta\theta}(i, j) = \text{tr} \left\{ C_{YY}^{-1} \left(\frac{\partial C_{YY}}{\partial \theta_i^*} \right) C_{YY}^{-1} \left(\frac{\partial C_{YY}}{\partial \theta_j^*} \right)^H \right\} \quad (24)$$

$$J_{\theta\theta^*}(i, j) = \text{tr} \left\{ C_{YY}^{-1} \left(\frac{\partial C_{YY}}{\partial \theta_i^*} \right) C_{YY}^{-1} \left(\frac{\partial C_{YY}}{\partial \theta_j^*} \right) \right\} \quad (25)$$

$$\text{where: } \frac{\partial C_{YY}}{\partial h_i^*} = \sigma_a^2 \mathcal{T}(h) \mathcal{T}^H \left(\frac{\partial h}{\partial h_i^*} \right) \text{ and } \frac{\partial C_{YY}}{\partial \sigma_v^2} = \frac{1}{2} I. \quad (26)$$

When the input constellation is real, the FIM has a similar expression [10].

C.1 FIM singularities

Identifiability here occurs from $C_{YY}(\theta) = \sigma_a^2 \mathcal{T}(h) \mathcal{T}^H(h) + \sigma_v^2 I$ as $m_Y(\theta) \equiv 0$. h can at best be identified up to a phase factor. Blind identifiability in the Gaussian model does not require the channel to be irreducible [5]. The zeros can be estimated, but it cannot be determined if the zeros are minimum phase or not; but if we know a priori that they are minimum phase for example, a reducible channel can be estimated up to a phase factor.

The real/complex channel is locally blindly identifiable if and only if the FIM is regular/1-singular. Note that locally a complex channel is identifiable up to a continuous phase factor but a real channel is locally identifiable strictly speaking.

The Gaussian FIM for a real/complex multichannel is regular/1-singular and the channel is locally blindly identifiable if the channel has no conjugate reciprocal zeros, *i.e.* there exists no $z_o \in \mathbb{R}/\mathbb{C}$ such that $\mathbf{H}(z_o) = \mathbf{H}(1/z_o^*) = 0$ (the burst length should also satisfy certain conditions).

For a complex channel h , under the previous conditions, the global FIM $\mathcal{J}_{\bar{\theta}_R \bar{\theta}_R}$ has one singularity as well as:

$$\mathcal{J}_{h_R h_R}(\bar{\theta}_R) = \mathcal{J}_{h_R h_R} - \mathcal{J}_{h_R \sigma_v^2} (\mathcal{J}_{\sigma_v^2 \sigma_v^2})^{-1} \mathcal{J}_{\sigma_v^2 h_R}. \quad (27)$$

$\mathcal{J}_{h_R h_R}^{-1}(\bar{\theta}_R)$ would be the unconstrained CRB for h if its estimation were regular. The null space of $\mathcal{J}_{h_R h_R}(\bar{\theta}_R)$ is spanned by

$$h_S = [-\text{Im}(h)^T \text{Re}(h)^T]^T = h_{S_2}. \quad (28)$$

The real FIM $\mathcal{J}_{\theta\theta}$ is regular under the local identifiability conditions, as well as $\mathcal{J}_{h_h}(\theta)$.

C.2 Regularized Blind CRBs

When the channel is complex, as in the deterministic case, we need to define a regularized CRB, by introducing some a priori knowledge on the parameters, allowing us to determine the ambiguous phase factor. We assume that the channel is (blindly) locally identifiable.

The estimation of h_R is considered under the constraint:

$$h_{S_2}^{oT} h_R = 0 \quad (29)$$

which leads to the constrained CRB for h_R :

$$CRB_{C, h_R} = \mathcal{J}_{h_R h_R}^+(\theta). \quad (30)$$

This linear constraint does not allow to estimate the phase factor completely and a sign ambiguity is left but not reflected in the FIM singularities as it is a discrete ambiguity. For MSE computation purposes, the sign ambiguity can be resolved by requiring $h_R^{oT} h_R > 0$, which together with (29) can be stated as $h^{oH} h > 0$.

When the channel is real, no regularization is necessary and the CRB is $\mathcal{J}_{hh}^{-1}(\theta)$. To compare the MSE for an estimator to this CRB, the knowledge of the right sign and right phase of the zeros (*e.g.* minimum phase in the reducible case) should be used.

V. CONCLUSION

In blind channel estimation under the deterministic or Gaussian symbol model, the estimation problem has to be augmented with constraints to remove singularities. We have introduced the notion of minimal constraints and shown how several intuitively attractive and hence popular constraint sets lead simply to the pseudo inverse of the FIM as constrained CRB. For the blind channel estimation problem, we have illustrated the connection between local identifiability problems and FIM singularities.

REFERENCES

- [1] K. Abed Meraim, E. Moulines, and P. Loubaton. Prediction Error Method for Second-Order Blind Identification. *IEEE Transactions on Signal Processing*, 45(3):694–705, March 1997.
- [2] G.B. Giannakis and S.D. Halford. Asymptotically Optimal Blind Fractionally-Spaced Channel Estimation and Performance Analysis. *IEEE Transactions on Signal Processing*, 45(7):1815–1830, July 1997.
- [3] S.M. Kay. *Fundamentals of Statistical Signal Processing Estimation Theory*. Prentice-Hall, Englewood Cliffs, NJ, 1993.
- [4] B. Hochwald and A. Nehorai. On Identifiability and Information-Regularity in Parameterized Normal Distributions. *Circuits, Systems and Signal Processing*, 16(1), 1997.
- [5] E. de Carvalho and D.T.M. Slock. Blind and Semi-Blind FIR Multichannel Estimation: Identifiability Conditions. Submitted to *IEEE Transactions on Signal Processing*.
- [6] Peter E. Caines. *Linear Stochastic Systems*. John Wiley & Sons, 1988.
- [7] J.D. Gorman and A.O. Hero. Lower Bounds for Parametric Estimation with Constraints. *IEEE Transactions on Information Theory*, 26(6):1285–1301, Nov. 1990.
- [8] T. Marzetta. On Simple Derivation of the Constrained Multiple Parameter Cramér–Rao Bound. *IEEE Transactions on Signal Processing*, 41(6):2247–2249, June 1993.
- [9] P. Stoica and B.C. Ng. On the Cramér–Rao Bound Under Parametric Constraints. *IEEE Signal Processing Letters*, 5(7):177–179, July 1998.
- [10] E. de Carvalho and D.T.M. Slock. Cramér–Rao Bounds for Semi-blind, Blind and Training Sequence based Channel Estimation. In *Proc. SPAWC 97 Conf.*, Paris, France, April 1997.
- [11] E. de Carvalho and D.T.M. Slock. Asymptotic Performance of ML Methods for Semi-Blind Channel Estimation. In *Proc. Asilomar Conference on Signals, Systems & Computers*, Pacific Grove, CA, Nov. 1997.
- [12] E. de Carvalho and D.T.M. Slock. Cramér–Rao Bounds for Blind Multichannel Estimation. Submitted to *IEEE Transactions on Information Theory*.
- [13] D.R. Morgan, J. Benesty, and M.M. Sondhi. On the Evaluation of Estimated Impulse Responses. *IEEE Signal Processing Letters*, 5(7):174–176, July 1998.