On Achivable Rates in a Multi-Antenna Broadcast Downlink

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Abstract

A Gaussian broadcast channel with r single-antenna receivers where the transmitter is equipped with t antennas and where both the transmitter and the receivers have perfect knowledge of the propagation channel is considered. This provides a very simple model for the downlink of a wireless system, but despite its apparent simplicity it is in general a non-degraded broadcast channel, for which the capacity region is not fully known. We propose a novel transmission scheme based on "ranked known interference". In brief, the transmitter decomposes the channel into an ordered (or ranked) set of interference channels for which the interference signal of the i-th channel is generated as a linear combination of the signals transmitted in channels $j < i$. In this way, known techniques of coding for non-causally known interference can be applied to make the interference in each channel harmless without further power penalty. We show that the proposed scheme is throughputwise asymptotically optimal for both low and high SNR. In the special case of 2-antenna and 2-users we propose a modification of the basic strategy achieving optimal throughput for all SNRs. Finally, the infinite-dimensional Rayleigh channel is considered and throughput closed-form expressions are provided in various cases. This analysis shows that a practical and sensible strategy to the downlink of a wireless system consists of hybrid TDMA and space-time multiplexing where only a subset of active users, whose optimal size depends on the available SNR, is served at any given channel use by using our ranked known interference scheme. TDMA is used for time-sharing between different active user subsets to give to all users the same average rate without penalty in the maximum throughput. Also, constant-power variable-rate coding achieves practically the same throughput of variable-power variable-rate coding (waterfilling power allocation).

Keywords: Gaussian broadcast channel, multiple-antenna systems.

1 Problem statement

We consider a system with one transmitter and r receivers. The transmitter has t antennas, while the receivers have one antenna each. The transmitter has to deliver to each receiver independent information, as in the downlink of a cellular system where the base station is equipped with an antenna array of t elements.[1] This channel is referred to in the following as the $t \times 1 : r$ Gaussian Broadcast Channel (GBC), in order to stress the fact that the r receivers must process their signals separately. If, on the contrary, the receivers are allowed to cooperate, the system boils down to the standard multiple-antenna single-user channel [13]. We shall refer to this case as the "cooperative system".

Let $\mathbf{x} \in \mathbb{C}^t$ denote the vector of modulation symbols transmitted in parallel from the t antennas in a given channel use (symbol interval), let $\mathbf{H} \in \mathbb{C}^{r \times t}$ denote the channel matrix, where $h_{i,j}$ is the complex channel gain from antenna j to antenna i, let $\mathbf{y} \in \mathbb{C}^r$ denote the vector of received signal samples at the r receivers and let $\mathbf{z} \in \mathbb{C}^r$ be the corresponding vector

[1] More in general, the t antennas might represent the collection of all base stations, each equipped with an antenna array. However, in this work we do not consider the effect of the spatial distribution of the transmit and receive antennas as for example in the cellular model of [11].For further results along this line see [12].

of noise samples, assumed to be circularly-symmetric complex Gaussian with i.i.d. components such that $z_i \sim \mathcal{N}_{\mathbb{C}}(0, N_0)$. Then, the $t \times 1 : r$ GBC is described by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \tag{1}$$

The channel matrix $\mathbf{H}$ is known to the transmitter and to all receivers and the input is constrained to satisfy trace$(\mathbf{\Sigma}_x) \leq \mathcal{E}$ where $\mathbf{\Sigma}_x \stackrel{\Delta}{=} E[\mathbf{x}\mathbf{x}^H]$ and $\mathcal{E}$ is the maximum allowed total transmit energy per channel use.

The $1 \times 1 : r$ GBC coincides with the classical *degraded* Gaussian broadcast channel, whose capacity region is well-known [4]. However, the $t \times 1 : r$ GBC for $t > 1$ is in general a non-degraded broadcast channel and cannot be cast in the framework of parallel Gaussian broadcast channels [5] since the receivers cannot cooperate. In this paper we are mainly concerned with the channel throughput R (or rate-sum), defined as the sum of all individual user rates for which the corresponding rate r-tuple is achievable. We shall consider the following scenarios: i) $\mathbf{H}$ is deterministic and fixed; ii) $\mathbf{H}$ is fixed during the transmission of each code word, but it is randomly selected according to a given probability distribution (composite channel).

In section 2 we outline the newly proposed transmission scheme. In section 3 we show that this scheme outperforms conventional zero-forcing (ZF) beamforming (see [1] and references therein), it is throughputwise asymptotically optimal for both low and high SNR, and in the 2-antennas 2-users case we demonstrate that a modified strategy is in fact optimal for all SNR. Finally, in section 4 we provide results for an independent Rayleigh channel in the large-system limit, i.e., when both r and t go to infinity with a given constant ratio α. The proofs of the results of this paper can be found in [2].

2 A new approach based on "ranked known interference"

In this section we present a new transmission strategy that combines linear signal processing at the transmitter with non-standard coding. The linear processing turns the original channel into an ordered set of interference channels, where the channels are given in a specific order and the interference in channel i is generated by the signals transmitted in channels $j < i$. Since all signals are generated at the same transmitting end, for each of these channels the interfering signal is known non-causally. Therefore, we can apply the results on capacity with interference known non-causally at the transmitter. Capacity and coding for Gaussian channels with Gaussian interference non-causally known at the transmitter is solved in [3], exploiting the result of [8]. More recently, a general coding technique based on *lattice precoding* that applies also to non-Gaussian interference and yields the same (optimal) result of [3] was presented in [6]. An uncoded version of this approach, where precoding is implemented as a Tomlinson-Harashima precoder [7] (a simple one-dimensional example of lattice precoding), has been independently proposed in [9].

Let $\mathbf{H} = \mathbf{G}\mathbf{Q}$ be a QR-type decomposition of $\mathbf{H}$ where $\mathbf{G} \in \mathbb{C}^{r \times m}$ is lower triangular and $\mathbf{Q} \in \mathbb{C}^{m \times t}$ has orthonormal rows (we define $m \stackrel{\Delta}{=} \min\{r, t\}$). The transmitted signal is obtained as $\mathbf{x} = \mathbf{Q}^H \mathbf{u}$. Let $g_{i,j}$ denote the (i, j)-th element of $\mathbf{G}$ and let $d_i = |g_{i,i}|^2$. The original channel is turned into the set of interference channels

$$y_i = g_{i,i} u_i + \sum_{j<i} g_{i,j} u_j + z_i, \qquad i = 1, \ldots, m \tag{2}$$

while no information is sent to users $m+1, \ldots, r$. Since $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, the covariance of $\mathbf{u}$ has the same trace constraint of $\mathbf{\Sigma}_x$. The signals u_i are all generated by the transmitter. Hence, they are all known non-causally by the encoder. As explained in [2], there exist coding schemes such that each user $i = 1, \ldots, m$ "sees" an interference-free channel, as if its signal $\mathbf{u}_i$ was alone. We refer to this scheme as *Ranked Known Interference* (RKI).

Proposition 1. The achievable throughput of the RKI scheme is given by

$$\left\{ \begin{array}{l} R = \sum_{i=1}^{m} \log(1 + d_i a_i) \\ \sum_{i=1}^{m} a_i = A \end{array} \right. \tag{3}$$

where $a_i \triangleq E[|u_i|^2]/N_0$ denote the SNR on the i-th channel and $A \triangleq \mathcal{E}/N_0$ denote the total *transmit* SNR. □

The RKI throughput can be optimized jointly with respect to the power allocation (the $a_i's$) and the user ordering. In fact, for an arbitrary permutation matrix $\mathbf{\Pi}$, the set of elements $\{d'_i : i = 1, \ldots, m\}$ resulting from the QR decomposition $\mathbf{\Pi H} = \mathbf{G}'\mathbf{Q}'$ is generally different from the set $\{d_i : i = 1, \ldots, m\}$ resulting from $\mathbf{H} = \mathbf{GQ}$.

3 Results for the deterministic channel

In this section we consider $\mathbf{H}$ given and constant. It is clear that the cooperative throughput $R^{\rm coop}$ (maximized w.r.t. the power allocation) is an upperbound to the throughput $R^{\rm gbc}$ of the $t \times 1 : r$ GBC, while the throughput of ZF and RKI, denoted by $R^{\rm zf}$ and by $R^{\rm rki}$, respectively, provide lower bounds. The RKI scheme yields generally a larger maximal throughput than conventional ZF beamforming:

Proposition 2. For any channel matrix $\mathbf{H}$, $R^{\rm rki-max} \geq R^{\rm zf-max}$. □

The following proposition makes this statement stronger in the limits for high and low SNR:

Proposition 3. For any channel matrix $\mathbf{H}$ with full row-rank r,

$$\lim_{A \to \infty} \left(R^{\rm coop-max} - R^{\rm rki-max} \right) = 0 \tag{4}$$

For any channel matrix $\mathbf{H}$,

$$\lim_{A \to 0} \frac{R^{\rm rki-max}}{R^{\rm zf-max}} = 1 \tag{5}$$

□

As a corollary of Proposition 2, we get that (for $\mathbf{H}$ of rank r) the RKI is asymptotically throughputwise optimal for large SNR, since the inequality $R^{\rm rki-max} \leq R^{\rm gbc} \leq R^{\rm coop-max}$ and the first limit of Proposition 2 imply that $\lim_{A \to \infty}(R^{\rm gbc} - R^{\rm rki-max}) = 0$.

Next, we find two upperbounds to $R^{\rm gbc}$ tighter than the trivial maximum cooperative throughput bound. The capacity region of a general broadcast channel depends only on the transition marginal probability assignments $p(y_i|\mathbf{x})$ and not on the whole joint transition probability $p(y_1, \ldots, y_r|\mathbf{x})$ [10, 4]. For the $t \times 1 : r$ GBC, this implies that the channels in the family defined by

$$\mathbf{y} = \mathbf{\Pi H x} + \mathbf{z} \tag{6}$$

where $\mathbf{H}$ is given, $\mathbf{\Pi}$ is any $r \times r$ permutation matrix, trace$(\mathbf{\Sigma}_x) \leq A$ and $\mathbf{z} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{\Sigma}_z)$ where $\mathbf{\Sigma}_z$ is any non-negative definite Hermitian matrix with diagonal elements equal to 1 (we refer to this constraint as the *unit-diagonal constraint*), have all the same broadcast capacity region (and therefore the same $R^{\rm gbc}$). Hence, a tighter cooperative throughput upperbound is obtained by choosing the worst-case (cooperative throughputwise) channel in the family defined by (6). We have:

Lemma 1. For any channel matrix $\mathbf{H}$,

$$R^{\rm gbc} \leq \max_{\mathbf{\Sigma}_x} \min_{\mathbf{\Pi}, \mathbf{\Sigma}_z} \left\{ \log\det\left(\mathbf{I} + \mathbf{\Sigma}_z^{-1} \mathbf{\Pi H \Sigma}_x \mathbf{H}^H \mathbf{\Pi}^H \right) \right\} \tag{7}$$

where minimization is over all $r \times r$ permutation matrices $\mathbf{\Pi}$ and over all noise covariances $\mathbf{\Sigma}_z$ with unit diagonal elements and maximization is over all input covariances with trace not larger than A. □

Assume that $\mathbf{H}$ has rank $k \leq m$ and, after a suitable row permutation, can be partitioned as $\mathbf{H} = [\mathbf{H}_1^T, \mathbf{H}_2^T]^T$ where $\mathbf{H}_1 \in \mathbb{C}^{k \times t}$ has rank k and $\mathbf{H}_2 \in \mathbb{C}^{(r-k)\times t}$. Then, any row of $\mathbf{H}_2$ can be expressed as a linear combination of rows of $\mathbf{H}_1$, i.e., we can write $\mathbf{H}_2 = \mathbf{B}\mathbf{H}_1$ where $\mathbf{B} = \mathbf{H}_2\mathbf{H}_1^+$. Moreover, partition the noise vector as $\mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T]^T$, and let $\mathbf{\Sigma}_{z1}$ and $\mathbf{\Sigma}_{z2}$ denote the $k \times k$ upper left and $(r-k)\times(r-k)$ lower right diagonal blocks of $\mathbf{\Sigma}_z$ (notice that both $\mathbf{\Sigma}_{z1}$ and $\mathbf{\Sigma}_{z2}$ must satisfy the unit diagonal constraint). Then, we have the following:

Lemma 2. With the above notation, if $\mathbf{\Sigma}_{z2} - \mathbf{B}\mathbf{\Sigma}_{z1}\mathbf{B}^H$ is non-negative definite, then

$$R^{\rm gbc} \leq I(\mathbf{x}; \mathbf{y}_1) \tag{8}$$

where $\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \mathbf{z}_1$. □

The above lemma can be applied to the simple choice $\mathbf{\Sigma}_{z1} = \mathbf{I}_k$ and $\mathbf{\Sigma}_{z2} = \mathbf{I}_{r-k}$. If $\|\mathbf{B}\|_2 = \sqrt{\rho(\mathbf{B}\mathbf{B}^H)} \leq 1$, then $R^{\rm gbc}$ is not larger than $R^{\rm coop-max}$ of the $t \times k$ channel defined by the matrix $\mathbf{H}_1$ (with i.i.d. Gaussian noise). From Proposition 3 we get that this throughput is asymptotically achieved for large SNR by the RKI scheme (in fact, $\mathbf{H}_1$ has full row-rank k). Then, as a consequence of Proposition 3 and Lemma 2, we have the following:

Corollary 1. Let $\mathbf{H}$ have rank $k \leq r$ and, after a suitable row permutation, assume that

$$\mathbf{H} = \left[\begin{array}{c} \mathbf{I}_k \\ \mathbf{B} \end{array}\right] \mathbf{H}_1 \tag{9}$$

where $\mathbf{H}_1 \in \mathbb{C}^{k\times t}$ has rank k and $\|\mathbf{B}\|_2 \leq 1$. Then, $\lim_{A\to\infty}(R^{\rm gbc} - R^{\rm rki-max}) = 0$. □

The following linear-algebra lemma (new to the autors' knowledge) shows that $\mathbf{H}_1$ is essentially unique up to row permutations:

Lemma 3. Let $\mathbf{H} \in \mathbb{C}^{r\times t}$ and assume that it can be decomposed as in Corollary 1. Let $\mathbf{H}_1' \in \mathbb{C}^{k\times t}$ and $\mathbf{H}_2' \in \mathbb{C}^{(r-k)\times t}$ be obtained by exchanging some rows of $\mathbf{H}_1$ with some rows of $\mathbf{H}_2 = \mathbf{B}\mathbf{H}_1$, such that that $\mathbf{H}_1'$ has rank k. Then, $\mathbf{B}' = \mathbf{H}_2'(\mathbf{H}_1')^+$ has 2-norm ≥ 1. □

Unfortunately, the partition of Corollary 1 is not always possible (it is immediate to find a counterexample for $r \geq 3$). Then, we cannot exclude the possibility that for such type of rank-deficient channel matrices $\mathbf{H}$ the RKI scheme is asymptotically suboptimal for large SNR.

3.1 The 2-antennas 2-users case

To fix the ideas and demonstrate results and techniques we consider in the details the example of $t = r = 2$. We have the following:

Proposition 4. For any 2×2 channel matrix $\mathbf{H}$,

$$\lim_{A\to 0} \frac{R^{\rm gbc}}{R^{\rm rki-max}} = 1 \tag{10}$$

□

The above proposition is proved by applying Lemma 1 where $\mathbf{\Pi}$ is the permutation that puts in first position the row of $\mathbf{H}$ with the largest 2-norm and with the choice

$$\mathbf{\Sigma}_z = \left[\begin{array}{cc} 1 & \frac{g_{2,1}^*}{g_{1,1}^*} \\ \frac{g_{2,1}}{g_{1,1}} & 1 \end{array}\right]$$

Since by construction $|g_{1,1}|^2 \geq |g_{2,2}|^2 + |g_{2,1}|^2$, this is a valid covariance matrix. In the case where the above inequality is strict the statement of Proposition 4 is actually stronger, since for $A \in [0, A_1]$, where A_1 is given by

$$A_1 = \frac{1}{|g_{1,1}|^2}\left(\frac{|g_{1,1}|^2 - |g_{2,1}|^2}{|g_{2,2}|^2} - 1\right) \tag{11}$$

the RKI scheme is (not only asymptotically) optimal.

As a corollary of Proposition 4 and Proposition 3, we have that the RKI and the ZF schemes are throughput-wise asymptotically optimal for the $2 \times 1 : 2$ GBC and low SNR. Moreover, since if $\mathbf{H}$ has rank 1 the partition (9) of Corollary 1 is always possible, the RKI scheme is asymptotically optimal for both high and low SNR for any $\mathbf{H}$. Because of these nice properties, we might be tempted to conjecture that the RKI scheme is actually optimal for all SNRs. The following modified strategy shows that this is actually not the case.

An optimal modified RKI strategy for the $2 \times 1 : 2$ GBC. We let $\mathbf{H} = \mathbf{G}\mathbf{Q}$ and construct the transmitted signal as $\mathbf{x} = \mathbf{Q}^H \mathbf{R}\mathbf{u}$, where $\mathbf{R}$ is upper triangular. The resulting two channels are

$$\begin{aligned} y_1 &= g_{1,1} r_{1,1} u_1 + g_{1,1} r_{1,2} u_2 + z_1 \\ y_2 &= (g_{2,1} r_{1,2} + g_{2,2} r_{2,2}) u_2 + g_{2,1} r_{1,1} u_1 + z_2 \end{aligned} \tag{12}$$

The signals u_1 and u_2 carry information for user 1 and 2, respectively, and the signal u_2 is constructed according the "coding for known interference" method (see [2]), by treating $g_{2,1} r_{1,1} u_1$ as known interference. All signals are Gaussian, and u_1, u_2 are asymptotically uncorrelated for large block length. Receiver 1 treats the signal $g_{1,1} r_{1,2} u_2$ as background noise, as this is *unknown* Gaussian interference (worst-case noise for Gaussian input). Receiver 2 is able to reliably decode the same rate as if the interference signal $g_{2,1} r_{1,1} u_1$ was not present. Therefore, the resulting throughput is given by

$$R^{\rm mod} = \log\left(1 + \frac{|g_{1,1} r_{1,1}|^2 a_1}{1 + |g_{1,1} r_{1,2}|^2 a_2}\right) + \log\left(1 + |g_{2,1} r_{1,2} + g_{2,2} r_{2,2}|^2 a_2\right) \tag{13}$$

(where "mod" stands for *modified RKI*) subject to the constraint

$$|r_{1,1}|^2 a_1 + (|r_{1,2}|^2 + |r_{2,2}|^2) a_2 = A$$

The above throughput can be maximized with respect to a_1, a_2 and the coefficients $r_{1,1}, r_{1,2}, r_{2,2}$. We reparameterize the problem by letting $b = |g_{2,1}/g_{2,2}|$, $z = |r_{2,2}/r_{1,2}|$, $p(z) = (b+z)^2/(1+z^2)$, $q(z) = 1/(1+z^2)$, $X_1 = |r_{1,1}|^2 a_1$ and $X_2 = (|r_{1,2}|^2 + |r_{2,2}|^2) a_2$. After some algebra, we can write

$$R^{\rm mod} = \log\left(1 + |g_{2,2}|^2 p(z) X_2 + \frac{1 + |g_{2,2}|^2 p(z) X_2}{1 + |g_{1,1}|^2 q(z) X_2} |g_{1,1}|^2 X_1\right) \tag{14}$$

with the constraint $X_1 + X_2 = A$. Obviously, since $\lim_{z \to \infty} p(z) = 1$ and $\lim_{z \to \infty} q(z) = 0$, the modified RKI strategy coincides with standard RKI for $z \to \infty$. Therefore, this is in fact a *generalization* of RKI and cannot yield worse results after maximizing with respect to the power allocation X_2, X_1 and the free parameter $z \in \mathbb{R}_+$.

By substituting $X_1 = A - X_2$ in (14), $R^{\rm mod}$ can be maximized with respect to X_2 over the interval $[0, A]$ for any value of z. The equation $\frac{\partial}{\partial X_2} R^{\rm mod} = 0$ has either none or one solution in the interval $[0, A]$. It can be shown that if there is no solution, the derivative is negative and the maximum is achieved by $X_2 = 0$ while if there is a solution it corresponds to a maximum. The case $X_2 = 0$ yields $R^{\rm mod} = \log(1 + |g_{1,1}|^2 A)$, which coincides with the maximum throughput of RKI and ZF in the range $A \in [0, A_1]$ where A_1 is given by (11). For $A > A_1$, there exists a range of $z \in \mathbb{R}_+$ for which the maximum is achieved by $0 < X_2 < A$. In this range of SNR, the modified RKI strategy is distinctly better than standard RKI. Unfortunately, the maximization with respect to z must be performed numerically.

With some more effort, we are able to prove that the modified RKI strategy is actually throughputwise optimal for all SNR. Remarkably, this is one of the rare examples of a non-degraded Gaussian broadcast channel for which the optimal throughput is known. The direct part of this statement is represented by the modified RKI strategy outlined above. The converse part is provided by the upperbound of Lemma 1. First, we choose the permutation $\mathbf{\Pi}$ such that $\mathbf{\Pi}\mathbf{H} = \mathbf{G}\mathbf{Q}$ with $|g_{1,1}|^2 \geq |g_{2,2}|^2 + |g_{2,1}|^2$, as in the proof of Proposition 4. Then, we notice that any signal covariance matrix can be parameterized as

$$\mathbf{\Sigma}_x = \mathbf{Q}^H \mathbf{R} \mathbf{\Lambda}_x \mathbf{R}^H \mathbf{Q}$$

where $\mathbf{\Lambda}_x = \text{diag}(P_1, P_2)$ and where $\mathbf{R}$ is upper triangular, while any noise covariance matrix satisfying the unit diagonal constraint can be written as

$$\mathbf{\Sigma}_z = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix}$$

where $|\rho|^2 \leq 1$. After some algebra, the upper bound of Lemma 1 can be written as

$$\begin{aligned} R^{\text{gbc}} \leq & \max_{z \geq 0, X_1 + X_2 = A} \min_{|\rho| \leq 1} \log \Big[1 + \frac{1}{1 - |\rho|^2} \big(|g_{2,2}|^2 p(z) X_2 + \\ & + (|g_{1,1}|^2 - 2|g_{2,1}||g_{1,1}||\rho| + |g_{2,1}|^2)(X_1 + q(z)X_2) + \\ & + X_2 q(z)(z^2 |g_{1,1}|^2 |g_{2,2}|^2 X_1 - 2|g_{1,1}||g_{2,2}||\rho| z - |g_{2,1}|^2) \big) \Big] \end{aligned} \quad (15)$$

where $p(z)$ and $q(z)$ have been defined previously. The minimization with respect to $|\rho| \in [0,1]$ yields $|\rho| = \frac{1}{2}\left(B - \sqrt{B^2 - 4}\right)$ where

$$B = \frac{(|g_{2,1}|^2 + |g_{1,1}|^2)X_1 + (|g_{2,2}|^2 p(z) + |g_{1,1}|^2 q(z))X_2 + q(z)z^2 |g_{1,1}|^2 |g_{2,2}|^2 X_1 X_2}{|g_{2,1}||g_{1,1}|(X_1 + q(z)X_2) + q(z)z|g_{1,1}||g_{2,2}|X_2}$$

is always ≥ 2 for all $z \in \mathbb{R}_+$ and $g_{1,1}, g_{2,1}, g_{2,2}$ such that $|g_{1,1}|^2 \geq |g_{2,2}|^2 + |g_{2,1}|^2$. Notice that for $X_2 = 0$ we get $|\rho| = |g_{2,1}/g_{1,1}|$, i.e., in the range $A \in [0, A_1]$ the bound coincides (obviously!) with that of Proposition 4. After substituting the above solution for $|\rho|$ and $X_1 = A - X_2$ into (15), the maximization with respect to X_2 and z must be carried out numerically. Interestingly, the maximum is achieved for the same pair (z, X_2) that maximizes R^{mod}, even though the upper bound does not coincides with R^{mod} for all (z, X_2).

Example 1. Fig. 1 shows $R^{\text{rki}-\max}$, $R^{\text{mod}-\max}$, $R^{\text{coop}-\max}$, the upperbound to R^{gbc} given by Proposition 4 and the upperbound obtained by the above application of Lemma 1 (denoted by RKI, MOD, COOP, UB1 and UB2, respectively) for a channel with $\mathbf{G}$ factor

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.2 \end{bmatrix}$$

$R^{\text{mod}-\max}$ and the upperbound UB2 coincide exactly for all SNR (thicker line). Notice that $R^{\text{rki}-\max}$ is optimal in the range $A \in [0, A_1]$ and strictly suboptimal for larger A. ◊

4 Results for the infinite-dimensional Rayleigh channel

In this section we focus on the composite channel when the channel matrix has i.i.d. entries $\sim \mathcal{N}_{\mathbb{C}}(0,1)$, and we consider the throughput achievable by the basic RKI strategy under various power constraints, assuming that no effort is made to optimize the user ordering. For finite t and r, closed-form expressions for the throughput are obtained in [2]. Here we focus on the large-system limits, i.e., we let $r \to \infty$ with $r/t = \alpha$, where $\alpha \geq 0$ is fixed and represents the channel *spatial* load expressed in "users per antenna".

In the composite channel setting, the channel is symmetric with respect to any user. Therefore, if a given ergodic throughput R is achievable for a given set of active users, by time-sharing with uniform probability over all possible user subsets (and orderings) every user can achieve the same per-user average rate R/r. As an effect of this "ergodic symmetrization", requiring equal average per-user rates involves no loss of optimality in the overall throughput. Moreover, as $r \to \infty$ the instantaneous throughput converges almost surely to the ergodic (or average) throughput [15, 14]. We let $\rho \stackrel{\Delta}{=} \frac{1}{r}R$ denote the normalized throughput, equal to the symmetric per-user rate achievable by time-sharing argument outlined above. It can be shown (see [2] for the details) that in the limit of large system the RKI normalized throughput is given by

$$\left\{ \begin{array}{l} \rho^{\rm rki} = \int_0^\mu \log\left(1 + [1-\alpha\nu]_+ a(\nu)\right)\, d\nu \\ \int_0^\mu a(\nu)\, d\nu = A/\alpha \end{array} \right. \tag{16}$$

where $a(\nu)$ is the transmit SNR of the $\lfloor \nu r \rfloor$-th signal and $\mu \stackrel{\Delta}{=} m/r = \min\{1, 1/\alpha\}$.

Uniform powers. Let $\kappa = k/r$ denote the fraction of active users. In general, $\kappa \in [0, \mu]$ since from Lemma 5 no more than $\lfloor \mu r \rfloor$ users can be served in parallel. For a given κ, we consider the uniform power allocation $a(\nu) = a$ for $\nu \leq \kappa$ and $a(\nu) = 0$ for $\nu > \kappa$. We obtain

$$\begin{aligned} \rho^{\rm rki-eq} &= \frac{1}{A}\left[(\kappa + A/\alpha)\log(\kappa + A/\alpha) - \right. \\ &\quad \left. - (\kappa + (1-\alpha\kappa)A/\alpha)\log(\kappa + (1-\alpha\kappa)A/\alpha)\right] - \kappa(\log(\kappa) - 1) \end{aligned} \tag{17}$$

This can be further maximized with respect to $\kappa \in [0, \mu]$.

Equal rates. Again, let $\kappa < \mu$ be the fraction of active users that must be served with equal rate $R_u = \log(1 + a_0)$. The resulting normalized throughput is given by $\rho^{\rm rki-cr} = \kappa R_u$ and where a_0 is the solution of

$$\int_0^\kappa \frac{a_0}{1 - \alpha\nu} d\nu = A/\alpha \tag{18}$$

yielding $a_0 = -A/\log(1 - \alpha\kappa)$. The resulting normalized throughput is given by

$$\rho^{\rm rki-cr} = \kappa \log\left(1 - \frac{A}{\log(1 - \alpha\kappa)}\right)$$

and can be further maximized with respect to $\kappa \in [0, \mu)$.

Maximum throughput. In this case, we want to maximize $\rho^{\rm rki}$ subject to the input constraint. The standard waterfilling solution yields

$$\rho^{\rm rki-max} = \left\{ \begin{array}{ll} \log\left[\frac{1}{\alpha}(A - \log(1-\alpha))\right] - (1/\alpha - 1)\log(1-\alpha) - 1 & \text{(case 1)} \\ \frac{1}{\alpha}\left(\log\xi + \frac{1}{\xi} - 1\right) & \text{(case 2)} \end{array} \right. \tag{19}$$

where "case 1" corresponds to the condition $\{\alpha < 1, A \geq \frac{\alpha}{1-\alpha} + \log(1-\alpha)\}$ and "case 2" to the complement condition and where ξ in case 2 is the solution of $\xi - \log\xi - 1 = A$.

ZF and cooperative throughput. For the sake of comparison, we calculate also the normalized throughput with ZF beamforming and with cooperative receivers in the large-system regime. We omit the details for the sake of space limitation (see [2] and references therein).

Results. Fig.2 shows the normalized throughput (or per-user rate) for all the cases discussed above versus the transmit SNR A, for $\alpha = 1.0$. In all cases, the throughput obtained by uniform power allocation with optimization of the fraction of active users (denoted by "EQ") is almost identical to that obtained by the optimal waterfilling power allocation (denoted by "MAX"). This is in accordance with the well-known fact of standard ISI channels, for which optimizing the transmission bandwidth and with a rectangular input power spectral density buys almost all the capacity achievable by waterfilling. RKI with constant per-user rate (denoted by "CR") performs slightly worse than with uniform (or waterfilling) power allocation. Finally, ZF performs quite worse than RKI (in all cases) even after optimizing the fraction of active users. Fig. 3 shows the optimal fraction of active users κ versus A for RKI-EQ, RKI-CR and ZF. The fraction of active users is an increasing function of the input SNR.

From these examples, we argue that a practical and sensible downlink system design consists of applying the RKI scheme by transmitting with constant-power variable-rate user codes and by selecting carefully the number of active users k according to the available transmit average power. Even if $r > k$ users are to be served, the system should allow only k active per channel use (say, in each slot), where k is optimally selected, and serve all r users equally by time-sharing. Therefore, the system works according to a hybrid TDMA (time-sharing) and "space-time multiplexing" given by the RKI scheme.

5 Conclusions

We proposed a new coding scheme based on *Ranked Known Interference* for the Gaussian broadcast channel with multiple antennas at the transmitter. The basic RKI scheme is asymptotically throughputwise optimal for both low and high SNR, and can be approximated in practice by some form of lattice precoding. We also exhibited a modified RKI scheme which is indeed optimal for all SNRs in the 2-users 2-antennas case. The modified RKI scheme can be applied for any t and r and might provide a handle to the study of individual achievable rates for the general $t \times 1 : r$ GBC, though there is little hope that this mechanism can determine the full capacity region.

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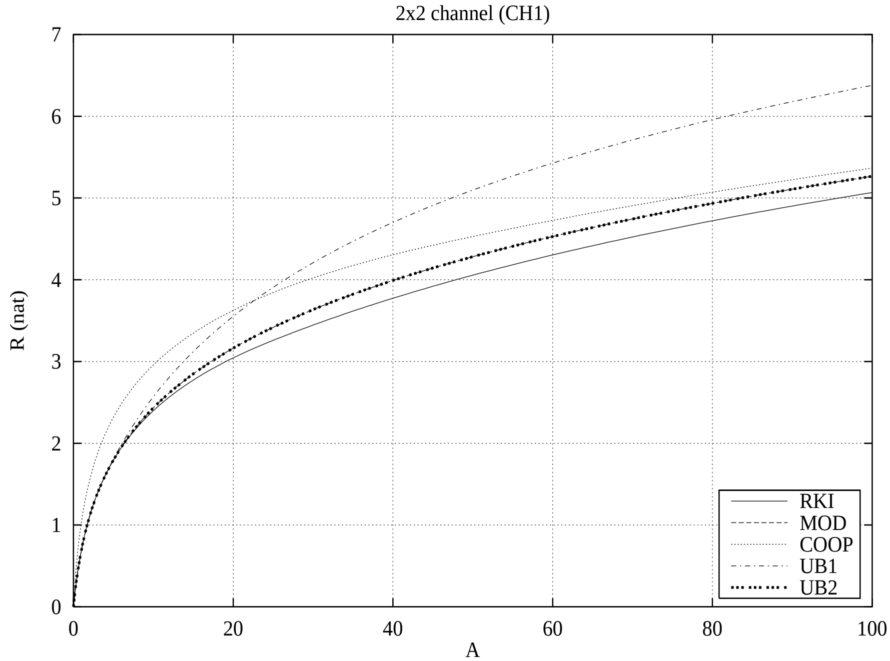

Figure 1: Channel matrix CH1: throughput (in nat) vs. A (not in dB) for RKI, cooperative (COOP), for the modified RKI scheme (MOD) and for the upperbound of Proposition 4 (UB1) and the upperbound of Lemma 1 (UB2).

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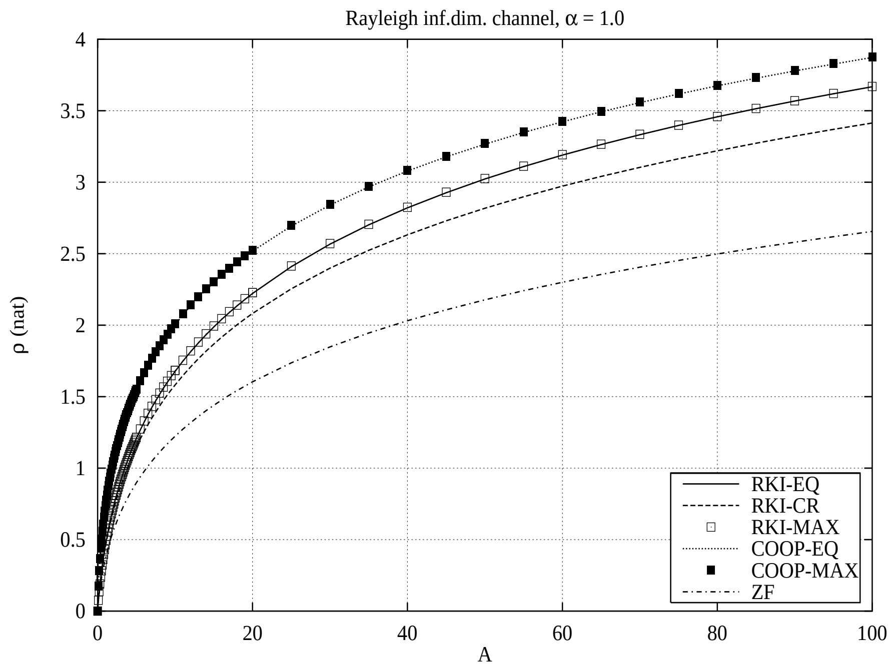

Figure 2: Normalized throughput versus transmit SNR for the Rayleigh infinite-dimensional channel with $\alpha = 1.0$.

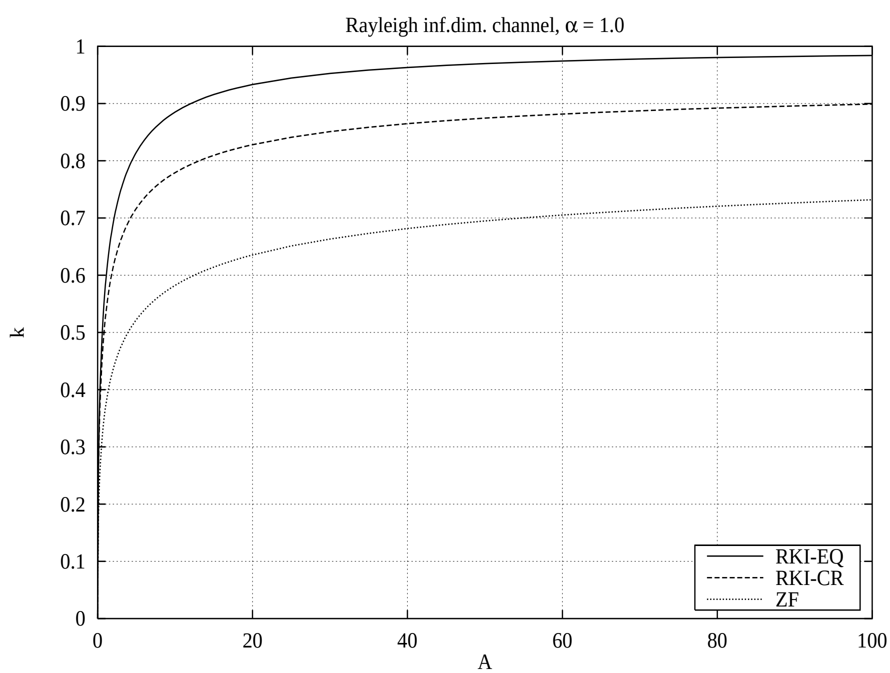

Figure 3: Optimal fraction of active users versus transmit SNR for the Rayleigh infinite-dimensional channel with $\alpha = 1.0$.