# The Optimal Throughput of some Wireless Multiaccess Systems

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November 7, 2000

**Abstract**

The throughput performance of some multiaccess wireless systems is compared on the basis of the average energy per successfully received information bit $E_b/N_0$. We assume that an infinite uncoordinated population of users access at random a block fading channel and that channel state information is available at the decoder only. To cope with multiaccess interference and fading, the users retransmit erroneously received packets that are then combined at the receiver to improve decoding. We prove that, for an optimal choice of the system parameters, some systems are not interference limited and have a similar behavior: at high $E_b/N_0$ they "self-orthogonalize", i.e., on the average only one user per degree of freedom is active, while at low $E_b/N_0$ they have the same throughput of conventional CDMA, achieved by infinite users per degree of freedom transmitting with vanishing rate.

## 1 Introduction

We present an information-theoretic comparison of some multiaccess wireless systems considered in a number of previous works ([3, 5, 4] and references therein). These systems are intrinsically quite different: some assume an infinite number of users with vanishing coding rate but with non-zero probability of accessing the channel while others assume "bursty" users with instantaneously non-vanishing coding rate but with vanishing probability of accessing the channel, some perform single user decoding while other do multi user decoding. Here, we do not question the validity of these (quite idealized) models, on the contrary, the related basic results are our starting point for comparison. In order to provide a *fair* performance measure, we compare the systems in terms of their maximum throughput (aggregate average bit/s/Hz) versus $E_b/N_0$.

We start with the system considered in [3], where infinite users access the channel at random and run the same decentralized Automatic retransmission ReQuest protocol in conjunction with packet combing (Hybrid-ARQ) as mean to combat MultiAccess Interference (MAI) and fading. After combining, packets are fed to a bank of Single User Decoders (SUD). Then, we turn to Direct Sequence Code Division MultipleAccess (DS-CDMA) with random spreading, linear detection, i.e., Single User Matched Filter (SUMF) and linear Minimum Mean Square Error filter (MMSE), and SUD [1, 8, 5]. The same Hybrid-ARQ schemes are applied in this setting to handle random user activity and fading. Finally, we consider the system analyzed in [4], where an infinite number of users access continuously the channel and a Joint Multi User Decoder (JMUD), or a Successive Interference Canceler (stripping) with SUD at each decoding step (SIC-SUD), decodes the largest possible subset of users on a slot-by-slot basis.

For all these systems, we are concerned with the throughput optimization with respect to the system parameters. The system throughput as a function of $E_b/N_0$ is given in parametric form and its optimization is often involved. We present analytic closed-form results and very rarely we shall resort to numerical calculation (never to computer simulation). As a byproduct of this analysis, we obtain insight on the optimal choice of system parameters in order to obtain maximum throughput.

We show that the unspread Hybrid-ARQ system outperforms SUMF DS-CDMA, which is throughputwise limited, but it is outperformed by MMSE DS-CDMA. All the SUD-based system have the same behavior in terms of throughput and of optimal system parameters. In the low $E_b/N_0$ regime, the optimized throughput is the same for all the systems and coincides with that of a SUMF DS-CDMA, achieved by an infinite number of users per degree of freedom transmitting at vanishing rate. In the high $E_b/N_0$ regime, while SUMF DS-CDMA is interference limited, the other systems are not. For this range of $E_b/N_0$ the optimized systems "self-orthogonalize", in the sense that optimal throughput is achieved by having on the average only one user per degree of freedom, i.e., one user per chip for the DS-CDMA and one active user per slot for the unspread system.

The paper is organized as follow: the system model is described in Section 2 and the summary of the results is presented in Section 3. In Section 4 the system performances are compared and in Section 5 we present our conclusions. Analytical details can be found in the Appendices.

## 2 System models

In this section we briefly review the wireless multiaccess general model for the three systems under analysis.

**Slotted channel.** We assume a complex channel of bandwidth $W$ Hz whose time axis is divided in slots of duration $T$ seconds. Every slot can accommodate packets of $L \approx WT$ independent complex dimensions, in the limit of $WT \gg 1$. [1] The channel is impaired by an additive white Gaussian background noise and by frequency-flat fading, which is assumed constant for the whole slot duration and independent for each slot and each user. The channel is accessed randomly and independently, with probability $p_t$ on every slot, by a population of $N_u$ users.

**Transmitter.** Each transmitter has an infinite sequence of information packets to encode, spread and transmit. Let $S$ be the length of the spreading sequence and $M$ be a given integer. User $k$ encodes its packet of $b$ data bits into a code word of length $LM/S$ complex symbols and spreads it, so that each data packet corresponds to a channel packet of $LM$ modulation symbols (chips) to send over the channel. Before actual transmission the channel packet is split into $M$ "chunks" to fit the slot duration. Note that in the sequel, depending on the context, we shall use the terms "complex symbol", "degree of freedom", "dimension" or "chip" to indicate a basic channel use, i.e., one second per Hz. The quantity $R \triangleq bS/L$ is referred to as the *elementary rate* and represents the number of data bit per coded symbol before spreading. The quantity $G \triangleq N_u\, p_t/S$ is referred to as *channel load* and represents the average number of active users per dimensions.

**Transmission protocol.** Each time a transmitter is active, it sends on the current slot the not-yet transmitted chunk of $L$ chips of the current channel packet. At the receiver, the sequence of slots where a user was active are collected, combined and used for decoding. Every time a new slot is received, decoding of all active users is attempted. Then, a positive ACKnowledgement (ACK) is sent to all users for which decoding has been successful, while a Negative ACKnowledgement (NACK) is sent to all users for which decoding has not been successful. The ACK/NACK feedback link is assumed to be error and delay free. When a user gets an ACK, it stops transmitting the current channel packet and the next time it is active it starts transmitting the first chunk of the next channel packet. On the contrary, when a user gets a NACK, the next time it is active it

[1] The assumption of $L \gg 1$ is not so unrealistic. In the 3G wireless system proposal [6], the chip rate is $W = 3.84$ Mchip/s and the slot duration is $T = 10/15$ ms, hence $WT = 2560$.

transmits the next chunk of the current channel packet. If successful decoding is not obtained after $M$ transmitted chunks, or after $N$ slots since the generation of the data packet, the packet is lost. $M$ and $N$ are referred to as the *rate constraint* and *delay constraint*, respectively. Three Hybrid-ARQ schemes are taken into account (see [3] and references therein): an ALOha-type scheme (ALO), where channel packets are made of the same basic code words of length $L$ repeated $M$ times, and where previously received chunks are discarded; a Repetition Time-Diversity scheme (RTD), where channel packets are also made of the same basic code words of length $L$ repeated $M$ times, but where previously received chunks are combined by *maximal ratio combining* before decoding; an INcremental Redundancy scheme (INR), where channel packets are effectively made of $M$ different segments of $L$ symbol each, and previously received chunks are all taken into account at each decoding attempt.

**Receiver.** The receiver for each user is formed by a chip matched filter (the chip pulse shape is assumed to be a Nyquist pulse), a sampler at chip rate, a linear filter, a packet combiner and a single user decoder. By stacking in a vector the $S$ chips referring to the same coded symbol, the discrete time model for the received signal is

$$\mathbf{y} = \sum_{k=1}^{K} c_k \mathbf{s}_k x_k + \mathbf{z} \tag{1}$$

where $\mathbf{z} \in \mathbb{C}^S$ is a circularly-symmetric complex Gaussian noise vector $\sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, N_0 \mathbf{I})$, $K$ is the number of active users during the current channel use, $\mathbf{s}_k \in \mathbb{C}^S$ is the spreading sequence of user $k$, and $x_k \in \mathbb{C}$ and $c_k \in \mathbb{C}$ are the transmitted modulation symbol and the fading amplitude of user $k$.

**User code, powers, decoders and channel statistics.** Throughout this work we assume that users employ Gaussian random codes, without any claim of optimality, and that the decoder is based on typical set decoding. We assume that the system is symmetric with respect to any user, i.e., that all users have the same power constraint ($\mathrm{E}[|x_k|^2] \leq E_k = E$ for all $k$), the same elementary rate $R$ and the same channel statistics. We consider simple AWGN ($c_k = 1$ for all users and over all slots) and independent Rayleigh fading (where the $c_k$'s are i.i.d., $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$ and the corresponding power gain $|c_k|^2$ is exponentially distributed, with cumulative distribution function (cdf) $F_{|c|^2}(x) = 1 - \mathrm{e}^{-x}$ for $x \geq 0$).

**Throughput versus $E_b/N_0$.** The system throughput $\eta$ is defined as the total average number of bit/modulation symbol correctly received. In [3] it is shown that for all protocols at hand the event of stopping the transmission of the current channel packet is a renewal event, the occurrence of which provides a random reward $\mathcal{R}$ equal to $R/S$ bit/s/Hz if decoding is successful or to 0 if the packet is lost. From the symmetry of the system with respect to any user and by applying the *renewal-reword* theorem, we obtain [3]

$$\eta = N_u \frac{\mathrm{E}[\mathcal{R}]}{\mathrm{E}[\mathcal{T}]} \tag{2}$$

where $E[\mathcal{T}]$ is the average inter-renewal time, i.e., the average time between two consecutive renewals. In particular, here we are concerned with the throughput of the unconstrained system $(M, N \to \infty)$, in this case the throughput is given by [3]

$$\eta = \frac{RG}{1 + \sum_{m=1}^{\infty} p(m)} \tag{3}$$

where $p(m) = \Pr\left(I_k(1) \leq R, \cdots, I_k(m) \leq R\right)$ are the probabilities of decoding failure with $m$ received chunks of the current code word and $I_k(m)$ denotes the accumulated mutual information between the output and transmitter $k$ after $m$ received slots (where user $k$ was active). For the INR, RTD and ALO protocols $I_k(m)$ is given by

$$I_k(m) = \begin{cases} \sum_{s \in \mathcal{S}_{k,m}} \log(1 + \mathrm{SINR}_{k,s}) & \text{INR} \\ \log(1 + \sum_{s \in \mathcal{S}_{k,m}} \mathrm{SINR}_{k,s}) & \text{RTD} \\ \log(1 + \mathrm{SINR}_{k,1}) & \text{ALO} \end{cases} \tag{4}$$

where $\mathrm{SINR}_{k,s}$ is the Signal over Interference plus Noise Ratio for user $k$ in slot $s$, and $\mathcal{S}_{k,m}$ is the set of $m$ slots where user $k$ was active.

Note that, as already pointed out in [3], the throughput (3) with INR protocol coincides with the *ergodic rate* of the underlying block-fading channel

$$\eta = G \; \mathrm{E}\left[\log(1 + \mathrm{SINR})\right] \tag{5}$$

while, with ALO protocol, it coincides with the *outage rate*

$$\eta = GR \; \Pr(\log(1 + \mathrm{SINR}) > R) \tag{6}$$

with $[1 - \Pr(\log(1 + \mathrm{SINR}) > R)] = p(1)$ being the outage probability.

Let $\gamma = E/N_0$ denote the transmit Signal to Noise Ratio (SNR) of each user. Since the users transmit for a fraction $p_t$ of the time and the average number of *received* information bits per modulation symbol is $S\eta/N_u$, the average energy per received information bit is $E_b = S\eta E/(p_t N_u) = \eta E/G$. Hence, the user SNR is related to $E_b/N_0$ by

$$\gamma = \frac{E_b}{N_0} \frac{\eta}{G} \tag{7}$$

Notice the difference between this definition of $E_b/N_0$ and the common definition used by coding theorists in a single-user channel. There, $E_b$ is the energy per *transmitted* information bit, irrespectively of error probability, i.e., on the fraction of erroneously received bits. Here, $E_b$ denotes energy per successfully delivered information bit at the receiver, which in a multiuser channel prone to collisions and packet loss is a more sensible definition.

For the sake of notation simplicity, in the rest of the paper we use natural logarithms. Hence, $\eta$ will be expressed in nat/s/Hz and we will use the notation $E_n/N_0$, instead of $E_b/N_0$, to indicate that $E_n$ is the energy per nat. All results can be readily translated in more usual bit/s/Hz vs. $E_b/N_0$ recalling that 1 nat=$\log_2$(e) bits and that $\log(2) = -1.5917$ dB.

**Unspread Hybrid-ARQ system.** This system is analyzed in [3]. In this case, $S = 1$ and $\mathbf{s}_k = 1$ for all $k$. We made the assumption of infinite population ($N_u \to \infty$), then for all finite $G$, the probability $p_t$ that a user transmit on any given slot is vanishing and the number of active users $K$ is a Poisson distributed random variable with probability mass function (pmf) $\Pr(K = k) = \mathrm{e}^{-G} G^k/k!$.

**DS-CDMA system with random spreading.** In this case, following [1, 8, 5, 2], we assume $S, N_u \to \infty$ and $N_u/S \to \alpha$, where $\alpha$ is the maximum number of active users per chips.[2] The user spreading sequences $\mathbf{s}_k$ are randomly generated with i.i.d. components drawn according to an arbitrary probability assignment with zero mean, variance $1/S$ and bounded forth moment. Transmission follows again the same Hybrid-ARQ scheme described before. The linear filter is either a SUMF or a linear MMSE filters [7]. The channel load is given by $G = p_t\alpha$, hence for all finite $G$ and $\alpha$, $p_t$ is non-vanishing. This system has been widely analyzed [1, 8, 5, 2] but never in an Hybrid-ARQ perspective and never properly compared with the Unspread Hybrid-ARQ system.

**Unspread system with joint decoding.** This system is described and analyzed in [4]. In this case, the system is analogous to the unspread Hybrid-ARQ system but users transmit and are decoded on a strict slot-by-slot basis. On each slot, the receiver applies either SIC-SUD or JMUD. In both cases, users are ranked in decreasing received SNR order. The SIC-SUD strips users once at a time starting from the strongest and the stripping process continues until it is possible to decode user reliably. The JMUD attempts to decode all $N_u$ users. If decoding is not successful, it treats

[2]For the DS-CDMA system with random spreading, the model can be seen as an approximation of a real system with long spreading and many users. In the 3G wireless system proposal, spreading sequencies up to 256 chips are envisaged [6].

the weakest user as noise and attempts the decoding of the remaining $N_u - 1$ users. It proceeds in this way until successful decoding of a subset of users occurs.

# 3 Summary of the results

In this section we report briefly and without derivations the optimized throughput expressions for the three systems under analysis and for a reference single user system. For each system we shall consider the ALO, the RTD and the INR protocols, with and without fading. The analytical details can be found in the Appendices.

## 3.1 The reference single user system

In this case there is not MAI and the SINR coincides with the SNR, given by

$$\mathrm{SNR}_k = \gamma |c_k|^2 \tag{8}$$

The optimization of the throughput (3) with respect to the channel load $G$, after the substitution of $\gamma = (\eta/G)/(E_n/N_0)$ is straightforward and gives $G = 1$. The optimization over the elementary rate $R$ (see Appendix A) yields to the following parametric expressions:

- ALO/RTD/INR without fading: for $\gamma \geq 0$

$$\begin{cases} \eta & = \log(1+\gamma) \\ \frac{E_n}{N_0} & = \frac{\gamma}{\eta} \end{cases} \tag{9}$$

- ALO with Rayleigh fading: for $R \geq 0$

$$\begin{cases} \eta & = R\mathrm{e}^{-\frac{\mathrm{e}^R - 1}{R\mathrm{e}^R}} \\ \frac{E_n}{N_0} & = \frac{R\mathrm{e}^R}{\eta} \end{cases} \tag{10}$$

- RTD with Rayleigh fading: for $R \geq 0$

$$\begin{cases} \eta & = \mathrm{e}^R + R - 1 \\ \frac{E_n}{N_0} & = \frac{1 + (R-1)\mathrm{e}^R}{\eta} \end{cases} \tag{11}$$

- INR with Rayleigh fading: for $\gamma \geq 0$

$$\begin{cases} \eta & = \mathrm{e}^{1/\gamma}\mathrm{Ei}(1/\gamma) \\ \frac{E_n}{N_0} & = \frac{\gamma}{\eta} \end{cases} \tag{12}$$

Fig. 1 shows the throughput $\eta$ vs. $E_n/N_0$ for the single user system.

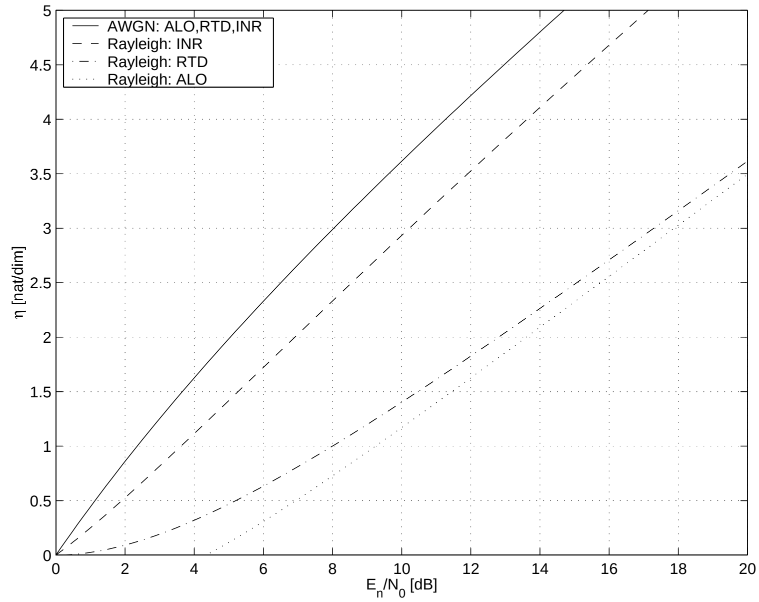


Figure 1: Throughput versus $E_n/N_0$ for a single user system.

**Remarks.** There exists a minimum value of $E_n/N_0$, referred to as $(E_n/N_0)_{\min}$, above which the system has non zero throughput. This value is $(E_n/N_0)_{\min} = 1 = 0$ dB for all the protocols but ALO in Rayleigh fading, for which $(E_n/N_0)_{\min} = \mathrm{e} = 4.3429$ dB. Interestingly, we shall see that those values are function of the protocol and the fading statistics, but not of the system, i.e., they are the same for all the systems under analysis.

Without fading, all the three protocols have the same throughput which coincides with the capacity of the single user channel and is also an upperbound to the throughput of any multiaccess system without power control.

### 3.2 Unspread Hybrid-ARQ system

For the unspread Hybrid-ARQ system, the SINR is given by

$$\mathrm{SINR}_k = \frac{\gamma|c_k|^2}{1 + \sum_{i=1,i\neq k}^{K} \gamma|c_i|^2} \tag{13}$$

and the throughput expressions are those obtained in [3].

- ALO without fading

$$\eta = \max_{K\geq 0} \left[ \log\left(1 + \frac{\gamma}{1+K\gamma}\right) \sum_{k=0}^{K} G\mathrm{e}^{-G}\frac{G^k}{k!} \right] \tag{14}$$

- RTD without fading

$$\eta = G \max_{R \geq 0} \frac{R}{1 + \sum_{m=1}^{\infty} \Pr\left[\sum_{i=1}^{m} \mathrm{SINR}_i \leq \mathrm{e}^R - 1\right]} \tag{15}$$

where $\mathrm{SINR}_i$ are i.i.d. $\forall i$ with pmf $\Pr(\mathrm{SINR}_i = \frac{\gamma}{1+k\gamma}) = \mathrm{e}^{-G} G^k / k!$ for $k \geq 0$.

- INR without fading

$$\eta = G \sum_{k \geq 0} \mathrm{e}^{-G} \frac{G^k}{k!} \log\left(1 + \frac{\gamma}{1 + k\gamma}\right) \tag{16}$$

- ALO with Rayleigh fading

$$\eta = G \max_{R \geq 0} R \exp\left(-\frac{\mathrm{e}^R - 1}{\gamma} - G(1 - \mathrm{e}^{-R})\right) \tag{17}$$

- RTD with Rayleigh fading.

  In this case the throughput is again given by (15), but $\mathrm{SINR}_i$ are i.i.d. $\forall i$ with cdf $F_{\mathrm{SINR}}(x) = [1 - \exp(-x/\gamma - Gx/(1+x))]$.

- INR with Rayleigh fading

$$\eta = G \int_0^{\infty} \exp\left(-\frac{\mathrm{e}^x - 1}{\gamma} - G(1 - \mathrm{e}^{-x})\right) \mathrm{d}x \tag{18}$$

Fig. 2 shows the throughput $\eta$ versus $E_n/N_0$ for the unspread Hybrid-ARQ system, with ALO and INR protocols only. The RTD case was not evaluated because of its complexity, due to the fact that for a given pair $(G, \gamma)$ the cdf of $\sum_{i=1}^{m} \mathrm{SINR}_i$ for all $m \geq 1$ needs to be computed. In [3] it is shown that RTD lies between ALO and INR.

Fig. 3 shows the inverse of optimal $G$, i.e., the number of degree of freedom per user, versus $E_n/N_0$ for the unspread Hybrid-ARQ system with ALO and INR protocol.

**Remarks.** The minimum values of $E_n/N_0$ are $(E_n/N_0)_{\mathrm{min}} = 1 = 0$ dB for all the protocols but ALO in Rayleigh fading, for which $(E_n/N_0)_{\mathrm{min}} = \mathrm{e} = 4.3429$ dB, the same values found for the single user case.

A parametric closed form expression of the optimized throughput can be found for the ALO protocol with Rayleigh fading only. The derivation is reported in Appendix B. In general, after the optimization of the throughput $\eta$ with respect to the elementary rate $R$, the throughput becomes a function of $E_n/N_0$ and the channel load $G$. By carring out the optimization over $G$, it emerges

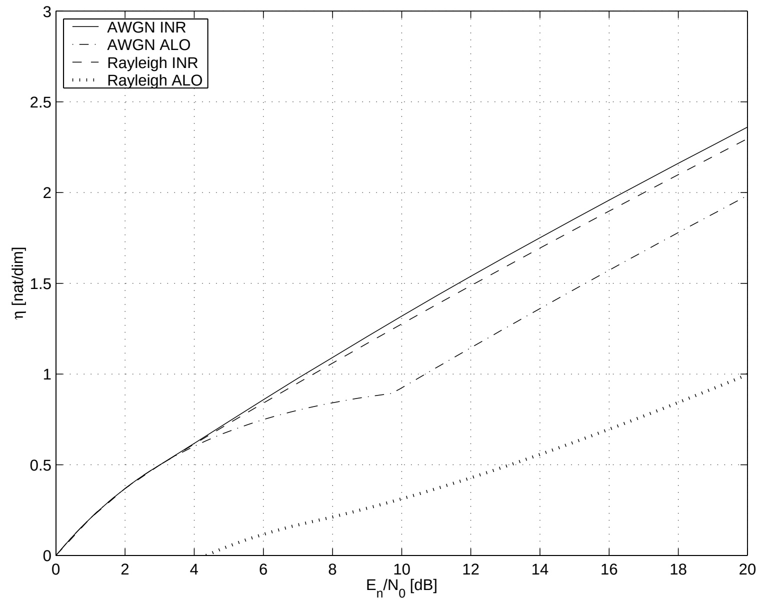


Figure 2: Throughput versus $E_n/N_0$ for an unspread Hybrid-ARQ system.

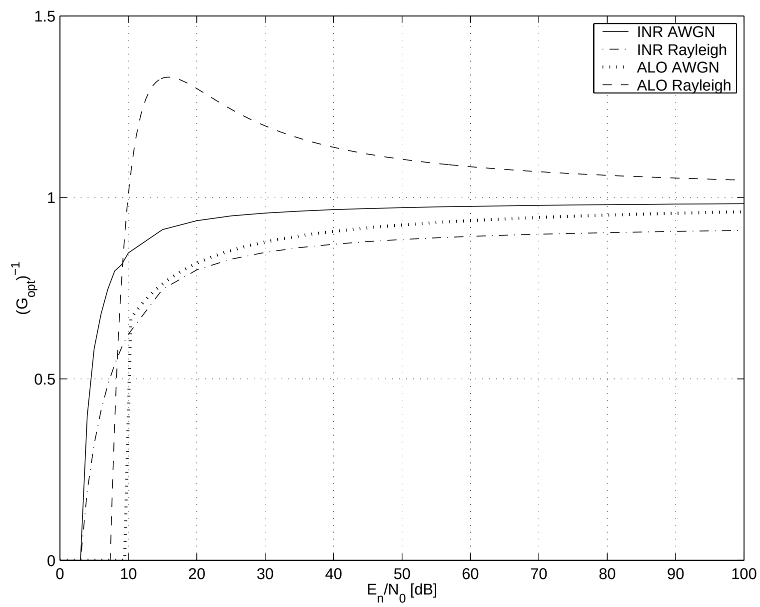


Figure 3: Inverse of optimal $G$ versus $E_n/N_0$ for an unspread Hybrid-ARQ system with ALO and INR protocol.

that there exists an certain interval of $E_n/N_0$, the interval $E_n/N_0 \in [(E_n/N_0)_{\min}, (E_n/N_0)_{\text{th}}]$, for which $\eta$ is maximized by $G \to \infty$ (see Appendix B). In this interval, the maximum throughput is attained by infinite number of users per degree of freedom transmitting at vanishing rate ($G \to \infty$ implies $R \to 0$) and it is given by

$$\eta = \left(\frac{E_n}{N_0}\right)_{\min}^{-1} - \left(\frac{E_n}{N_0}\right)^{-1} \tag{19}$$

In the interval $E_n/N_0 > (E_n/N_0)_{\text{th}}$, the throughput is maximized by a finite $G$ and non-zero $R$ and furthermore, as $E_n/N_0 \to \infty$, optimal $G$ tends to $G \to 1$. This means that maximum throughput is obtained by having on the average only one active user per degree of freedom transmitting at non-vanishing rate. The introduction of the parameter $(E_n/N_0)_{\text{th}}$, makes unambiguous the expressions "low $E_n/N_0$", that refers to the interval $E_n/N_0 \in [(E_n/N_0)_{\min}, (E_n/N_0)_{\text{th}}]$, and "large $E_n/N_0$", that refers to the interval $E_n/N_0 > (E_n/N_0)_{\text{th}}$. In the case of ALO with Rayleigh fading, we have $(E_n/N_0)_{\text{th}} = 2\text{e} = 7.3532$ dB.

The curve for ALO without fading was obtained via the numerical technique described in Appendix C. It presents a change in slope at $(E_n/N_0)_{\text{th}} = 9.7305$ dB due to the fact that optimization over $K$ in (14) gives either $K = 0$ for $E_n/N_0 > (E_n/N_0)_{\text{th}}$ or $K \to \infty$ for $E_n/N_0 \leq (E_n/N_0)_{\text{th}}$. This means that for low $E_n/N_0$ users must encode their messages at vanishing rate and transmit all the time ($K \to \infty$), while for large $E_n/N_0$ users must encode their messages with non-vanishing rate and transmit (on the average) one at a time. This effect of self-orthogonalization of the optimized system is shown in Fig. 3.

The same numerical technique described in Appendix C was used to obtain the curves for the INR protocol with and without fading. In both cases the $E_n/N_0$ threshold is $(E_n/N_0)_{\text{th}} = 2 = 3.0103$ dB and optimal $G$ tends to $G \to 1$ as $E_n/N_0 \to \infty$. Again, this effect of self-orthogonalization is shown in Fig. 3.

### 3.3 DS-CDMA system with random spreading

This system has been analyzed in [1, 8, 5]. The SINR is given by

$$\text{SINR}_k = A|c_k|^2 \tag{20}$$

where $A$ is a deterministic constant, given by the unique non-negative solution of

$$A = \frac{\gamma}{1 + G\gamma} \quad \text{for SUMF} \tag{21}$$

$$A = \frac{\gamma}{1 + G\gamma\,\text{E}\left[\dfrac{|c|^2}{1 + A|c|^2}\right]} \quad \text{for MMSE} \tag{22}$$

The system is "single-user" like, in the sense that the SINR is function only of the fading experienced by the user itself, while the random nature of the system, i.e., random activity, fading and spreading sequencies, is taken into account by the deterministic (in the limit for large $N_u$) constant $A$. For this reason, we can apply the results obtained in Appendix A for the "single-user like" case for what concerns the maximization over the elementary rate $R$. The maximization over $G$ is more complicated because $A$ is also function of $G$. The throughput of all the protocols can be put in the form

$$\eta = \max_{A \geq 0} \frac{g(A) - \left(\frac{E_n}{N_0}\right)^{-1}}{f(A)} \tag{23}$$

where $f(A)$ is given by

$$f(A) = \begin{cases} 1 & \text{SUMF} \\ \mathrm{E}\left[\frac{|c|^2}{1+A|c|^2}\right] & \text{MMSE} \end{cases} \tag{24}$$

and where, in the case of Rayleigh fading, $g(A)$ is given by

- ALO/RTD/INR without fading

$$g(A) = \frac{1}{A}\log(1+A) \tag{25}$$

- ALO with Rayleigh fading

$$g(A) = \left. \mathrm{e}^{-x - \frac{\mathrm{e}^x - 1}{x\mathrm{e}^x}} \right|_{x\mathrm{e}^x = A} \tag{26}$$

- RTD with Rayleigh fading

$$g(A) = \left. \mathrm{e}^{-x} \right|_{1+(x-1)\mathrm{e}^x = A} \tag{27}$$

- INR with Rayleigh fading

$$g(A) = \frac{1}{A}\mathrm{e}^{1/A}\mathrm{Ei}(1/A) \tag{28}$$

Fig. 4 shows the throughput $\eta$ versus $E_n/N_0$ for DS-CDMA with random spreading.

Fig. 5 shows the inverse of optimal $G$ versus $E_n/N_0$ for the DS-CDMA with MMSE. We did not report the curves for DS-CDMA with SUMF because in this case optimal $G$ is $G \to \infty$ for all protocols for every fading statistics.

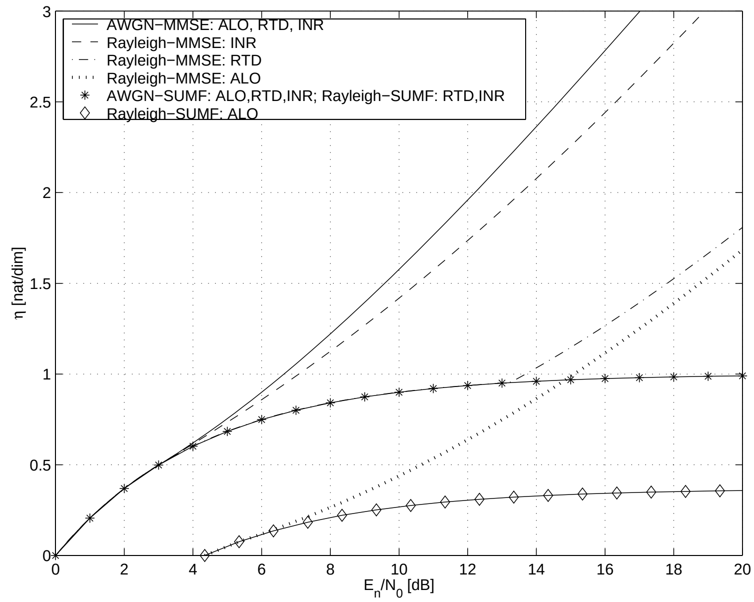


Figure 4: Throughput versus $E_n/N_0$ for a DS-CDMA system with random spreading.

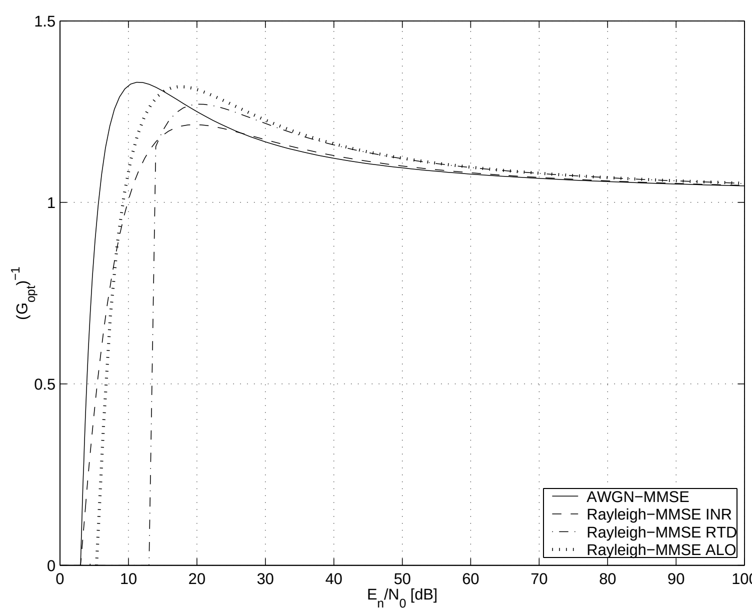


Figure 5: Inverse of optimal $G$ versus $E_n/N_0$ for an MMSE DS-CDMA system with random spreading.

**Remarks.** In the case of SUMF, the optimization over $A$ is straightforward and gives, for all the protocols, with and without fading, $A = 0$. This is because the numerator $g(A)$ is always a continuous non increasing function of $A$ for all $A \geq 0$. Note that $A = 0$ for all $E_n/N_0$ implies that optimal channel load is $G \to \infty$ for all $E_n/N_0$, meaning that $\eta$ is maximized by infinite users per chip transmitting with vanishing coding rate.

For MMSE (see Appendix D), we have $(E_n/N_0)_{\rm th} = 2{\rm e} = 7.3532$ dB for ALO with Rayleigh fading, $(E_n/N_0)_{\rm th} = 13.18$ dB for RTD with Rayleigh fading, $(E_n/N_0)_{\rm th} = 2 = 3.0105$ dB for INR with Rayleigh fading and all the protocols without fading. For $E_n/N_0 > (E_n/N_0)_{\rm th}$ there exists a finite $G$ optimizing the throughput with $G \to 1$ as $E_n/N_0 \to \infty$. This effect of self-orthogonalization of the optimized system is shown in Fig. 5.

## 3.4 Unspread system with joint decoding

The system is analyzed in [4] in the case $N_u \to \infty$ and $p_t = 1$. To cast this system into our previous definitions, the system is governed by an ALO protocol, since decoding is performed on a slot basis, and channel load is infinite, i.e., $G = N_u \to \infty$.

### 3.4.1 Results for SIC-SUD.

The users are sorted in decreasing received SNR order and the strongest is decoded first, by considering the others as noise. Its decoded message is re-modulated and subtracted from the overall received signal. Then, the second strongest users is decoded, re-modulated and subtracted, and so on until it is possible to decode users reliably. Let $y$ denote the probability that a randomly chosen user is in the set of users that can be decoded reliably. Then, the throughput as a function of $y$ is given by the equation [4]

$$\eta = y \inf_{x \in ]0,y]} \frac{F^{-1}_{|c|^2}(1-x)}{(E_n/N_0)^{-1}\eta^{-1} + \beta(x,1)} \tag{29}$$

where

$$\beta(a,b) \triangleq \int_a^b F^{-1}_{|c|^2}(1-x)\mathrm{d}x = \int_{F^{-1}_{|c|^2}(1-b)}^{F^{-1}_{|c|^2}(1-a)} x \mathrm{d}F_{|c|^2}(x) \tag{30}$$

Intuitively, the fraction to be minimized in (29) represents the rate of user in position "$N_u x$" (in the list of users ranked according to their received SNR) among the set of decoded users which have position "$1, 2, \cdots, N_u y$", when all the other users "$N_u x + 1 \cdots N_u$" are treated as noise.

Fig. 6 shows the throughput $\eta$ versus $E_n/N_0$ for the system with joint decoding, both SIC-SUD and JMUD.

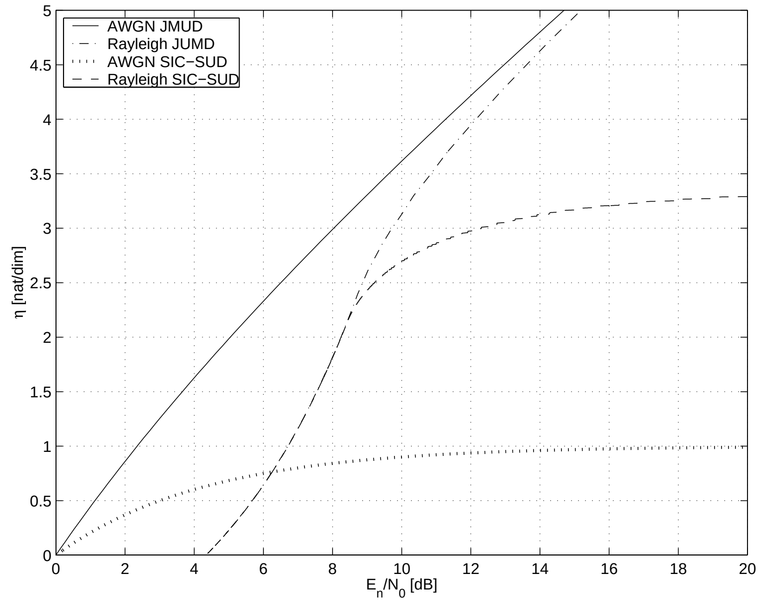


Figure 6: Throughput versus $E_n/N_0$ for a system with joint decoding.

**Remarks.** Without fading SIC-SUD is equivalent to DS-CDMA with SUMF, i.e.,

$$\eta = 1 - \left(\frac{E_n}{N_0}\right)^{-1} \tag{31}$$

because the condition for successful decoding for the first user coincides with that of DS-CDMA with SUMF and, since all the users are received with identical power, either all or none of them can be decoded.

With Rayleigh fading optimization cannot be carried out in closed form, so we used numerical optimization. Also in this case, as before for the AWGN case, the SIC-SUD is throughputwise limited, in fact the limit of (29) for large $E_n/N_0$ is

$$\eta = \max_{u \geq 0} \mathrm{e}^{-u} \inf_{\theta \geq u} \frac{\theta}{1 - \mathrm{e}^{\theta}(1+\theta)} = 3.3509 \tag{32}$$

obtained for $u = 0$.

### 3.4.2 Results for JMUD.

The users are again sorted in decreasing received SNR order. The receiver attempt to decode all $N_u$ users. If the equal-rate point falls inside the capacity region for the $N_u$ users, then decoding is successful. If not, the weakest user (user $N_u$) is treated as noise and the receiver considers the capacity region for the remaining $N_u - 1$ users. If the equal-rate point is not inside it, the two weakest users are treated as noise, and so on until it is possible to decode a subset of the $N_u$ users. For the JMUD receiver the throughput is (see [4])

$$\eta = \inf_{x \in ]0,1]} \frac{1}{x} \log \left( 1 + \frac{\beta(y(1-x), y)}{(E_n/N_0)^{-1}\eta^{-1} + \beta(y,1)} \right) \tag{33}$$

Intuitively the logarithm to be minimized in (33) represents the most stringent constraint, in terms of aggregated rate, for which the equal rate point falls inside the capacity region defined by the set of the best "$N_u y$" users, when treating the rest as noise.

Fig. 6 shows the throughput $\eta$ versus $E_n/N_0$ for the system with joint decoding, both SIC-SUD and JMUD.

**Remarks.** Without fading JMUD is equivalent to the unfaded single user system. To show this, let's first consider the case of finite $G = N_u$ and then take the limit for $G = N_u \to \infty$. For a given $N_u$ let $R = \frac{1}{N_u} \log(1 + N_u \gamma)$ be the elementary rate of the users. Since all the users transmit (and are received) with identical power, the equal rate point is on the dominant face of the capacity region for these $N_u$ users, hence the throughput is

$$\eta = N_u\, R = \log(1 + N_u \gamma) = \log(1 + (E_n/N_0)\eta) \tag{34}$$

Being (34) valid for all finite $N_u$, then, by continuity, it is valid for $N_u \to \infty$. Also for JMUD without fading, we have $(E_n/N_0)_{\min} = 1 = 0$ dB. Note that all the users are always decoded, i.e., $y = 1$, and the for all $E_n/N_0 \geq (E_n/N_0)_{\min}$ the optimal $G$ is indeed $G \to \infty$, which achieves the maximum possible throughput for a multiaccess system without power control.

With Rayleigh fading, the optimization of (33) cannot be carried out in closed form and hence we used numerical evaluation. As already pointed out in [4], for high $E_n/N_0$ the throughput (33) tends to

$$\eta \to \log \left( 1 + \frac{(E_n/N_0)\eta}{1+\delta} \right) \tag{35}$$

for some $\delta > 0$ and outage probability vanishes as $\sqrt{\delta/(\eta\, E_n/N_0)}$. This means that at high $E_n/N_0$ ALO-JMUD approaches unfaded single user performance, i.e., asymptotically there is no loss in performances with respect to the optimal system. It is interesting to note that, in Rayleigh fading channel, also with JMUD $(E_n/N_0)_{\min} = \mathrm{e} = 4.3429$ dB as it is for SUD-based systems.

# 4 Comparison

In this section we present a comparison among the systems and protocols analyzed in the previous sections. We claim that our comparison is fair. However, it is worth reminding that from a practical

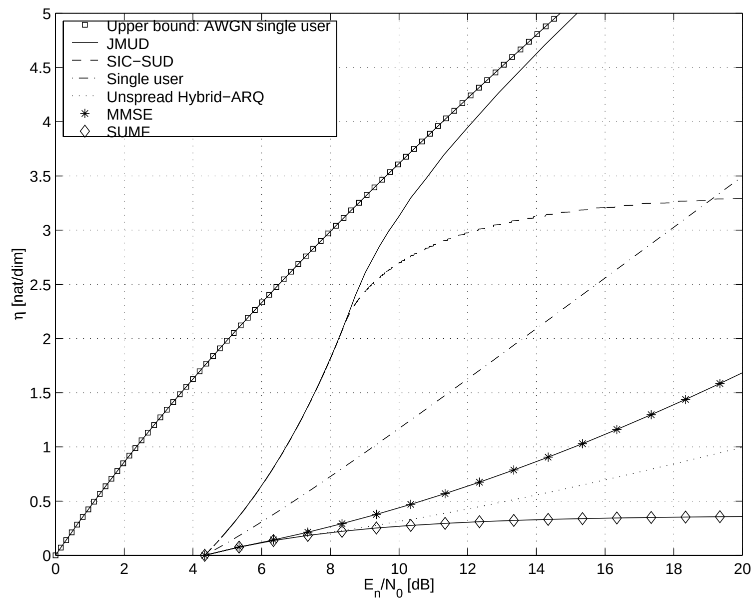


Figure 7: Throughput versus $E_n/N_0$ for ALO with Rayleigh fading.

point of view there are other quantities of great interest, average delay and probability of packet loss for example, that were not considered in this work.

In the following we refer to Fig. 7,which shows the throughput curves versus $E_n/N_0$ of all the analyzed systems with ALO protocol in Rayleigh fading (we did not reported the cases without fading to make the picture more readable) and to Fig. 8, which shows the throughput curves versus $E_n/N_0$ of all the analyzed systems with INR without fading (again, the curves referring to the Rayleigh fading case were not added to make the picture more readable).

**Minimum $E_n/N_0$.** We have identified two values of $(E_n/N_0)_{\min}$ under which the throughput is zero. One is $(E_n/N_0)_{\min} = \mathrm{e} = 4.3429\,\mathrm{dB}$ for ALO with Rayleigh fading, the other is $(E_n/N_0)_{\min} = 1 = 0\,\mathrm{dB}$ for ALO, RTD, INR without fading and RTD and INR with Rayleigh fading. This shows the benefit of packet combining over discarting previous transmission in a faded environment.

**Low $E_n/N_0$ regime.** For low $E_n/N_0$ all the protocols, regardless the fading statistic, have the same throughput which corresponds to vanishing elementary rate $R$ and infinite channel load $G$. This means that users must be active all the time, in the unspread case, or be infinitely more than the spreading gain, in the DS-CDMA system. Fading does not impair the system performance, compared to the unfaded case, because the available power is so little that fading fluctuations do not matter. In this regime the simplest system can be implemented without loosing in throughput,

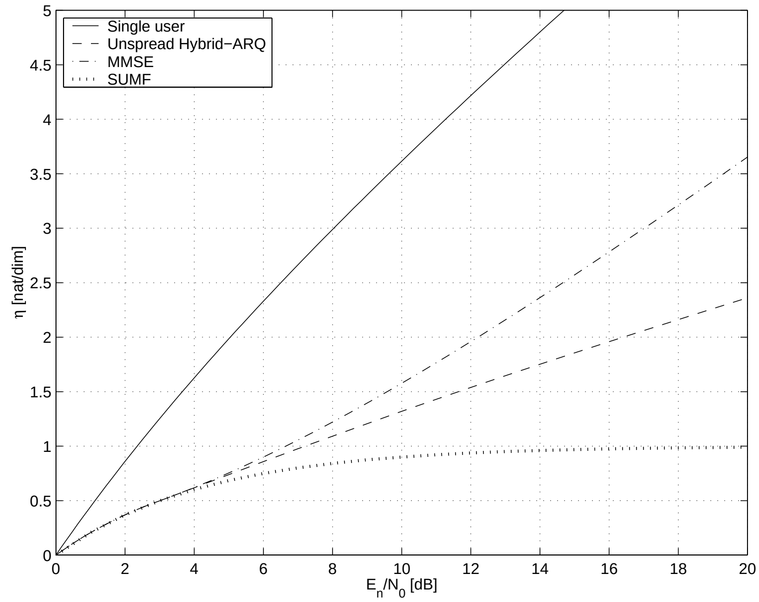


Figure 8: Throughput versus $E_n/N_0$ for INR without fading.

in fact DS-CDMA with SUMF does not loose in performance with respect to more complicated schemes as DS-CDMA with MMSE. For the ALO with Rayleigh fading, the curves of Fig. 7 show that JMUD is superior to DS-CDMA and unspread system. In this regime SIC-SUD is equivalent to JMUD and again the simpler system has the same performance that the more complex one.

**High $E_n/N_0$ regime.** The differences in performance among systems and protocols are visible at high $E_n/N_0$. For all SUD-based systems but DS-CDMA with SUMF, the optimal elementary rate $R$ grows with $E_n/N_0$, the channel load $G \to 1$ and the throughput grows linearly with $\log(E_n/N_0)$. This means that when the available power is high, the best strategy consists of encoding at non vanishing rate and having, on average, only one user per slot, in the unspread system, or one user per chip, in the DS-CDMA system. This does not hold for DS-CDMA with SUMF for which optimal $G$ is always $G \to \infty$ and the throughput is limited to 1 in the case without fading and to $\mathrm{e}^{-1}$ in the case with Rayleigh fading. The unspread system behaves always better than DS-CDMA with SUMF but is outperformed by DS-CDMA with MMSE. At $E_n/N_0 = 18$ dB, for example, the difference between unspread system and DS-CDMA with MMSE is about 1 nat/dim.

Without fading JMUD is optimal, in the sense that it has the same performance as unfaded single user case, while with Rayleigh fading it approaches the unfaded single user performance for $E_n/N_0 \to \infty$. SIC-SUD is highly suboptimal and it is limited in terms of maximum throughput.

**Performance in a faded environment.** Fading does not improve the throughput performance of this system since CSI is not available at the transmitter, i.e., no power control is possible. The single user unfaded case upperbounds all the systems. This bound is reached by ALO-JMUD without fading and approached by ALO-JMUD with fading in an high $E_n/N_0$ regime.

# 5 Conclusions

A comparison between three different multiaccess strategies is presented in a scenario characterized by random activity of an infinite population of uncoordinated users. Different retransmission protocols and combing techniques are considered in presence of block fading and additive noise. To make the comparison fair, the system throughput is optimized with respect to all the system parameters and expressed as function of $E_b/N_0$. The best performance is obtained by joint decoding even without packet combing. Among SUD-based systems, MMSE DS-CDMA outperforms the system without spreading, while the SUMF DS-CDMA is heavily suboptimal and interference limited.

We showed that at low $E_b/N_0$ all SUD-based system are equivalent to DS-CDMA with SUMF, suggesting that practical system operating in this region do not need to be complex and that users must transmit continuously with vanishing rate. At high $E_b/N_0$ the best strategy, for SUD-based system, is having on the average one active user per degree of freedom transmitting at non-vanishing rate, this makes the system not interference limited. In practice, a call admission control scheme should keep the channel load $G$ close to its optimum value, depending on the operating $E_b/N_0$.

# A Throughput optimization for the single-user like system

In a single-user like case, the SINR can be expressed as

$$\mathrm{SINR} = A|c|^2 \tag{36}$$

where $|c|^2$ is the (random) fading power and $A$ is a system (deterministic) constant. For an actual single user system, $A$ coincides with the SNR, but in general it depends on other system parameters, as for the case of DS-CDMA with random spreading. The probabilities of decoding failure $p(m)$ at the denominator of (3) are

$$p(m) = \begin{cases} F_1\left(\frac{\mathrm{e}^R-1}{A}\right)^m & \text{ALO} \\ F_m\left(\frac{\mathrm{e}^R-1}{A}\right) & \text{RTD} \\ \Pr[\sum_{s=1}^m \log(1+\mathrm{SINR}_s) \le R] & \text{INR} \end{cases} \tag{37}$$

where we define the cdf

$$F_m(x) = \Pr\left[\sum_{s=1}^{m} |c_s|^2 \le x\right] = \begin{cases} 1_{\{x \ge m\}} & \text{without fading} \\ 1 - \sum_{j=0}^{m-1} \frac{x^j}{j!} \mathrm{e}^{-x} & \text{Rayleigh fading} \end{cases} \tag{38}$$

Note that $F_1(x) \stackrel{\Delta}{=} F_{|c|^2}(x)$.

## A.1 Result for ALO

In this case, the $p(m)$'s form a geometric series, hence

$$\eta = RG\,[1 - p(1)] = \left.\begin{cases} RG\,1_{\{x<1\}} & \text{without fading} \\ RG\,\mathrm{e}^{-x} & \text{Rayleigh fading} \end{cases}\right|_{x=(\mathrm{e}^R-1)/A} \tag{39}$$

The maximization over $R$ gives

$$\eta = \left.\begin{cases} G\,\log(1+A) & \text{without fading} \\ G\,A\,\mathrm{e}^{-x-\frac{\mathrm{e}^x-1}{x\mathrm{e}^x}} & \text{Rayleigh fading} \end{cases}\right|_{x\mathrm{e}^x=A} \tag{40}$$

## A.2 Result for RTD

Without fading the denominator of (3) yields

$$1 + \sum_{m \ge 1} p(m) \;=\; \sum_{m \ge 0} 1_{\{m<x\}} = 1 + \lfloor x \rfloor \tag{41}$$

while with Rayleigh fading we get

$$\begin{aligned} 1 + \sum_{m \ge 1} p(m) &= 1 + \sum_{m \ge 1}\left[1 - \sum_{j=0}^{m-1} \frac{x^j}{j!}\mathrm{e}^{-x}\right] = 1 + \sum_{m \ge 1}\sum_{j=m}^{\infty} \frac{x^j}{j!}\mathrm{e}^{-x} \\ &= 1 + \sum_{j=1}^{\infty} j\frac{x^j}{j!}\mathrm{e}^{-x} = 1 + x \end{aligned} \tag{42}$$

By substitution of (41) and (42) in (3) we obtain

$$\eta = \left.\begin{cases} \dfrac{RG}{1+\lfloor x \rfloor} & \text{without fading} \\ \dfrac{RG}{1+x} & \text{Rayleigh fading} \end{cases}\right|_{x=(\mathrm{e}^R-1)/A} \tag{43}$$

The maximization over $R$ gives

$$\eta = \left.\begin{cases} G\,\log(1+A) & \text{without fading} \\ G\,A\mathrm{e}^{-x} & \text{Rayleigh fading} \end{cases}\right|_{1+(x-1)\mathrm{e}^x=A} \tag{44}$$

### A.3 Result for INR

The optimization over $R$ yields $R \to \infty$ (see [3]) and the throughput is

$$\eta = \mathrm{E}[\log(1 + \mathrm{SINR})] \tag{45}$$

hence

$$\eta = \begin{cases} G \log(1 + A) & \text{without fading} \\ G\, \mathrm{e}^{1/A} \mathrm{Ei}(1/A) & \text{Rayleigh fading} \end{cases} \tag{46}$$

where $\mathrm{Ei}(x) \triangleq \int_x^\infty \mathrm{e}^{-t}/t \, \mathrm{d}t$ is the exponential integral function.

**Remark on the optimization over $G$.** For an actual single user system $A = \gamma$ and $G \in [0, 1]$, hence the optimization over $G$ is trivial and gives for all the protocols with every fading statistics $G = 1$. In general, $G$ can be a function of the other system parameters, hence the maximization in more involved and must be carried out case by case.

## B Throughput optimization for the Hybrid-ARQ system: ALO with Rayliegh fading

Suppose the following implicit equation is given

$$H(x, y) = 0 \tag{47}$$

that locally can be put in explicit form as

$$y = f(x) \tag{48}$$

The derivative of $f(\cdot)$ can be obtained by solving the following system, which involves the differential of (47) and (48),

$$\begin{cases} \mathrm{d}x H_x + \mathrm{d}y H_y = 0 \\ \mathrm{d}y = f_x \mathrm{d}x \end{cases} \tag{49}$$

where $H_x \triangleq \frac{\partial H}{\partial x}$, and get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{H_x}{H_y} \tag{50}$$

(for more details see [9], Sections 2.10 and 2.11).

In the case at hand, we have

$$H(G,R,\eta,E_n/N_0) = -\eta + RG\exp\left(-G\frac{\mathrm{e}^R-1}{E_n/N_0\,\eta} - G(1-\mathrm{e}^{-R})\right) \tag{51}$$

and locally

$$\eta = f(G,R,E_n/N_0) \tag{52}$$

Applying (50) we obtain that the derivative of (52) with respect to $G$ is

$$\frac{\partial f}{\partial G} = -\frac{H_G}{H_\eta} = \frac{\eta}{1-G\dfrac{\mathrm{e}^R-1}{E_n/N_0\,\eta}}\left[\frac{1}{G} - \frac{\mathrm{e}^R-1}{E_n/N_0\,\eta} - (1-\mathrm{e}^{-R})\right] \tag{53}$$

By solving (53) equal to zero with respect to $G$, we get

$$G = \frac{1}{\dfrac{\mathrm{e}^R-1}{E_n/N_0\,\eta} - (1-\mathrm{e}^{-R})} \tag{54}$$

Equation (54) can be view as a parametric definition of $G$. By substitution of (54) in (51) and writing explicitly $\eta$ with respect to $E_n/N_0$, we get

$$\eta = \frac{\mathrm{e}^{-1}R}{1-\mathrm{e}^{-R}} - \frac{\mathrm{e}^R}{E_n/N_0} \tag{55}$$

Note that (55) is positive $\forall R \geq 0$ iff $E_n/N_0 > \mathrm{e}$. The optimization of (55) with respect to $R$ gives

$$E_n/N_0 = \mathrm{e}\frac{\mathrm{e}^R(1-\mathrm{e}^{-R})^2}{1-\mathrm{e}^{-R}(R+1)} \tag{56}$$

The limit of (56) for $R \to 0$ is $E_n/N_0 = 2\mathrm{e}$, in fact for $E_n/N_0 \in [\mathrm{e}, 2\mathrm{e}]$ the function (55) in monotonic and has it maximum for $R = 0$. Hence the final expression is

$$\begin{cases} E_n/N_0 \in [\mathrm{e}, 2\mathrm{e}] \\ \eta = \mathrm{e}^{-1} - (E_n/N_0)^{-1} \end{cases}$$
$$\begin{cases} R \geq 0 \\ E_n/N_0 = \mathrm{e}\dfrac{\mathrm{e}^R(1-\mathrm{e}^{-R})^2}{1-\mathrm{e}^{-R}(R+1)} \\ \eta = \frac{\mathrm{e}^{-1}R}{1-\mathrm{e}^{-R}} - \frac{\mathrm{e}^R}{E_n/N_0} \end{cases} \tag{57}$$

Note that $R \to 0$ in (54) means $G \to \infty$.

# C Throughput optimization for the Hybrid-ARQ system: numerical technique

In general, after the maximization of $\eta$ with respect to $R$, we have

$$\begin{cases} \eta & = f(G,\gamma) \\ \frac{E_n}{N_0} & = \frac{G\,\gamma}{f(G,\gamma)} \end{cases} \tag{58}$$

where $f(G,\gamma)$ is a function that depends on the protocol and on the cdf of the fading. Equations in (58) define $\eta$ as a function of $E_n/N_0$ in a parametric form that depends on two parameters: the channel load $G$ and the average SNR $\gamma$. In principle, we could let $G$ and $\gamma$ vary in all $R_+^2$, for every pair $(G,\gamma)$ plot the corresponding point $(\eta, E_n/N_0)$ in a cartesian plane and then take the closure of the obtained set of points. Numerically this in not a well define problem, unless we are reasonably sure having taken enough "good" pairs $(G,\gamma)$ that are on the closure of the set $(\eta, E_n/N_0)$.

The procedure just discussed is equivalent to fixing a value of $E_n/N_0$, searching for the all the pairs $(G,\gamma)$ that give that $E_n/N_0$ and among all these pairs take the one that gives the maximum throughput $\eta$. In formulas, we define

$$\mathcal{A}_x = \left\{ (G,\gamma) \in R_+^2 : \frac{G\,\gamma}{f(G,\gamma)} = x \right\} \tag{59}$$

to be the set of pairs $(G,\gamma)$ yielding $E_n/N_0 = x$. Then, the problem of finding the closure of the set defined by (58) is equivalent to

$$\eta = \frac{1}{x} \max_{(G,\gamma)\in\mathcal{A}_x} G\gamma \tag{60}$$

Assuming we are able to compute $\mathcal{A}_x$ for every $x$, the formulation (60) is much more appealing.

Let apply (60) to the case of ALO without fading. The the throughput formula is

$$\eta = \log\left(1 + \frac{\gamma}{1+K\gamma}\right) \sum_{k=0}^{K} G\mathrm{e}^{-G} \frac{G^k}{k!} \tag{61}$$

(see (14)), i.e., every user encodes its messages with elementary rate $R = \log\left(1 + \gamma/(1+K\gamma)\right)$, that allows for successful decoding with all sets of less than $K$ simultaneous interferring users. The optimization of (61) over $K$, i.e. chosing the best elementary rate for a given pair $(G,\gamma)$, cannot be carried out in closed form, but assuming known the best value of $K$ we can apply the method in (60) and write

$$\eta = \frac{1}{x} \max_{(G,\gamma)\in\mathcal{A}_x^{(K)}} G\gamma \tag{62}$$

$$\mathcal{A}_x^{(K)} = \left\{ (G,\gamma) \in R_+^2 : \sum_{k=0}^{K} \mathrm{e}^{-G} \frac{G^k}{k!} \frac{1}{\gamma} \log\left(1 + \frac{\gamma}{1+K\gamma}\right) = \frac{1}{x} \right\} \tag{63}$$

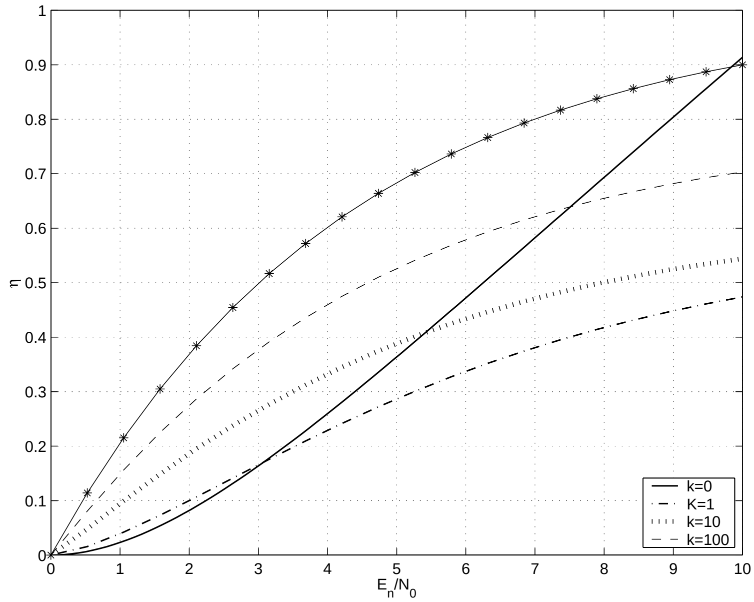


Figure 9: $\eta$ as a function of $E_n/N_0$ for different values of $K$

We can solve (63) for $K \geq 0$ and plot curves of $\eta$ indexed by $K$: for each value of $E_n/N_0$ we choose the point of the curve with attains maximum $\eta$ and that gives us also the optimum $K$. Fig. 9 shows the curves obtained for different values of $K$. It turns out that the best value of $K$ is either zero or infinity. In the regime of low $E_n/N_0$ the optimum $K$ tends to $K \to \infty$ and the optimum $G$ tends to $G \to \infty$ as $\gamma \to 0$. This means that users must encode their messages at vanishing rate and transmit all the time. The resulting optimized throughput in this region is $\eta = 1 - (E_n/N_0)^{-1}$. For large $E_n/N_0$, the throughput is maximum for $K = 0$ and $G \to 1$ as $\gamma \to \infty$, This means that users must encode their messages with non-vanishing rate, which allows correct decoding only if there are no collisions, and transmit very rearly so that on average there is one active user per slot. The resulting parametric throughput expression in this region is $\eta = G\mathrm{e}^{-G}\log(1+\gamma)$.

# D Throughput optimization for the DS-CDMA system with random spreading

In general, in the CDMA with random spreading, the throughput can be written as

$$\eta = \frac{g(A) - (E_n/N_0)^{-1}}{f(A)} \tag{64}$$

where $f(A)$ is given in (24) and $g(A)$ in (25) for the case without fading, in (26), in (27) and in (28) for ALO, RTD and INR, respectively, for the case with Rayleigh fading

For all the case above, but RTD with Rayleigh fading, the optimization with respect to $A$ is quite simple since $\eta$ is either a decreasing function of $A$ or it has just one local maximum. In the former case, the optimal value is $A = 0$, in the latter is some positive $A$. We can find the range of $E_n/N_0$ for wich the optimum is achieved by $A = 0$ as the solution of

$$\left.\frac{\partial \eta}{\partial A}\right|_{A=0} \leq 0 \tag{65}$$

In the case without fading and INR with Rayleigh fading, this gives $(E_n/N_0) \leq (E_n/N_0)_{\text{th}} = 2$. In the case of ALO with Rayleigh fading, this gives $(E_n/N_0)_{\text{th}} = 2\text{e}$. In the case of RTD with Rayleigh fading, $\eta$ is either decreasing function of $A$ or it has got a local maximum and a local minimum. In the former case optimal $A$ is again $A = 0$, in the latter case the optimal $A$ is $A > 0$ iif the local maximum of $\eta$ is actually the absolute maximum. In this case, numerically we found $(E_n/N_0)_{\text{th}} = 13.18\,\text{dB}$.

In all cases, in the range $(E_n/N_0) > (E_n/N_0)_{\text{th}}$, the optimization over $A$ gives the following parametric equations

$$\begin{cases} \eta & = \left(\frac{\mathrm{d}g(A)}{\mathrm{d}A}\right) / \left(\frac{\mathrm{d}f(A)}{\mathrm{d}A}\right) \\ (E_n/N_0)^{-1} & = g(A) - f(A)\eta \end{cases} \tag{66}$$

for $A \geq 0$, while in the range $(E_n/N_0)_{\min} \leq (E_n/N_0) < (E_n/N_0)_{\text{th}}$, the throughput is given by $\eta = 1 - (E_n/N_0)^{-1}$.

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