# Massive MIMO Stochastic Geometry and Analysis of Beamforming with Partial CSIT

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***Abstract*—We consider coordinated beamforming (BF) for the Multi-Input Single-Output (MISO) Interfering Broadcast Channel (IBC) under imperfect channel state information at the transmitter(s) (CSIT). We start from a BF design which optimizes a Massive MISO limit upper bound of the ergodic capacity, termed Expected Signal and Interference Power Weighted Sum Rate (ESIP-WSR). We extend a recently introduced large system analysis (LSA) for beamformers with partial CSIT, by a stochastic geometry inspired randomization of the channel covariance eigen spaces, leading to much simpler analytical results. These depend only on some essential channel characteristics such as the numbers of antennas and users, channel rank and eigenvalue profile, and (channel estimate) signal to noise ratio (SNR). We analyze the spectral efficiency behavior at extreme SNR regions which provide insights (through the SNR offset) into the characteristics of the various channel estimates and suboptimal BFs compared to ESIP-WSR BF with Linear Minimum Mean Squared Error (LMMSE) channel estimates. Furthermore, simulations validate the superior performance of ESIP-WSR BF compared to the sub-optimal BFs with different channel estimates and also the accuracy of the large system approximations derived herein. Our analysis is focused on constant channel estimation regime which is indicative of the finite rate feedback channels and pilot contamination regime.**

## I. Introduction

In this paper, Tx may denote transmit/transmitter/transmission, Rx may denote receive/receiver/reception, BF may denote beamforming/beamformer. In a Massive Multi-Input Multi-Output MIMO (MaMIMO) system [1], the overhead associated with the acquisition of channel state information at the Tx (CSIT) is quite high. Indeed, in Massive Multiple-Input Single-Output (MaMISO) systems, the received interference and possibly signal powers converge to their expected value (channel hardening effect) due to the law of large numbers. The large system analysis (LSA) becomes an important topic to consider since Monte-Carlo simulations involving large numbers of antennas and user equipments (UEs) become cumbersome in a MaMIMO system. Also, simulations do not allow to see immediately how performance depend on various system parameters. A major breakthrough in the LSA of MIMO systems came in [2], where Wagner et al. develop results in random matrix theory to obtain deterministic equivalents for the signal-to-interference-plus-noise ratio (SINR) and thus the rate expression for regularized zero forcing (R-ZF) precoding under partial channel knowledge. The very recent work [3] extends the LSA results in [2] to a Rician fading channel with perfect CSIT. However, to simplify the analysis, the authors therein consider identical correlation matrix for all the users in the system which is impractical in MaMIMO or mmWave system. Also, worth mentioning is that, a very recent work [4] deals with the deterministic equivalents of the upper and lower bounds to the ergodic capacity and not on the asymptotic tightness of the approximations.

In [5] we have extended the LSA of [2] to a scenario with users having different channel covariance matrices and BF techniques with partial CSIT. However, due to the abundance of different covariance matrices, the resulting deterministic analysis does not allow for much insight. The multi-antenna stochastic geometry aspect introduced here reduces such LSA analysis back to the simplicity of the case of multiple of identity covariance matrices. Under stochastic geometry regime, the random positions of users and scatterers lead to antenna array responses at random angles. As a result of this randomness of angles and antenna array responses, and due to limited angular support, the multipath channels live in subspaces that are of limited dimension and uniformly randomly oriented in array response space.

### A. Contributions of this paper

- The analysis presented in the paper provides accurate spectral efficiency (SE) expressions under realistic channel estimation quality which are useful at any operating SNR. Compared to our previous work [6], [7], we derive simplified sum rate expressions at low and high SNR for the various BFs (ESIP-WSR, naive and expected weighted sum mean squared error (EWSMSE) ) for the various channel estimates, which clearly shows the SNR offset for the sub-optimal BFs compared to the proposed ESIP-WSR BF. Moreover, our analysis differs from the previous work since we focus on a constant channel estimation error (CCEE) regime.
- We furthermore provide certain illustrative examples which are special cases of the ESIP-WSR BF such as perfect channel CSIT case and covariance only CSIT (CoCSIT) scenario where only the channel covariance information is known at the base stations (BSs). We show that we can obtain analytical expressions for the implicit equations which need to be solved as part of the LSA, which in fact provide analytical insights into the system behavior. We also provide simplified sum rate expressions at high SNR for the CoCSIT case and obtain the rate offset with respect to the perfect CSIT case under CCEE.
- With CCEE, it is observed that the EWSMSE and naive BFs with Linear Minimum Mean Squared Error (LMMSE) saturate at high SNR, explained by the derived SNR offset. The SNR offset for the EWSMSE or naive BFs shows that at high SNR, the interference power also increases along with the SNR, since no ZF to the interfering channels happens. However, the ESIP-WSR design does not exhibit a saturation.

## II. MISO IBC Signal Model

[1]We consider here an Interfering Broadcast Channel (IBC) with $C$ cells. We shall consider a system-wide numbering of the users, so $K$ is total number of users in the system. User $k$ is served by BS $b_k$. $\mathbf{h}_{k,b_i}$ is the $M_{b_i} \times 1$ channel from BS $b_i$ to user $k$. For notational convenience, we use an abbreviated notation for the direct channels (channel from BS $b_k$ to the serving user $k$), i.e., $h_{k,b_k}$ will be denoted as $h_k$. Note that similar rule applies to other variables associated with the direct channel. The received signal at user $k$ in cell $b_k$ is

$$y_k = \underbrace{\mathbf{h}_k^H \, \mathbf{g}_k \, x_k}_{\text{signal}} + \underbrace{\sum_{\substack{i \neq k \\ b_i = b_k}} \mathbf{h}_k^H \, \mathbf{g}_i \, x_i}_{\text{intracell interf.}} + \underbrace{\sum_{j \neq b_k} \sum_{i: b_i = j} \mathbf{h}_{k,j}^H \, \mathbf{g}_i \, x_i}_{\text{intercell interf.}} + v_k \quad (1)$$

where $x_k$ is the intended (white, unit variance) scalar signal stream, The Rx signal (and hence the channel) is assumed to be scaled so that we get for the noise $v_k \sim \mathcal{CN}(0,1)$. BS $c$ serves $K_c = \sum_{i:b_i=c} 1$

[1]Notation: In the following, boldface lower-case and upper-case characters denote vectors and matrices respectively. The operators $\mathrm{E}(\cdot)$, $tr(\cdot)$ , $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^T$ represent expectation, trace , conjugate, conjugate transpose and transpose, respectively. $\mathrm{diag}(\cdot)$ represents the diagonal matrix formed by the elements $(\cdot)$. A circularly complex Gaussian random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Theta}$ is distributed as $\boldsymbol{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Theta})$. $\mathbf{V}_{max}(\mathbf{A}, \mathbf{B})$ or $\mathbf{V}_{max}(\mathbf{A})$ represents (normalized) dominant generalized eigenvector of $\mathbf{A}$ and $\mathbf{B}$ or (normalized) dominant eigenvector of $\mathbf{A}$ respectively and $\lambda_{max}(\mathbf{A})$ being the max eigenvalue. $\mathbf{I}_N$ represents an identity matrix of size $N$. $\boldsymbol{e}_i$ is $i^{th}$ column of the identity matrix. We define the function, $(x)^+ = \max\{0, x\}$.

users. The $M_{b_k} \times 1$ spatial Tx filter or beamformer (BF) is $\mathbf{g}_k$. The Tx power constraint at BS $c$ is, $\sum_{i:b_i=c} \|\mathbf{g}_i\|^2 \leq P_c$.

### A. Channel and CSIT Model

For simplicity, we omit all the user indices $k$. Each non-zero mean MISO channel is modeled according to Karhunen-Loeve representation [8] as

$$\mathbf{h} = \mathbf{C}\mathbf{D}^{1/2}\mathbf{c}\,,\ \boldsymbol{R}_{\mathbf{hh}} = \mathbf{C}\mathbf{D}\mathbf{C}^H, \tag{2}$$

where $\boldsymbol{R}_{\mathbf{hh}}$ is the covariance matrix and $\mathbf{c} \sim \mathcal{CN}(0, \mathbf{I}_L)$ are the Rayleigh fading multipath gains in the eigen domain. Here $\mathbf{C}$ is the $M \times L$ eigenvector matrix of the reduced rank channel covariance $\boldsymbol{R}_{\mathbf{hh}}$ with diagonal eigenvalue matrix $\mathbf{D}$. This reduced rank covariance matrix of user channels typically occurs in realistic MaMISO channels due to the limited angular spread of the multipath components [9]. The rank corresponds to an equivalent number of linearly independent multipath components. The total sum rank across all user channels from BS $c$, $\sum_{k=1}^{K} L_{k,c}$ is assumed to be less than $M_c$, where $L_{k,c}$ is the channel rank between user $k$ and BS $c$.

Since the focus of this paper is to study the effect of CCEE, we assume that we are given a deterministic Least-Squares (LS) channel estimate (which results after correlation with the uplink (UL) pilot sequences)

$$\widehat{\mathbf{h}}_{LS} = \mathbf{h} + \widetilde{\mathbf{h}}, \tag{3}$$

where $\mathbf{h}$ is the true MISO channel, and the error is modeled as circularly symmetric white Gaussian noise $\widetilde{\mathbf{h}} \sim \mathcal{CN}(0, \widetilde{\sigma}^2\mathbf{I})$. The variance of the error $\widetilde{\sigma}^2$ is given a priori. Now, assuming the channel covariance subspace is known, the LMMSE channel estimate can be obtained as

$$\widehat{\mathbf{h}} = \mathbf{C}\left(\widetilde{\sigma}^2\mathbf{D}^{-1} + \mathbf{I}\right)^{-1}\mathbf{C}^H\widehat{\mathbf{h}}_{LS} = \mathbf{C}\widehat{\mathbf{D}}^{1/2}\widehat{\mathbf{c}}, \tag{4}$$

where $\widehat{\mathbf{D}} = (\widetilde{\sigma}^2\mathbf{D}^{-1} + \mathbf{I})^{-1}\mathbf{D}$ and $\widehat{\mathbf{c}} = \mathbf{D}^{-1/2}(\sigma^2\mathbf{D}^{-1} + \mathbf{I})^{-1/2}\mathbf{C}^H\widehat{\mathbf{h}}_{LS}$ with $\boldsymbol{R}_{\widehat{\mathbf{c}}\widehat{\mathbf{c}}} = \mathbf{I}$. The estimation error covariance

$$\boldsymbol{R}_{\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}} = \mathbf{C}\widetilde{\mathbf{D}}\mathbf{C}^H = \mathbf{C}[\mathbf{D} - (\widetilde{\sigma}^2\mathbf{D}^{-1} + \mathbf{I})^{-1}\mathbf{D}]\mathbf{C}^H. \tag{5}$$

Further exploiting the orthogonality property of the LMMSE channel estimate, we can write $\mathbf{S} = \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}\left(\mathbf{h}\mathbf{h}^H\right) = \widehat{\mathbf{h}}\widehat{\mathbf{h}}^H + R_{\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}} = \mathbf{C}\mathbf{W}_L\mathbf{C}^H$, where $\mathbf{W}_L = \widehat{\mathbf{D}}^{1/2}\widehat{\mathbf{c}}\widehat{\mathbf{c}}^H\widehat{\mathbf{D}}^{1/2} + \widetilde{\mathbf{D}}$. For the convenience of analysis in the following sections, we define the following quantities, $\mathbf{C}^H\mathbf{h}_{LS} = \widehat{\mathbf{d}} = \mathbf{d} + \widetilde{\mathbf{d}}$, where $\mathbf{d} = \mathbf{C}^H\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{D})$ and $\widetilde{\mathbf{d}} \sim \mathcal{CN}(\mathbf{0}, \widetilde{\sigma}^2\mathbf{I}_L)$. We also investigate the subspace channel estimator (SV) [7], the effect of limiting CCEE to the covariance subspace (LMMSE without weighting). It is defined as, $\widehat{\mathbf{h}}_S = \mathbf{C}\mathbf{C}^H\widehat{\mathbf{h}}_{LS} = \mathbf{h} + \boldsymbol{P}_C\widetilde{\mathbf{h}}_{LS}$ , $\boldsymbol{R}_{\widetilde{\mathbf{h}}_S\widetilde{\mathbf{h}}_S} = \widetilde{\sigma}^2\mathbf{C}\mathbf{C}^H$.

In this paper, we analyze the scenario where the channel estimation quality remains constant with SNR. It is difficult to meet the required CSIT quality particularly in the frequency division duplexed (FDD) systems. At the UE, downlink (DL) training can be used to obtain the CSIT. But obtaining CSIT in the UL requires feedback from the UE due to the lack of channel reciprocity. This leads to the finite rate feedback model [10], where each UE feedbacks the estimated channel information through finite number of bits. Motivated by this, we also consider the case of CCEE in the uplink. Even though in this paper, we do not explicitly consider the pilot contamination effects in the channel estimation phase as in [11], the CCEE scenario considered herein can also be interpreted as representing the case of pilot contamination assuming UL powers are less than or not proportional to that of the DL Tx power.

### B. Beamforming with Partial CSIT

Since the CSIT is imperfect, we look at the optimization of expected weighted sum rate (EWSR). We observed that ESIP-WSR represents an upper bound to the MaMIMO ergodic capacity. Three types of BF design with partial CSIT can be analyzed, rewriting (1)

$$y_k = \widehat{\mathbf{h}}_k^H\,\mathbf{g}_k\,x_k + \widetilde{\mathbf{h}}_k^H\,\mathbf{g}_k\,x_k + \sum_{i=1,\neq k}^{K}(\widehat{\mathbf{h}}_{k,b_i}^H\,\mathbf{g}_i\,x_i + \widetilde{\mathbf{h}}_{k,b_i}^H\,\mathbf{g}_i\,x_i) + v_k \tag{6}$$

From the law of total expectation, we formulate the BF design with a sum power constaint at each BS ($P_c$) as follows

$$EWSR = \mathrm{E}_{\widehat{\mathbf{h}}}\max_{\mathbf{g}} EWSR(\mathbf{g}),\ \text{with}\ \sum_{i=1,b_i=c}^{K_i}\|\mathbf{g}_i\|^2 \leq P_c,\ \text{where}$$

$$EWSR(\mathbf{g}) = \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}WSR(g) = \sum_{k=1}^{K}u_k\ \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}\ln(s_k/s_{\overline{k}}) = \tag{7}$$

$$\mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}\sum_{k=1}^{K}u_k\ln(1+\frac{|\mathbf{h}_k^H\mathbf{g}_k|^2}{s_{\overline{k}}}) \overset{(a)}{\approx} \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}\sum_{k=1}^{K}u_k\ln(1+\frac{|\mathbf{h}_k^H\mathbf{g}_k|^2}{\mathrm{E}_{\mathbf{h}}\,s_{\overline{k}}})$$
$$\overset{(b)}{\leq}\sum_{k=1}^{K}u_k\ \ln(1+\frac{\mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}|\mathbf{h}_k^H\,\mathbf{g}_k|^2}{\mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}\,s_{\overline{k}}})$$
$$= \sum_{k=1}^{K}u_k\ \ln(r_{\overline{k}}^{-1}r_k) = ESIP-WSR(\mathbf{g}) \tag{8}$$

where $u_k$ are the rate weights, $\mathbf{g}$ represents the collection of BFs $\mathbf{g}_k$. Transition $(a)$ is due to the MaMISO limit ($K \to \infty$) and $(b)$ is due to the concavity of $\ln(.)$ and Jensen's inequality. This leads to the ESIP-WSR upper bound. The (channel dependent) interference plus noise power is $s_{\overline{k}}$ and $s_k$ is the total received power, with conditional expectations $r_{\overline{k}}$, $r_k$:

$$s_{\overline{k}} = 1 + \sum_{i\neq k}|\mathbf{h}_{k,b_i}^H\mathbf{g}_i|^2,\ s_k = s_{\overline{k}} + |\mathbf{h}_k^H\mathbf{g}_k|^2,\ \gamma_k = \frac{s_k}{s_{\overline{k}}} - 1$$
$$r_{\overline{k}} = \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}s_{\overline{k}} = 1 + \sum_{i\neq k}\mathbf{g}_i^H\mathbf{S}_{k,b_i}\mathbf{g}_i,$$
$$r_k = \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}}s_k = r_{\overline{k}} + \mathbf{g}_k^H\mathbf{S}_k\mathbf{g}_k,\ \mathbf{S}_k = \mathbf{C}_k\mathbf{W}_k\mathbf{C}_k. \tag{9}$$

Further, we can show that [7], we get the optimized BF w.r.t. partial CSIT, in the stochastic geometry MaMIMO regime as follows, with an interference leakage aware water filling (ILA-WF) for user powers ($\mu_{b_k}$ is the Lagrange multiplier)

$$\mathbf{g}'_k = \frac{\mathbf{g}''_k}{\|\mathbf{g}''_k\|},\ \mathbf{g}''_k = \boldsymbol{\Gamma}_k^{-1}\mathbf{C}_k\boldsymbol{v}_k,\ \boldsymbol{v}_k = \mathbf{V}_{max}(\mathbf{W}_k),\ \mathbf{g}_k = \mathbf{g}'_k\sqrt{p_k}$$
$$\boldsymbol{\Gamma}_k = \mathbf{A}_k + \mu_{b_k}\,\mathbf{I},\quad \beta_i = u_i(\frac{1}{r_{\overline{i}}} - \frac{1}{r_i}),$$
$$\text{ILA-WF:}\ p_k = (\frac{u_k}{\sigma_k^{(2)} + \mu_{b_k}} - \frac{1}{\sigma_k^{(1)}})^+,$$
$$\sigma_k^{(1)} = \mathbf{g}'^H_k\mathbf{B}_k\mathbf{g}'_k, \sigma_k^{(2)} = \mathbf{g}'^H_k\mathbf{A}_k\mathbf{g}'_k, \tag{10}$$

where $\mathbf{A}_k = \sum_{i=1,\neq k}^{K}\beta_i\mathbf{S}_{i,b_k}, \mathbf{B}_k = r_{\overline{k}}^{-1}\mathbf{S}_k$. Also, note that the BF expression above (10) is quite generic and holds for all the cases 0)-4) described below, with different $\mathbf{A}_k$ for each case.

0) **Perfect CSIT:** This corresponds to the case when we replace $\widetilde{\mathbf{h}}_k = 0, \forall k$ in the optimizing function $EWSR(\mathbf{g})$. For the perfect CSIT case, we can obtain

$$\mathbf{A}_k = \sum_{i\neq k}\beta_i\mathbf{h}_{i,b_k}\mathbf{h}_{i,b_k}^H. \tag{11}$$

1) **Naive BF EWSR:** In the optimizing function $EWSR(\mathbf{g})$, just replace $\mathbf{h}$ by $\widehat{\mathbf{h}}$ in a perfect CSIT approach, i.e., ignore $\widetilde{\mathbf{h}}$ everywhere. For naive BF

$$\mathbf{A}_k = \sum_{i\neq k}\beta_i\widehat{\mathbf{h}}_{i,b_k}\widehat{\mathbf{h}}_{i,b_k}^H. \tag{12}$$

2) **EWSMSE BF [12]**: It accounts for covariance CSIT in the interference terms, but also associates the signal $\widetilde{\mathbf{h}}$ term with the interference. EWSMSE, also called the "use and forget lower bound" in [13], can indeed be shown to be a lower bound for EWSR. For EWSMSE, we just have to replace

$$\mathbf{A}_k = \sum_{i\neq k}\beta_i\mathbf{S}_{i,b_k} - \beta_k\mathbf{C}_k\widetilde{\mathbf{D}}_k\mathbf{C}_k^H. \tag{13}$$

For EWSMSE criterion, it is suboptimal in that it exploits the rate-MSE (mean squared error) relation to transform weighted sum rate (WSR) in weighted sum MSE (WSMSE) but the order of expectation and optimization over weights is reversed, to simplify the cost function.

3) **EWSR upper bound ESIP-WSR:** It also accounts for covariance CSIT in the interference term but, unlike EWSMSE, associates the signal $\widetilde{\mathbf{h}}$ term with the signal power.

4) **Covariance CSIT (CoCSIT):** CoCSIT represents the case when only the channel covariance information (of all the users in the system) is known at the BS, i.e the knowledge of $\mathbf{C}$ and $\mathbf{D}$. For the CoCSIT case, we obtain

$$\mathbf{A}_k = \sum_{i\neq k} \beta_i \mathbf{C}_{i,b_k}\mathbf{D}_{i,b_k}\mathbf{C}_{i,b_k}^H,\ \boldsymbol{v}_k = \boldsymbol{e}_{i,max} \tag{14}$$

In fact, (11) tells us that BF is along only one of the unitary vectors (or dominant eigen vector) in the covariance subspace $\mathbf{C}_k$, which leads to a reduction signal power which accounts for the rate offset for the CoCSIT case, see Section IV-E.

## III. Asymptotic Analysis: Stochastic Geometry MaMIMO Regime

In this section, we analyze the asymptotic SE behaviour of MaMIMO systems using ESIP-WSR BFs solved using ergodic capacity formulation under partial CSIT. In the following sections, we use an abuse of notation for convenience, when we refer to $\xrightarrow[a.s]{M\to\infty}$, we refer to the almost sure convergence in the large system limit where $M, K \to \infty$ with $L$ being finite, and the ratio $\frac{KL}{M} = \kappa \in (0,1)$. Also, whereas the BF optimization with partial CSIT concerns $EWSR(\mathbf{g}) = \mathrm{E}_{\mathbf{h}|\widehat{\mathbf{h}}} WSR(\mathbf{g})$, the large system analysis actually analyzes the resulting ergodic rate $\mathrm{E}_{\widehat{\mathbf{h}}} max_g E_{\mathbf{h}|\widehat{\mathbf{h}}} WSR(\mathbf{g})$ or approximations thereof.

To motivate the large system analysis results we provide, we illustrate it with a simple example of perfect CSIT case. First, we consider the computation of the deterministic equivalent of the signal power part, $|\mathbf{h}_k^H\mathbf{g}_k|^2$. Using Lemma 4 from [2], the term $\mathbf{h}_k^H\mathbf{g}_k$ becomes $\mathbf{h}_k^H\mathbf{\Gamma}_k^{-1}\mathbf{h}_k \xrightarrow[a.s]{M\to\infty} \mathrm{tr}\{\boldsymbol{Q}\mathbf{\Gamma}_k^{-1}\}$, where $\boldsymbol{Q} = \boldsymbol{R}_{\mathbf{h}_k\mathbf{h}_k}$. In $\mathrm{tr}\{\boldsymbol{Q}\mathbf{\Gamma}_k^{-1}\}$, $\mathbf{\Gamma}_k$ can be considered as a sum of $K$ terms (bringing in term $k$ has negligible effect as $K\to\infty$). Further we look at the deterministic equivalent of the SINR and rate, which is based on the following theorem.

**Theorem 1** ( [2, Theorem 1]). *Let $\boldsymbol{Q}_M \in \mathcal{C}^{M\times M}$ be a Hermitian deterministic matrix and $\mathbf{A}_M = \mathbf{X}_M\mathbf{D}\mathbf{X}_M^H$, with $\mathbf{X}_M$ contains $K$ independent columns with covariance matrix $\mathbf{\Theta}_i$ for $i^{th}$ column. $\mathbf{D}$ is a diagonal matrix, with $i^{th}$ diagonal element being $d_i$. Also, assume that $\boldsymbol{Q}_M$, $\mathbf{\Theta}_i$ have uniformly bounded spectral norms. Then, for any $z>0$*

$$\begin{aligned}
&\tfrac{1}{M}\,tr\{\boldsymbol{Q}_M(\mathbf{A}_M+z\mathbf{I}_M)^{-1}\} - \tfrac{1}{M}\,tr\{\boldsymbol{Q}_M\mathbf{T}(z)\} \xrightarrow[a.s]{M\to\infty} 0, \textit{ with},\\
&\mathbf{T}(z) = \left(\tfrac{1}{M}\sum_{i=1}^{K}\tfrac{d_i\mathbf{\Theta}_i}{1+e_i(z)} + z\mathbf{I}_M\right)^{-1}, \textit{ where},\\
&e_i(z) = e_i^{(\infty)}(z) \textit{ is defined as the unique positive solution of}\\
&e_i(z) = \tfrac{1}{M}\,tr\{d_i\mathbf{\Theta}_i\left(\tfrac{1}{N}\sum_{i=1}^{K}\tfrac{d_i\mathbf{\Theta}_i}{1+e_i(z)} + z\mathbf{I}_M\right)^{-1}\}.
\end{aligned} \tag{15}$$

**Theorem 2.** *In Theorem 1, let $\boldsymbol{Q}_k = \mathbf{C}_k\mathbf{D}_k\mathbf{C}_k^H \in \mathcal{C}^{M_{b_k}\times M_{b_k}}$ be a Hermitian deterministic matrix and $\mathbf{\Gamma}_k = \sum_{i=1}^{K}\mathbf{C}_{i,b_k}\mathbf{V}_{i,b_k}\Lambda_{i,b_k}\mathbf{V}_{i,b_k}^H\mathbf{C}_{i,b_k}^H$, with $\mathbf{C}_{i,b_k}\mathbf{V}_{i,b_k}$ contains $L_{i,b_k}$ independent columns with covariance matrix $\mathbf{\Theta}_{i,b_k} = \frac{1}{M}\mathbf{I}_M$ for $r^{th}$ column. $\mathbf{\Lambda}_{i,b_k}$ is a diagonal matrix, with $r^{th}$ diagonal element being $\lambda_{i,b_k}^{(r)}$. Then, for any $z>0$*

$$\begin{aligned}
&\tfrac{1}{M_{b_k}}\,tr\{\boldsymbol{Q}_k(\mathbf{\Gamma}_k+z\mathbf{I}_M)^{-1}\} - \tfrac{1}{M_{b_k}}\,tr\{\mathbf{D}_k\}e_c \xrightarrow[a.s]{M\to\infty} 0,\\
&\textit{with, } b_k=c,\ e_c, \textit{ is defined as the unique positive solution of}\\
&e_c = (\tfrac{1}{M_{b_k}}\sum_{i=1}^{K}\sum_{r=1}^{L_{i,c}}\tfrac{\beta_i\lambda_{i,c}^{(r)}}{1+\beta_i\lambda_{i,c}^{(r)}e_c} + z)^{-1}.
\end{aligned} \tag{16}$$

The fixed point solution of $e$ can be found by iterating, see [2]. Note that [5] consider extensions to the partial CSIT case with general covariance matrices for channels and their estimates and errors, leading to cumbersome solutions. Now here comes in our MaMIMO Stochastic Geometry assumptions which simplify the large system results in [5]. Considering again the perfect CSIT case (signal power part) and applying Theorem 1, we arrive at the simplified result in Theorem 2 which forms the basis of the LSA results for all BF and channel estimator combination. For the purpose of the LSA here, an $M_{b_i}\times L_{i,b_i}$ matrix $\mathbf{C}_i$ which is Haar (uniformly random semi-unitary) can be replaced by a matrix of i.i.d elements when $L_{i,b_i}$ remains finite.

Also, it is to be emphasized that there is no trace of matrices in (2) compared to (15). Also, there is only one $e$ to be computed per BS, instead of one $e$ per every user as in (15). This also explains the reasoning behind the simplified results in this paper compared to [5]. We also define

$$x_c = \tfrac{e_c^2}{M_c}\sum_{i=1}^{K}\sum_{r=1}^{L_{i,c}}\tfrac{\beta_i^2\lambda_{i,c}^{(r),2}}{(1+\beta_i\lambda_{i,c}^{(r)}e_c)^2},\ \ e_c' = \tfrac{e_c^2}{1-x_c}. \tag{17}$$

The LSA of the ESIP-WSR BF starts with an eigenvalue decomposition of $\mathbf{W}_{i,c}$, $\mathbf{W}_{i,c} = \mathbf{V}_{i,c}\mathbf{\Lambda}_{i,c}\mathbf{V}_{i,c}^H$. We remark that $\mathbf{C}_{k,b_i}'$, which is the product of a Haar matrix and a unitary matrix remains Haar, whose entries can be modeled and thus Theorem 2 remains applicable. The computation of the eigenvalues $\lambda_{i,c}^{(r)}$ is based on a rank-2 approximation of the eigenvalue decomposition (EVD), which is described in detail in [7]. The eigenvalue matrix $\mathbf{\Lambda}_{i,c}$ is approximated as

$$\mathbf{\Lambda} \;=\; \widetilde{\sigma}^2\mathbf{D}(\widetilde{\sigma}^2\mathbf{I}+\mathbf{D})^{-1} + \mathrm{tr}\{\mathbf{D}(\mathbf{I}+\widetilde{\sigma}^2\mathbf{D}^{-1})^{-1}\}\,\boldsymbol{e}_1\boldsymbol{e}_1^H. \tag{18}$$

## IV. Simplified Sum Rate Expressions with Different BF and Channel Estimators

### A. Sum Rate Analysis at any SNR

Even though in general, the true channel eigenvalues may be distinct, it is illustrative to consider an extreme case where the eigenvalues are all equal. We discuss the simplified sum rate expressions for naive, EWSMSE and ESIP-WSR BFs for LMMSE/SV/LS channel estimators under multi cell (C cells), with identical parameters, $\widetilde{\sigma}_{k,c}^2 = \widetilde{\sigma}^2$, $L_{k,c} = L$, $\mathbf{D}_{k,c} = \frac{\eta_{k,c}}{L}\mathbf{I}$, $P_c = P, \forall c$ and $M_c = M, \forall k, c$. Number of users in cell $c$ is denoted as $K_c = K/C, \forall c$. For ESIP-WSR BF with LMMSE channel estimate, substituting these values in (18), we obtain

$$\lambda_{k,c}^{(1)} = \zeta_{k,c} + \lambda_{k,c}^{(2)},\quad \zeta_{k,c} = \frac{\eta_{k,c}^2}{L\widetilde{\sigma}^2+\eta_{k,c}},\quad \lambda_{k,c}^{(2)} = \frac{\widetilde{\sigma}^2\eta_{k,c}}{L\widetilde{\sigma}^2+\eta_{k,c}}, \tag{19}$$

and rest of the eigenvalues $\lambda_{k,c}^{(r)} = \lambda_{k,c}^{(2)}, \forall r = 2,...,L$. In the case of naive BF with LMMSE channel estimate, there will only be one eigenvalue and that will be $\zeta_{k,c}$. For the SV channel estimator, the eigenvalues are $\lambda_{k,c}^{(1)} = (\eta_{k,c}+L\widetilde{\sigma}^2)+\widetilde{\sigma}^2$, $\lambda_{k,c}^{(r)} = \widetilde{\sigma}^2, \forall r\neq 1$. Similarly for the naive BF with SV channel estimator, the only one eigenvalue is, $\lambda_{k,c}^{(1)} = (\eta_{k,c}+L\widetilde{\sigma}^2)$. For ESIP-WSR BF with LS only channel estimate, the only eigenvalue will be $\lambda_{k,c}^{(1)} = (\eta_{k,c}+M\widetilde{\sigma}^2)$.

For the case of identical eigenvalues for all users ($\eta_{k,c} = \eta$), we denote the eigenvalues (which are the same for all the $\mathbf{W}_{k,c}$) are of the form $\lambda_{k,c}^{(1)} = \lambda_\zeta + \lambda_2\, \lambda_{k,c}^{(r)} = \lambda_2\, \forall r>1$, where $\zeta_1, \zeta_2$ are defined below. Further we can write the equation for solving $e_c$ as

$$\begin{aligned}
&\tfrac{1}{e_c} = \tfrac{K}{M}\tfrac{\beta\lambda_1}{1+\beta\lambda_1 e_c} + \tfrac{KL}{M}\tfrac{\beta\lambda_2}{1+\beta\lambda_2 e_c} + \mu_c,\ \zeta = \tfrac{\eta^2}{L\widetilde{\sigma}^2+\eta},\\
&\lambda_2 = \tfrac{\widetilde{\sigma}^2\eta}{L\widetilde{\sigma}^2+\eta},\ \lambda_1 = \zeta+\lambda_2.
\end{aligned} \tag{20}$$

**Theorem 3.** *In the large system limit, the quantities $\gamma_k - \overline{\gamma}_k \xrightarrow[a.s]{M\to\infty} 0$, $r_k - \overline{r}_k \xrightarrow[a.s]{M\to\infty} 0$, $r_{\overline{k}} - \overline{r}_{\overline{k}} \xrightarrow[a.s]{M\to\infty} 0$ where $\overline{\gamma}_k$ is the deterministic equivalent of the SINR. Further we can show that, since the logarithm is a continuous function, by applying the continuous mapping* *theorem, it follows from the almost sure convergence of* $\gamma_k$ *that,* $R_k - \overline{R}_k \xrightarrow[a.s]{M\to\infty} 0$*, where* $R_k = \ln(1+\gamma_k)$ *is the rate of user* $k$*, with* $\overline{R}_k = \ln(1+\overline{\gamma}_k)$*. The deterministic limits for the ESIP-WSR BF with LMMSE and SV channel estimates are obtained as*

$$\overline{\gamma}_{k,L}^{(Opt)} = \frac{p_k(1-x_c^{(L,\,Opt)})\,\eta_{k,c}}{\frac{1}{M_{b_i}}\sum_{i\neq k} p_i \frac{\eta_{k,b_i}}{L_{k,b_i}} tr\{\mathbf{B}_{k,b_i}^{-2}\}+1},\ b_k = c$$

$$\overline{\beta}_k = u_k(\tfrac{1}{\overline{r}_k} - \tfrac{1}{\overline{r}_k}), \text{where},\ x_c^{(L,\,Opt)} = \frac{e_c^2}{M_c}\sum_{i=1}^{K}\sum_{r=1}^{L_{i,c}} \frac{\beta_i^2 \lambda_{i,c}^{(r),\,2}}{(1+\beta_i\lambda_{i,c}^{(r)} e_c)^2}. \tag{21}$$

*Similarly for the naive BF, we obtain the deterministic equivalent for SINR as*

$$\overline{\gamma}_{k,L}^{(N)} = \frac{(1-x_c^{(L,\,N)})\,\frac{\eta_{k,c}^2 p_k}{(\eta_{k,c}+\tilde{\sigma}_k^2 L_{k,c})}}{\frac{1}{M}\sum_{i=1}^{K} p_i \eta_{k,b_i}+1},\ x_c^{(L,\,N)} = \frac{e_c^2}{M_c}\sum_{i=1}^{K} \frac{\beta_i^2\lambda_{i,c}^{(1),\,2}}{(1+\beta_i\lambda_{i,c}^{(1)} e_c)^2}. \tag{22}$$

***Proof:*** *See [7] for detailed derivations.*

In the above SINR expression, the quantities $(1-x_c^{(L,\,Opt)}), (1-x_c^{(L,\,N)})$ represent the loss in signal power due to the amount of ZF happening at any SNR, which varies from no ZF at very low SNR to ZF to all the paths ($\sum_{i=1}^{K} L_{k,c}$ of them, case of CCEE) to which $\mathbf{g}_k$ cause interference. The details of this analysis will be dealt in the following sections. Also, note that from (21), we can conclude that for the SV channel estimator the signal power gets reduced by a factor $\frac{\eta_{k,c}}{\eta_{k,c}+\tilde{\sigma}_k^2 L_{k,c}}$ compared to the LMMSE. This is attributed to the absence of weighting which is present in the case of LMMSE channel estimator. We consider below certain special cases for which the implicit equation of $e_c$ can be analytically solved.

### B. Extreme SNR Regimes: Naive BF

**Corollary 3.1.** *For the naive BF with LMMSE channel estimate (or with LS or SV estimator),* $\frac{1}{e_c} = \frac{K}{M}\frac{\beta\lambda_1}{1+\beta\lambda_1 e_c} + \mu_c$*, after some algebraic manipulations, it can be shown to be the solution of a quadratic equation and the positive* $e_c$ *can be obtained as*

$$e_c = \frac{-(\mu_c + \beta\lambda_1(\alpha-1)) + \sqrt{(\mu_c+\beta\lambda_1(\alpha-1))^2 + 4\beta\lambda_1\mu_c)}}{2\beta\lambda_1\mu_c}. \tag{23}$$

*At extreme SNR regions (where* $\mu_c \propto 1/P$*), it can be deduced that* $\lim_{P\to 0} e_c = 0$*,* $\lim_{P\to\infty} e_c = \infty$*. Further by substituting for* $e_c$ *in* (17) *leads to* $x_c^{(LS,\,N)} = x_c^{(L,\,N)} = x_c^{(S,\,N)} = \frac{K}{M}$ *at high SNR and* $x_c^{(LS,\,N)} = x_c^{(L,\,N)} = x_c^{(S,\,N)} = 0$ *at low SNR for the naive BFs.*

### C. Extreme SNR Regimes: Perfect CSIT Case

**Corollary 3.2.** *For WSR based BF design with perfect CSIT, which represents a special case of ESIP-WSR BF considered in this paper* ($\mathbf{D} = \mathbf{0}$)*, the implicit equation for* $e_c$ *gets simplified as,* $\frac{1}{e_c} = \frac{K}{M}\frac{\beta\eta}{1+\beta\eta e_c} + \mu_c$*. Note that there is only one eigenvalue corresponding to the true rank one channel vector which is* $\eta$*. Hence we obtain a positive solution by solving the resulting quadratic equation*

$$e_c = \frac{-(\mu_c + \beta\eta(\frac{\kappa}{L}-1)) + \sqrt{(\mu_c+\beta\eta(\frac{\kappa}{L}-1))^2 + 4\beta\eta\mu_c)}}{2\beta\eta\mu_c}. \tag{24}$$

*Again, at extreme SNR regions , it can be deduced that* $\lim_{P\to 0} e_c = 0$*,* $\lim_{P\to\infty} e_c = \infty$*. Further substituting these values in* (17) *leads to the ZF dimension of* $K$ *(interfering user channels) and hence the rate expression can be written as (*$SNR$ *represents Tx SNR, which is* $P$*)*

$$\overline{R} = K\ln\left((1-\frac{K}{M})\rho\frac{C}{K}\right),\ \rho = SNR\,\eta. \tag{25}$$

### D. Extreme SNR Regimes: CoCSIT Case

**Corollary 3.3.** *In the case of CoCSIT, the implicit equation for* $e_c$ *gets simplified as,* $e_c^{-1} = \kappa\beta\eta(1+\beta\eta e_c)^{-1} + \mu_c$ *and a positive solution obtained have the same form as* (23)*, with* $\lambda_1 = \eta$*. At extreme SNR regions, it can be shown that* $\lim_{P\to 0} e_c = 0$*,* $\lim_{P\to\infty} e_c = \infty$*. Further by substituting for* $e_c$ *in* (17) *leads to* $x_c^{(C)} = \frac{KL}{M}$ *at high SNR and* $x_c^{(C)} = 0$ *at low SNR for the naive BFs. For the CoCSIT case, the sum rate can be obtained as,* $\overline{R}_{CoCSIT} = K\ln\left((1-\frac{KL}{M})SNR\frac{C\eta}{KL}\right)$*. This represents a rate offset (per-user) of* $\ln\frac{M-K}{M-KL} + \ln L$ *w.r.t the perfect CSIT.*

In Table I and Table II, we provide the simplified sum rate expressions at high SNR and low SNR, respectively for various BF and channel estimator combination. The simplified sum rate expressions follow directly from the high SNR expressions for the quantity $e_c$ as in [7] and hence the details are skipped here. For the low SNR analysis, we observe that the sum rate can be written as follows

$$\overline{R} = \sum_{c=1}^{C}\ln(1+\chi_c\rho_c) \overset{a}{\approx} \sum_{c=1}^{C}\chi_c\rho_c,\ \text{where},\ \rho_c = \eta_{k,c}P, \tag{26}$$

where in (a), we made the approximation $\ln(1+x)\approx x$, when $x \ll 1$ and $\chi$ represents the SNR offset for various BFs.Also, $\eta_{k,c} = \arg\max_i \eta_{i,c}$, representing the channel attenuation associated with the strongest user. With $\eta_{k,c} = \eta, \forall c$, the rate becomes $\overline{R} \approx C\chi\rho, \rho = \eta P$.

### E. High SNR Analysis under CCEE

We observe that the sum rate expressions at high SNR can be expressed as

$$\overline{R} = \sum_{k=1}^{K}\ln(1+\omega_k\rho_{k,c}), \tag{27}$$

where $\omega = \frac{z}{1+yP}$ represents the rate offset, where $z, y$ varies w.r.t the channel estimator and the type of BF design. For those BF which saturates at high SNR, the saturation level is represented as $\frac{zP}{1+yP} \approx \frac{z}{y}$. Under CCEE, the ESIP-WSR BF does pathwise zero forcing and hence the reduction in signal power is $(1-\frac{KL}{M})$. With LMMSE channel estimate, since the estimation error is also reduced to the covariance subspace, ZF to the covariance subspace of the interfering channels imply that the interference power gets reduced to zero. Hence, $KL$ spatial dimensions are used to suppress the inter-cell and intra-cell interference. However, for the LS channel estimate, since the estimation error is present in the entire $M$ dimensional space, interference power still remains. For the naive BF, where the estimation error is not considered in the BF design, ZF to all the interfering user channel estimates ($(K-1)\approx K$ of them) does happen and hence the signal power reduction due to ZF is $(1-\frac{K}{M})$. For the naive BF also, the interference power still remains and the sum rate saturates at high SNR. This explains the drastic improvement in performance between ESIP-WSR BF with LMMSE/SV channel estimate compared to the ESIP-WSR BF with LS channel estimate and naive BFs.

## V. Simulation Results

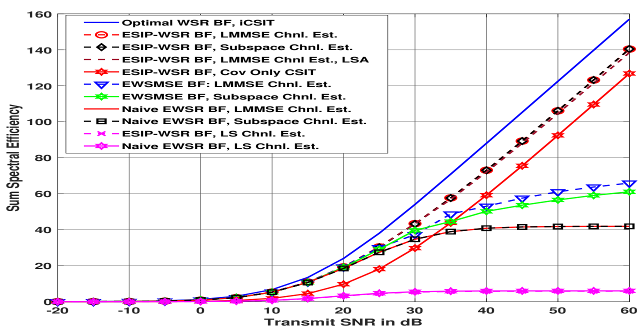
In this section, we present the Ergodic Sum Rate Evaluations for BF design for the various channel estimates. Monte Carlo evaluations of ergodic sum rates are done with the following parameters: $C$, number of cells. $K_c$, number of (single-antenna) users in cell $c$ and $K = \sum_c K_c$. $M$, number of transmit antennas in each cell. We consider a path-wise or low rank channel model as in section II-A, with $L$ = number of paths = channel covariance rank. The elements of the eigenvalue matrix $\mathbf{D}$ is generated from an exponential distribution with mean 1. Further, all the entries are scaled such that $\text{tr}\{\mathbf{D}\} = 1$. The eigenvectors, $\mathbf{C}$ of user channel covariance matrix are generated as random unitary matrices. We do evaluate the sum rate performance under CCEE regime. Notations: in the figures, iCSIT refers to the optimal BF design for the instantaneous (or perfect) CSIT case [14]. "LSA" refers to Large System Approximation. In all the figures, we compare the various BF designs such as ESIP-WSR, EWSMSE and

TABLE I: High SNR Rate Offset for Various BFs ($\mathbf{D}_{k,c} = \frac{\eta}{L}\mathbf{I}_L$) under CCEE

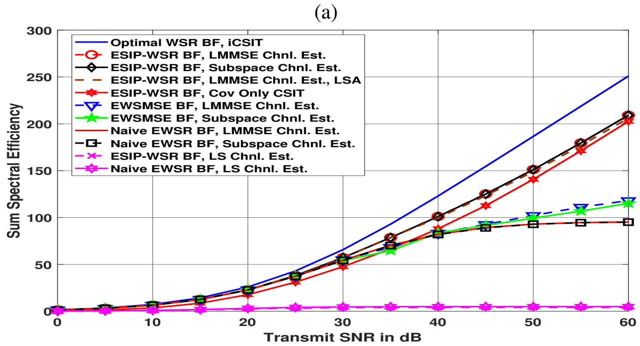
| $\omega\rho$ | naive | EWSMSE | ESIP-WSR | Perfect CSIT | CoCSIT |
|---|---|---|---|---|---|
| LS | $\frac{(1-\frac{K}{M})\frac{\eta}{\eta+\widetilde{\sigma}^2 M}}{\widetilde{\sigma}^2 C\frac{\rho}{\eta}+1}\rho$ | $\frac{(1-\frac{K}{M})\frac{\eta}{\eta+\widetilde{\sigma}^2 M}}{\widetilde{\sigma}^2 C\frac{\rho}{\eta}+1}\rho$ | $\frac{(1-\frac{K}{M})\frac{\eta}{\eta+\widetilde{\sigma}^2 M}}{\widetilde{\sigma}^2 C\frac{\rho}{\eta}+1}\rho$ | $(1-\frac{K}{M})\frac{C}{K}\rho$ | $(1-\frac{KL}{M})\frac{C}{KL}\rho$ |
| LMMSE/SV | $\frac{(1-\frac{K}{M})\frac{\eta}{(\eta+\widetilde{\sigma}^2 L)}}{\frac{C\rho}{M}+1}\rho$ | $\frac{(1-\frac{KL}{M})\frac{\eta}{(\eta+\widetilde{\sigma}^2 L)}}{\frac{C\rho}{M}+1}\rho$ | $(1-\frac{KL}{M})\frac{\eta}{\eta+\widetilde{\sigma}^2 L}\rho$ | $(1-\frac{K}{M})\frac{C}{K}\rho$ | $(1-\frac{KL}{M})\frac{C}{KL}\rho$ |

TABLE II: Low SNR Rate Offset for Various BFs ($\mathbf{D}_{k,c} = \frac{\eta}{L}\mathbf{I}_L$)

| $\chi$ | naive | EWSMSE | ESIP-WSR |
|---|---|---|---|
| LS | $\frac{\eta}{(\eta+\widetilde{\sigma}^2 M)}$ | $\frac{\eta}{(\eta+\widetilde{\sigma}^2 M)}$ | $\frac{\eta}{(\eta+\widetilde{\sigma}^2 M)}$ |
| LMMSE/Subspace | $\frac{1}{L}$ | $\frac{1}{L}$ | $\frac{1}{L}$ |




(a)




(b)

Fig. 1: a) EWSR for $C = 1$ cell, $K_1 = K = 15$ users, $M = 100$, $L = 4$, $\widetilde{\sigma}^2 = 0.1$. b) EWSR for $C = 4$ cell, $K_i = 7, \forall i$, So, $K = 28$ users, $M = 64$, $L = 2$, $\widetilde{\sigma}^2 = 0.1$.

naive under different channel estimates. For the multi-cell simulations, we multiply the inter-cell channels by a random scalar factor ($< 1$) to represent the attenuation in channel power for inter-cell channels from any BS.

### A. CCEE

The CCEE regime looks the most interesting scenario in terms of the superior performance improvement of ESIP-WSR based BF design compared to the very suboptimal schemes such as naive or EWSMSE BFs. The naive and EWSME BFs are observed to saturate at high SNR as seen in Figure 1. In the same figure, we also compare the performance of CoCSIT based BF with the ESIP-WSR BF. From the Figure 1:a) we deduce that there is a sum rate offset of 17 bits/sec/Hz for the CoCSIT compared to the perfect CSIT which is very close to the rate offset predicted by the large system approximations in Section IV-E. The BFs with LMMSE and SV channel estimators converge to the same performance at high SNR in the simulations which is also analytically proved in the paper.

## VI. Conclusion

In this paper, we introduced a stochastic geometry inspired randomization of the channel covariance eigen spaces of the $K$ different users and analyzed the large system behavior. In particular, we focused on a spatial correlation regime in which the ratio of the sum of the ranks of the channels from a BS to the number of antennas $M$ remains a constant. Numerical simulations suggest that the large system approximations are accurate even for finite values of $M, K$. We provided simple and elegant expressions for the sum rate at high and low SNR, providing useful analytical insights into the SNR offsets between different suboptimal BFs which match with our simulations.

## Acknowledgements

EURECOM's research is partially supported by its industrial members: ORANGE, BMW, Symantec, SAP, Monaco Telecom, iABG, and by the projects MASS-START (French FUI), DUPLEX (French ANR) and EU ITN project SPOTLIGHT.

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