CRAMÉR-RAO BOUND FOR JOINT ANGLE AND DELAY ESTIMATORS BY PARTIAL RELAXATION

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ABSTRACT

Novel Fisher-Information Matrix (FIM) and Cramér-Rao Bound (CRB) expressions for the problem of the "partially relaxed" Joint Angle and Delay Estimation (JADE) are derived and analyzed in this paper. In particular, exact closed form expressions of the CRB on the Angles and Times of Arrival of multiple sources are presented. Furthermore, interesting asymptotic and desirable properties are demonstrated, such as high SNR behaviour and lower bound expressions on the CRBs of Angles and Times of Arrival of multiple sources. Computer simulations are also given to visualize CRB behaviour in regimes of interest.

***Index terms*—** Fisher-Information Matrix (FIM), Cramér-Rao Bound (CRB), JADE, Partial Relaxation, AoA

1. INTRODUCTION

Localization has been a challenging topic over the past 70 years. Applications include seismology, radar, sonar, communications [1], etc. One way to achieve this task is to compute the Angle-of-Arrival (AoA) between an anchor point and the intended user. Many techniques were proposed for this purpose, such as MUSIC[2] and ESPRIT[3]. Asymptotic studies were conducted on the variances of these method, in which they attain the CRB with uncorrelated sources and high SNR (or high number of antennas) [4]. The Joint Angle and Delay Estimation (JADE) [11] parametrizes each source through its AoA and its Time-of-Arrival (ToA). Even though more parameters are to be estimated, this allows to resolve more sources [6]. As a result, many methods were developed to solve the JADE problem, such as shift-invariant ones in [7], single-snapshot methods [8], mutual coupling agnostic methods [9], etc.

Recently, the partial relaxation (PR) framework has been introduced as a novel framework for the Angle-of-Arrival (AoA) problem in [5]. In precise, the maximum likelihood cost function is partially relaxed to include one parametrized source, through its AoA and other un-parametrized sources. This relaxation results in new cost functions that are able to resolve AoAs in a reliable and computationally efficient manner.

This paper derives and analyzes the Cramér-Rao Bound (CRB) of the partially-relaxed JADE problem. Indeed, the traditional CRB of the JADE problem was presented in [11] and analyzed in [7]. The contributions of this paper are the following: (i) The Fisher-Information-Matrix (FIM) and CRB of the partially relaxed JADE problem are derived. (ii) Exact closed form expressions are given on the FIM/CRB of the AoAs and ToAs, (iii) Some interesting asymptotic properties are revealed, i.e. lower bounds on the CRBs of the AoAs/ToAs are given. (iv) the cross-correlation CRB between ToA and AoA vanishes exponentially with linear increase of number of subcarriers/antennas.

This paper is divided as follows: Section 2 presents the system model. A possible formulation of the partially relaxed JADE problem is presented in Section 3. Section 4 presents the Cramér-Rao Bound for Times and Angles of Arrivals. We reveal some important properties regarding the CRBs of ToAs/AoAs. Computer simulations are given in Section 6 to support the properties given in Section 5. The paper is concluded in Section 7. **Notations:** Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^\top$ and $(.)^\mathsf{H}$ denote the transpose and Hermitian operators. $\mathrm{Re}(z), \mathrm{Im}(z)$ denote the real and imaginary parts of z, respectively. $\otimes$ is the Kronecker product.

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2. SYSTEM MODEL

Consider an OFDM symbol consisting of M subcarriers, and centered at a carrier frequency f_c (usually 2.4/5 GHz) that has been transmitted through a rich multipath channel of q taps, and received via an array of N antennas. If we parametrize the i^{th} multipath component by a Direction-of-Arrival (DoA) θ_i and Time-of-Arrival (ToA) τ_i, then the ℓ^{th} received OFDM symbol could be expressed as

$$\mathbf{x}(\ell) = \mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})\boldsymbol{\gamma}(\ell) + \mathbf{n}(\ell) \tag{1}$$

where $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathbb{C}^{MN\times q}$ and $\boldsymbol{\gamma}(\ell) \in \mathbb{C}^{q\times 1}$ are defined as

$$\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau}) = \begin{bmatrix}\mathbf{h}(\theta_1,\tau_1) & \dots & \mathbf{h}(\theta_q,\tau_q)\end{bmatrix} \tag{2}$$

$$\boldsymbol{\gamma}(\ell) = \begin{bmatrix}\gamma_1(\ell) & \dots & \gamma_q(\ell)\end{bmatrix} \tag{3}$$

and $\mathbf{n}(\ell) \in \mathbb{C}^{MN\times 1}$ is additive circularly complex noise, i.e. $\mathbf{n}(\ell) \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_{MN})$. The vector $\mathbf{x}(\ell)$ is indexed as follows

$$\mathbf{x}(\ell) = \begin{bmatrix}\mathbf{x}_1^\top(\ell) & \dots & \mathbf{x}_N^\top(\ell)\end{bmatrix}^\top \tag{4}$$

and

$$\mathbf{x}_n(\ell) = \begin{bmatrix}x_{n,1}(\ell) & \dots & x_{n,M}(\ell)\end{bmatrix}^\top \tag{5}$$

where $x_{n,m}(\ell)$ represents the data on the m^{th} subcarrier received by the n^{th} antenna in the ℓ^{th} frame. The problem is to estimate $\boldsymbol{\tau},\boldsymbol{\theta}$ given all observations $\mathbf{x}(1)\dots\mathbf{x}(L)$.

3. JADE BY PARTIAL RELAXATION

The optimal criterion in the presence of white Gaussian noise $\{\mathbf{n}(\ell)\}_{\ell=1\dots L}$ is to solve the Deterministic Maximum Likelihood (DML) cost, i.e.

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\tau}},\hat{\mathbf{G}}) = \underset{\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{G}}{\arg\min}\left\|\mathbf{X} - \mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})\mathbf{G}\right\|^2 \tag{6}$$

where

$$\mathbf{X} = \begin{bmatrix}\mathbf{x}(1) & \dots & \mathbf{x}(L)\end{bmatrix} \tag{7}$$

$$\mathbf{G} = \begin{bmatrix}\boldsymbol{\gamma}(1) & \dots & \boldsymbol{\gamma}(L)\end{bmatrix} \tag{8}$$

Traditional beamformers aim at maximizing the above assuming one source at a time in $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})$. This leads to a simple and fast implementation of the final criterion that aims at finding $\boldsymbol{\tau},\boldsymbol{\theta}$, through peak finding, such as MUSIC. However, this is suboptimal due to existence of multiple sources, when focusing on one. Said differently, equation (6) leads to the following cost

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\tau}}) = \underset{\boldsymbol{\theta},\boldsymbol{\tau}}{\arg\min}\left\|\boldsymbol{\mathcal{P}}^{\perp}_{\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})}\hat{\mathbf{R}}\right\|^2 \tag{9}$$

where $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^{\mathsf{H}}$ is the empirical covariance matrix. Now, it is clear that one should jointly focus on all AoAs and ToAs when solving the JADE problem. Unfortunately, the cost in (9) is highly complex and may not be implementable in most applications. One might resort to suboptimal techniques such as MUSIC/ESPRIT. Another alternative is to "partially relax" the parametric structure of the interfering sources when looking in direction θ at time τ, i.e.

$$\underset{\theta,\tau,\mathbf{B}}{\arg\min}\left\|\boldsymbol{\mathcal{P}}^{\perp}_{[\mathbf{h}(\theta,\tau)\ \mathbf{B}]}\hat{\mathbf{R}}\right\|^2 \tag{10}$$

In the above cost, we parameterize only one column in terms of the times and angles of arrivals, whereas the other $q-1$ columns, captured by an term $\mathbf{B}$, are relaxed to have an arbitrary structure. The matrix $\mathbf{B}$ could be seen as an interference term in which $q-1$ sources contribute to, when beamforming at the remaining one source. For example, in the neighbourhood of (θ_1,τ_1), the matrix $\mathbf{B}$ will play the role of an unstructured approximation of the last $q-1$ columns of $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})$.

In this paper, we derive the Fisher-Information Matrix (FIM) and the Cramér-Rao Bound (CRB) assuming the above partially relaxed JADE model. The paper in [12] presents methods that are capable of estimating $\boldsymbol{\theta},\boldsymbol{\tau}$ by making use of the partially relaxed cost above.

4. CRAMÉR-RAO BOUND FOR TIMES AND ANGLES OF ARRIVAL

The Fisher-Information Matrix (FIM) measures the quantity of information embedded in random parameters. We find it useful to partition the FIM into smaller block FIMs to separate the nuisance from parameters of interest as follows

$$\mathbf{I}_{\boldsymbol{\beta}\boldsymbol{\beta}} = \begin{bmatrix} I_{\theta\theta} & I_{\theta\tau} & \mathbf{I}_{\theta\boldsymbol{\epsilon}} & \mathbf{I}_{\theta\boldsymbol{\eta}} \\ I_{\tau\theta} & I_{\tau\tau} & \mathbf{I}_{\tau\boldsymbol{\epsilon}} & \mathbf{I}_{\tau\boldsymbol{\eta}} \\ \mathbf{I}_{\boldsymbol{\epsilon}\theta} & \mathbf{I}_{\boldsymbol{\epsilon}\tau} & \mathbf{I}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} & \mathbf{I}_{\boldsymbol{\epsilon}\boldsymbol{\eta}} \\ \mathbf{I}_{\boldsymbol{\eta}\theta} & \mathbf{I}_{\boldsymbol{\eta}\tau} & \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\epsilon}} & \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\eta}} \end{bmatrix} \tag{11}$$

$$I_{\beta_i\beta_j} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\mathsf{tr}\left\{\boldsymbol{\Pi}\frac{\partial\mathbf{H}^{\mathsf{H}}}{\partial\beta_i}\mathcal{P}^{\perp}_{\mathbf{H}}\frac{\partial\mathbf{H}}{\partial\beta_j}\right\}\right) \tag{12}$$

where

$$\boldsymbol{\Pi} = \mathbf{P}\mathbf{H}^{\mathsf{H}}\mathbf{R}^{-1}\mathbf{H}\mathbf{P} \tag{13}$$

and $\mathbf{P}$ represents the source covariance matrix, namely

$$\mathbf{P} = \mathbb{E}\Big[\boldsymbol{\gamma}(\ell)\boldsymbol{\gamma}^{\mathsf{H}}(\ell)\Big] \tag{14}$$

Using straightforward manipulations, we can say that

$$I_{\theta\theta} = \frac{2L}{\sigma^2}\boldsymbol{\Pi}_{11}\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}^{\perp}_{\mathbf{H}}\mathbf{d}_\theta \tag{15}$$

$$I_{\tau\tau} = \frac{2L}{\sigma^2}\boldsymbol{\Pi}_{11}\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}^{\perp}_{\mathbf{H}}\mathbf{d}_\tau \tag{16}$$

$$I_{\theta\tau} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\boldsymbol{\Pi}_{11}\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}^{\perp}_{\mathbf{H}}\mathbf{d}_\tau\right) \tag{17}$$

where $\mathbf{d}_\theta = \frac{d\mathbf{a}(\theta)}{d\theta} \otimes \mathbf{c}(\tau)$ and $\mathbf{d}_\tau = \mathbf{a}(\theta) \otimes \frac{d\mathbf{c}(\tau)}{d\tau}$. Now, taking a look at the $(i,j)^{th}$ entry at the following block matrices, we have

$$[\mathbf{I}_{\theta\boldsymbol{\epsilon}}]_{i,j} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\operatorname{tr}\left\{\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{18}$$

$$[\mathbf{I}_{\theta\boldsymbol{\eta}}]_{i,j} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\operatorname{tr}\left\{j\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{19}$$

$$[\mathbf{I}_{\tau\boldsymbol{\epsilon}}]_{i,j} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\operatorname{tr}\left\{\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{20}$$

$$[\mathbf{I}_{\tau\boldsymbol{\eta}}]_{i,j} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\operatorname{tr}\left\{j\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{21}$$

In compact matrix form, the above could be expressed as

$$\mathbf{I}_{\theta\boldsymbol{\epsilon}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\boldsymbol{\Pi}_{21}^{\mathsf{H}} \otimes (\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{22}$$

$$\mathbf{I}_{\theta\boldsymbol{\eta}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(j\boldsymbol{\Pi}_{21}^{\mathsf{H}} \otimes (\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{23}$$

$$\mathbf{I}_{\tau\boldsymbol{\epsilon}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\boldsymbol{\Pi}_{21}^{\mathsf{H}} \otimes (\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{24}$$

$$\mathbf{I}_{\tau\boldsymbol{\eta}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(j\boldsymbol{\Pi}_{21}^{\mathsf{H}} \otimes (\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{25}$$

$$\mathbf{I}_{\boldsymbol{\epsilon\epsilon}} = \mathbf{I}_{\boldsymbol{\eta\eta}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\boldsymbol{\Pi}_{22} \otimes (\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{26}$$

$$\mathbf{I}_{\boldsymbol{\epsilon\eta}} = \frac{2L}{\sigma^2} \operatorname{Re}\left(j\boldsymbol{\Pi}_{22} \otimes (\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E})\right) \tag{27}$$

The Cramer-Rao bound is the inverse of the FIM. In our setting, we can say that

$$\mathbf{C}_{\boldsymbol{\beta\beta}} = \mathbf{I}_{\boldsymbol{\beta\beta}}^{-1} = \begin{bmatrix} C_{\theta\theta} & C_{\theta\tau} & \mathbf{C}_{\theta\boldsymbol{\epsilon}} & \mathbf{C}_{\theta\boldsymbol{\eta}} \\ C_{\tau\theta} & C_{\tau\tau} & \mathbf{C}_{\tau\boldsymbol{\epsilon}} & \mathbf{C}_{\tau\boldsymbol{\eta}} \\ \mathbf{C}_{\boldsymbol{\epsilon}\theta} & \mathbf{C}_{\boldsymbol{\epsilon}\tau} & \mathbf{C}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{C}_{\boldsymbol{\epsilon\eta}} \\ \mathbf{C}_{\boldsymbol{\eta}\theta} & \mathbf{C}_{\boldsymbol{\eta}\tau} & \mathbf{C}_{\boldsymbol{\eta\epsilon}} & \mathbf{C}_{\boldsymbol{\eta\eta}} \end{bmatrix} \tag{28}$$

By using the matrix block inversion formula, we can say that

$$\begin{bmatrix} C_{\theta\theta} & C_{\theta\tau} \\ C_{\tau\theta} & C_{\tau\tau} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\theta\theta} & I_{\theta\tau} \\ I_{\tau\theta} & I_{\tau\tau} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_{\theta\boldsymbol{\epsilon}} & \mathbf{I}_{\theta\boldsymbol{\eta}} \\ \mathbf{I}_{\tau\boldsymbol{\epsilon}} & \mathbf{I}_{\tau\boldsymbol{\eta}} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{I}_{\boldsymbol{\epsilon\eta}} \\ \mathbf{I}_{\boldsymbol{\eta\epsilon}} & \mathbf{I}_{\boldsymbol{\eta\eta}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I}_{\theta\boldsymbol{\epsilon}} & \mathbf{I}_{\theta\boldsymbol{\eta}} \\ \mathbf{I}_{\tau\boldsymbol{\epsilon}} & \mathbf{I}_{\tau\boldsymbol{\eta}} \end{bmatrix}^{\mathsf{H}} \tag{29}$$

The involved block matrices could be written as

$$\begin{bmatrix} \mathbf{I}_{\theta\boldsymbol{\epsilon}} & \mathbf{I}_{\theta\boldsymbol{\eta}} \\ \mathbf{I}_{\tau\boldsymbol{\epsilon}} & \mathbf{I}_{\tau\boldsymbol{\eta}} \end{bmatrix} = \mathbf{K}_1\left(\boldsymbol{\Pi}_{21}^{\mathsf{H}} \otimes \begin{bmatrix} \operatorname{Re}(\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) & -\operatorname{Im}(\mathbf{d}_\theta^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) \\ \operatorname{Re}(\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) & -\operatorname{Im}(\mathbf{d}_\tau^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) \end{bmatrix}\right)\mathbf{K}_2 \tag{30}$$

$$\begin{bmatrix} \mathbf{I}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{I}_{\boldsymbol{\epsilon\eta}} \\ \mathbf{I}_{\boldsymbol{\eta\epsilon}} & \mathbf{I}_{\boldsymbol{\eta\eta}} \end{bmatrix} = \mathbf{K}_2\left(\boldsymbol{\Pi}_{22} \otimes \begin{bmatrix} \operatorname{Re}(\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) & -\operatorname{Im}(\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) \\ -\operatorname{Im}(\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) & \operatorname{Re}(\mathbf{E}^{\mathsf{H}}\mathcal{P}_{\mathbf{H}}^{\perp}\mathbf{E}) \end{bmatrix}\right)\mathbf{K}_2 \tag{31}$$

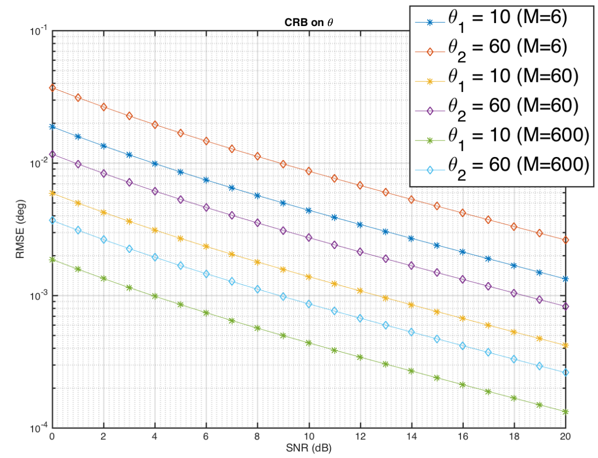

Figure 1: CRB $C_{\theta\theta}$

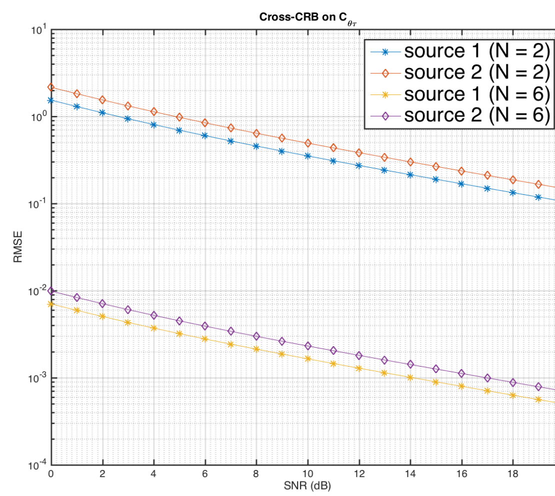

Figure 2: CRB $C_{\theta\tau}$

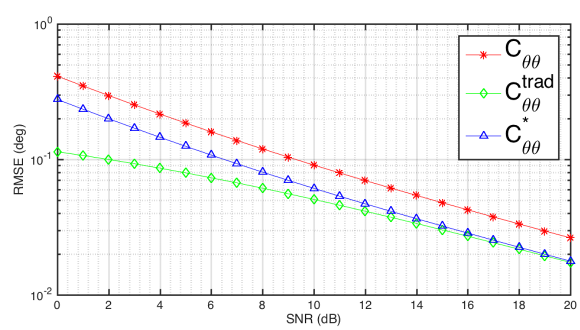

Figure 3: The Traditional JADE CRB vs the Partially Relaxed JADE CRB ($M = 5, N = 2$)

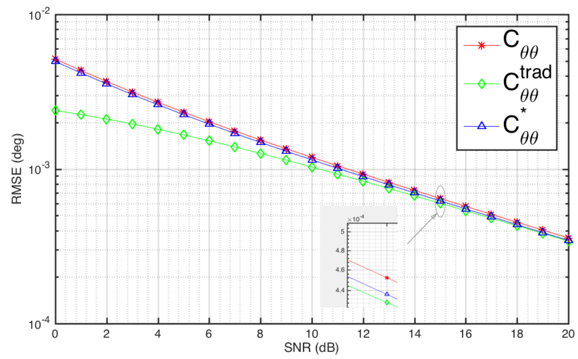

Figure 4: The Traditional JADE CRB vs the Partially Relaxed JADE CRB ($M = 100, N = 2$)

where $\mathbf{K}_1, \mathbf{K}_2$ are suitable permutation matrices. Using the following identities

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \tag{32}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD} \tag{33}$$

and noting that the quantity $\mathbf{\Pi}_{21}^{\mathsf{H}}\mathbf{\Pi}_{22}^{-1}\mathbf{\Pi}_{21}$ is a scalar. After some straightforward arrangements, we can write (29) as

$$\begin{bmatrix} C_{\theta\theta} & C_{\theta\tau} \\ C_{\tau\theta} & C_{\tau\tau} \end{bmatrix} = \frac{\sigma^2}{2\alpha L} \begin{bmatrix} \mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta & \mathrm{Re}(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) \\ \mathrm{Re}(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) & \mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau \end{bmatrix}^{-1} \tag{34}$$

where $\alpha = \mathbf{\Pi}_{11} - \mathbf{\Pi}_{21}^{\mathsf{H}}\mathbf{\Pi}_{22}^{-1}\mathbf{\Pi}_{21}$ is the Schur's complement of $\mathbf{\Pi}$ w.r.t its block matrix $\mathbf{\Pi}_{22}$. Finally, the CRBs of the parameters of interest are given as

$$C_{\theta\theta} = \frac{\sigma^2}{2\alpha L} \frac{\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau}{(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta)(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) - \mathrm{Re}^2(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)} \tag{35}$$

$$C_{\tau\tau} = \frac{\sigma^2}{2\alpha L} \frac{\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta}{(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta)(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) - \mathrm{Re}^2(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)} \tag{36}$$

$$C_{\theta\tau} = -\frac{\sigma^2}{2\alpha L} \frac{\mathrm{Re}(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)}{(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta)(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) - \mathrm{Re}^2(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)} \tag{37}$$

Notice that when the cross-term $\mathrm{Re}(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) = 0$, the CRB on θ, $C_{\theta\theta}$ aligns with the expression in [10].

5. PROPERTIES AND RESULTS

In this section, we discuss some useful insights related to the derived CRBs. First and foremost, we note that the cross-correlation CRB term, $C_{\theta\tau}$ vanishes in the large regime (either in space or frequency). This is easily seen as the term

$$\mathrm{Re}(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) \longrightarrow 0 \tag{38}$$

for large M given a fixed N, or vice versa. Even more, this regime allows us to lower bound the CRBs on θ and τ, i.e.

$$C_{\theta\theta} > C_{\theta\theta}^* \tag{39}$$

$$C_{\tau\tau} > C_{\tau\tau}^* \tag{40}$$

where

$$C_{\theta\theta}^* = \frac{\sigma^2}{2\alpha L} \left(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta\right)^{-1} \tag{41}$$

$$C_{\tau\tau}^* = \frac{\sigma^2}{2\alpha L} \left(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau\right)^{-1} \tag{42}$$

Secondly, it is worth noting that the traditional CRB of the Joint Angle and Delay Estimation problem [11] serves as a lower bound on $C_{\theta\theta}^*$ and $C_{\tau\tau}^*$, i.e.

$$C_{\theta\theta} > C_{\theta\theta}^* > C_{\theta\theta}^{\mathsf{trad}} \tag{43}$$

$$C_{\tau\tau} > C_{\tau\tau}^* > C_{\tau\tau}^{\mathsf{trad}} \tag{44}$$

where $C_{\theta\theta}^{\mathsf{trad}}, C_{\tau\tau}^{\mathsf{trad}}$ are extracted from the following quantity

$$\mathsf{CRB}(\theta, \tau) = \frac{\bar{\sigma}}{2} \sum_{\ell=1}^{L} \mathrm{Re}(\boldsymbol{B}_\ell^{\mathsf{H}} \mathbf{F}^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{F} \boldsymbol{B}_\ell) \tag{45}$$

where $\bar{\sigma}$ is the estimation noise variance and $\mathbf{F} = \begin{bmatrix} \mathbf{d}_\theta & \mathbf{d}_\tau \end{bmatrix}$ and $\boldsymbol{B}_\ell = \mathbf{I}_2 \otimes \mathrm{diag}\{\boldsymbol{\gamma}(\ell)\}$. Note that $C_{\theta\theta}^{\mathsf{trad}}, C_{\tau\tau}^{\mathsf{trad}}$ is attained only for large N or M and at high SNR for uncorrelated sources, i.e. when $\mathbf{\Gamma}$ is diagonal.

6. COMPUTER SIMULATIONS

In this section, computer simulations are presented to visualize the behaviour of the partially relaxed CRB in different scenarios.

In the first simulation, we plot the square root of the CRB on θ on a dB scale, that is $10\log_{10}\sqrt{C_{\theta\theta}}$ for different number of subcarriers by keeping all other parameters fixed. We have set $N = 2$, $L = 100$, $\Delta_f = 3.125$ MHz, $\tau_1 = 10$ nsec and $\tau_2 = 30$ nsec, $\theta_1 = 10°$ and $\theta_2 = 60°$. As expected, we see that the CRB $C_{\theta\theta}$ on both θ_1, θ_2 decreases linearly on a logarithmic scale, by increasing the number of subcarriers at the same rate, as shown in Fig. 1.

On the other hand, we fix the number of subcarriers to $M = 2$ and slightly change the number of antennas from $N = 2$ to $N = 6$ as depicted in Fig. 2. We can observe that the cross-CRB $C_{\theta\tau}$ decreases massively (order $\sim O(10^{2.5})$).

In the third scenario, we study the behaviour of the three CRBs mentioned in the previous section, i.e. the CRB of the partially relaxed JADE problem $C_{\theta\theta}$, its lower bound in the large N, M regime $C_{\theta\theta}^*$ and the CRB of the traditional

JADE problem $C_{\theta\theta}^{\text{trad}}$. Notice that $C_{\theta\theta}^{*}$ converges towards $C_{\theta\theta}^{\text{trad}}$ at high SNR given $M = 5$ subcarriers and $N = 2$ antennas as depicted in Fig. 3. As the number of subcarriers increase, we see that $C_{\theta\theta}$ and $C_{\theta\theta}^{*}$ coincide at any SNR with constant difference and that both converge towards the traditional CRB, $C_{\theta\theta}^{\text{trad}}$.

7. CONCLUSIONS

In this paper, we have extended the CRB of the partial relaxation framework to the case of joint angle and delay estimation (JADE). The exact closed form expressions of the Fisher Information Matrix (FIM), as well as the Cramér-rao Bound (CRB) are derived. Some interesting asymptotic results are then presented, which reveals desired properties and results of the partial relaxation framework, in the context of Joint Angle and Delay Estimation (JADE). Finally, the results are then analyzed through computer simulations.

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