Maximizing Diversity on Block–Fading Channels

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Abstract

This work considers the achievable diversity for coded systems appropriately characterized by a *block–fading* channel model. We are primarily interested in cases where the number of uncorrelated fading channel realizations(blocks) F, is small, so that ideal interleaving assumptions do not hold. This is usually the case in mobile radio systems which employ *coded slow frequency–hopping* such as the GSM system and its derivatives. We show that the diversity order is limited to a value less than or equal to F which depends on the code rate and the size of the signaling constellation. We report on the results of code searches for rate $1/n$ convolutional codes for simple AM constellations, which show the minimum complexity needed to achieve maximum diversity. We also present computer simulations of some codes in order to determine the effect of code complexity on the frame and bit error–rate performance.

1 Introduction

This work deals with so–called *block–fading channels*, which are models appropriate in very slowly fading situations (i.e. low mobile speed) and assume that the channel state is stationary for blocks of N symbols. In most applications like GSM [1] or IS54 [2] N is quite large and there is a constraint on the interleaving depth due to a maximum processing delay requirement. Even in the absence of such delay constraints, there may be a maximum number of channel realizations (for instance FDMA slots in GSM). Both amount to the same thing, namely that the number of blocks over which coding is performed, F, is small.

Most work dealing with code design for fading channels assumes an ideal interleaving situation [3, 4] which, in the context of block–fading channels, is equivalent to letting F tend to infinity. Codes designed in this fashion may be ineffective when applied to a system where F is small. An important exception is the work of Lapidoth in [5] where the construction of binary codes *matched* to the depth of the interleaver (or number of blocks) is addressed for an *erasure–channel* model of a fading channel. Another application of a block–fading model is for fast frequency–hopping systems where N is small (on the order of a few symbols). The work of Kaplan, *et al.* in [6] considers coding for these systems *without* a constraint on the number of frequencies over which the signal can hop. This amounts again to an ideal interleaving situation, where the fast frequency–hopping pattern takes the place of the interleaver.

2 System Model and Performance Measures

Consider transmission scheme in Fig. 1. The information bits are coded/modulated into F blocks of length N symbols. We consider codewords having length NF and information rate R bits/dim, which are denoted by $\mathbf{c} = (c_{0,0}\ c_{0,1}\ \cdots\ c_{0,N-1}\ c_{1,0}\ \cdots\ c_{F-1,N-1}) = (\mathbf{c}_0\ \cdots\ \mathbf{c}_{F-1})$. When the blocks are long (i.e. N is large), the coded symbols are formed by a combination of either a block or convolutional encoder and an interleaver. The interleaver serves to spread the information evenly over the F blocks so that very high complexity codes are not needed. Except when explicitly stated otherwise, we will consider the interleaver as part of the encoder.

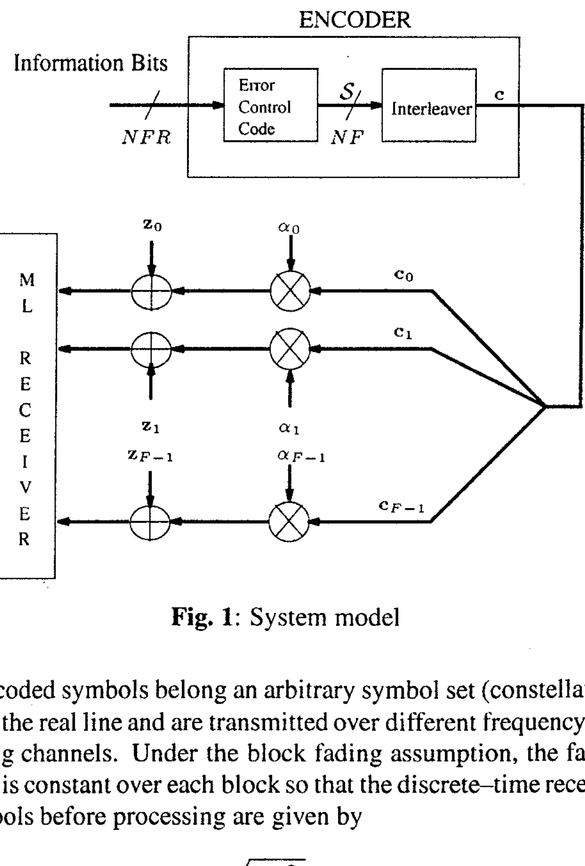

Fig. 1: System model

The coded symbols belong an arbitrary symbol set (constellation) $\mathcal{S}$ on the real line and are transmitted over different frequency–flat fading channels. Under the block fading assumption, the fading level is constant over each block so that the discrete–time received symbols before processing are given by

$$r_{f,n} = \sqrt{\alpha_f \mathcal{E}_s} c_{f,n} + z_{f,n}, \tag{1}$$

for $n = 0, 1, \cdots, N-1$, $f = 0, 1, \cdots, F-1$ where α_f is the random unit–mean strength of the f^{th} channel, $\mathcal{E}_s$ is the energy

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per coded symbol and $z_{f,n}$ is a real zero–mean Gaussian random variable with variance $N_0/2$.

Coding across different channel realizations provides a certain amount diversity, which counters the effects of fading. In what follows we will assume that the F channel realizations are uncorrelated. In system such as GSM, for instance, blocks modulate $F = 4$ (half–rate) or $F = 8$ (full–rate) carriers whose spacing is larger than the coherence bandwidth, resulting in virtually uncorrelated blocks. Moreover, for reasonable mobile speeds, the channel is stationary during the block, so that the block fading assumption holds. The practical advantages of such a system are firstly that reliable coherent communication is possible. Secondly and more importantly, the amount of diversity is independent of the rate of channel variation, since it is a result of exploiting frequency–selectivity. For mobile telephony, this is crucial since the majority of calls are made at low speed. In the IS54 standard coding is performed across $F = 2$ TDMA frames so that the blocks start to become less correlated for high mobile speeds.

The received signal is processed by a maximum–likelihood decoding rule as

$$\arg\min_{m=0,\cdots,2^{FNR}-1} \sum_{f=0}^{F-1}\sum_{n=0}^{N-1} \left| r_{f,n} - \sqrt{\alpha_f \mathcal{E}_s} c_{f,n} \right|^2 . \quad (2)$$

The pairwise error probability (PEP) between two codewords $\mathbf{c}^{(a)}$ and $\mathbf{c}^{(b)}$ conditioned a set of F channel strengths is given by

$$\Pr\left(\mathbf{c}^{(a)} \to \mathbf{c}^{(b)} \middle| \{\alpha_f\} \right) = \mathrm{Q}\left(\sqrt{d^2(a,b)\frac{\mathcal{E}_s}{2N_0}} \right), \quad (3)$$

where $d^2(a,b)$ is the Euclidean distance between the code sequences weighted by the channel strengths and is given by

$$d^2(a,b) = \sum_{i=0}^{F-1} \alpha_f d^2(\mathbf{c}_f^{(a)}, \mathbf{c}_f^{(b)}), \quad (4)$$

We assume now that the $\sqrt{\alpha_f}$ are Rayleigh distributed so that the random variable $z = d^2(a,b)\mathcal{E}_s/2N_0$ has characteristic function

$$\Phi_z(s) = \prod_{i=0}^{d_{\mathrm{H}}^F - 1} \frac{1}{1 - sd^2(\mathbf{c}_i^{(a)}, \mathbf{c}_i^{(b)})\mathcal{E}_s/2N_0} \quad (5)$$

where d_{H}^F is the number of non–zero $d^2(\mathbf{c}_f^{(a)}, \mathbf{c}_f^{(b)})$. By using the Chernov bound on the Q–function, $\mathrm{Q}(x) \le \frac{1}{2}e^{-x^2/2}$, we may bound the PEP as

$$\begin{aligned} \Pr\left(\mathbf{c}^{(a)} \to \mathbf{c}^{(b)} \right) &\le \frac{1}{2}\mathrm{E}_z\left(e^{-\frac{z}{2}}\right) \\ &= \frac{1}{2}\Phi_z\left(-\frac{1}{2}\right) \quad (6) \\ &< \frac{1}{2}\left(\frac{4N_0}{\chi\mathcal{E}_s}\right)^{d_{\mathrm{H}}^F} . \quad (7) \end{aligned}$$

We see that the upper–bound to the pairwise error probability is specified asymptotically by d_{H}^F which is slope of the probability of error vs. SNR on a log-log scale, and χ for the SNR gain factor. The slope is commonly referred to as the diversity order. In order to maximize the asymptotic performance the goal of any coding system is therefore to maximize d_{H}^F.

The limitations of the frequency–flat Rayleigh fading model are clear, since we have made no assumptions regarding multipath. In medium-band systems like GSM where the multipath induces intersymbol interference, a sub–optimal receiver is used which first equalizes the F channels with a soft–output algorithm (e.g. soft–output Viterbi equalization [7]). These outputs are then deinterleaved and passed to a Viterbi decoder to retrieve the information bits. In this case, the channel strengths are not Rayleigh distributed. In narrow-band systems such as IS-54, the channel is almost always ISI–free, and either a very simple equalizer or none at all is needed prior to deinterleaving/decoding. Here, the Rayleigh assumption is valid. In wide-band systems with little ISI, equalization is also not required and some of the multipath can be exploited with a RAKE receiver prior to decoding. Here, the diversity is a product of d_{H}^F and the number of resolvable paths. The gain factor χ also involves the average strengths of the resolved components. For a more complete discussion of the effect of multipath see [8]. In general, we can say that the performance will lie between those of unresolved and resolved systems for which d_{H}^F is the diversity factor due to coding.

3 Maximum Code Diversity

This section addresses the issue of determining maximum code diversity (d_{H}^F) for a given number of uncorrelated blocks and information rate. For binary modulation, such codes may or may not exhibit maximum (free) Hamming distance, and, in general, $d_{\mathrm{H}}^F \le d_{\mathrm{free}}$. A simple example is the rate 1/2 binary convolutional code with binary modulation employed in the full–rate GSM standard shown in Fig.2. The output bits are interleaved over 8 blocks transmitted on widely spaced carriers. The minimum free Hamming distance path ($d_{\mathrm{free}} = 7$) (after deinterleaving) is shown along with the blocks in which each bit were transmitted. It is clear that this path achieves $d_{\mathrm{H}}^8 = 5$. It turns out that this is also the minimum diversity path for this code and, moreover, that there is no other code which achieves a higher diversity with binary modulation and $R = 1/2$ bits/dim.

3.1 Maximum Diversity Bound

In order to determine an upper–bound on the minimum pairwise d_{H}^F, it is convenient to group together the N symbols which are transmitted in the same block, and view them as a super-symbol over $\mathcal{S}^N$. The codeword is then a vector of length F super-symbols. Using this interpretation, d_{H}^F is simply the Hamming distance in $\mathcal{S}^N$. This reduces the analysis to one of non–binary block codes with a fixed block length F, and therefore all traditional bounding techniques apply. The most appropriate bound in this case is the Singleton bound [9] which in this context can be expressed as

$$d_{\mathrm{H}}^F \le 1 + \left\lfloor F\left(1 - \frac{R}{\log_2|\mathcal{S}|}\right)\right\rfloor . \quad (8)$$

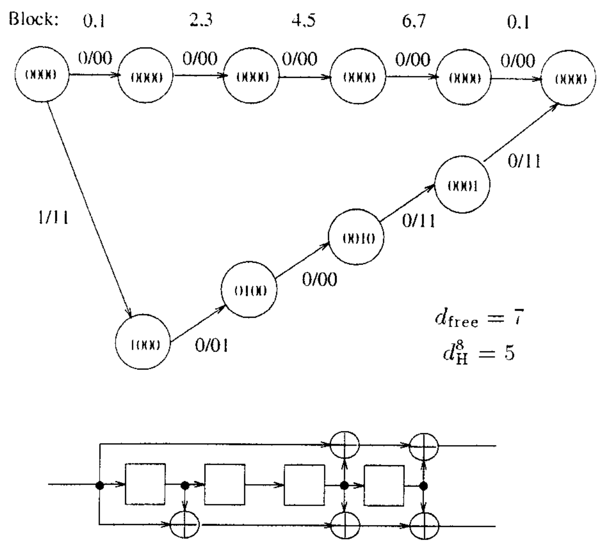

Fig. 2: Minimum diversity/weight error event for full–rate GSM.

The reason for its importance is that we will see that it can almost always be met with reasonable complexity.

The first interesting result of this analysis is that the shape of the constellation is not important with regard to the code diversity since it is a completely algebraic measure of the performance. Secondly, and more importantly, we see that a small amount of constellation expansion yields a significant diversity increase. Take for example transmission at $R = .5$ bits/dim over $F = 8$ blocks as in full–rate GSM . With binary modulation ($|\mathcal{S}| = 2$), the maximum pairwise diversity is 5, which incidentally is what is achieved by the coding scheme used in GSM. With quaternary modulation we see that it can be increased to 7. On the downside, for high code rates (> 2 bits/dim) very large symbol alphabets are required to achieve high asymptotic diversity. For example, with $F = 8$ and $R = 3$ bits/dim, a 16-point constellation can only achieve a diversity of $d_{\text{H}}^{8} = 3$. To achieve $d_{\text{H}}^{8} = 7$ a constellation with 4096 points is needed.

3.2 Convolutional Codes

The Singleton bound is also applicable to convolutional codes, since they can always be interpreted as very long block codes. In fact, in systems like GSM the convolutional codes are used in a block fashion by appending trailing zeros to the information sequence, and a one–shot decoding of the entire block is performed. It is worthwhile to perform a code search using d_{H}^{F} as a primary performance criterion rather than d_{free}. At the same time we determine the number of states needed to achieve the maximum diversity indicated by the Singleton bound. We have performed a search for maximum diversity rate 1/4, 1/3 and 1/2 codes for a varying numbers of frequencies and states. Because of space limitations we could not list all the codes which that were found. Table 1 lists some of the codes related to the simulations described in the next section. The code search first maximized the diversity order and then χ_{min}. The generator polynomials are shown using the convention given in [10]. In general, we have found that for low spectral efficiency (<2 bits/dim) fairly simple codes achieve maximum diversity. Those highlighted in bold type in the Table achieve the Singleton bound. As an example, for the case of $R = .5$ bits/dim with $F = 8$, we can achieve maximum diversity ($d_{\text{H}}^{F} = 5$) with an eight–state code, and moreover, it turns out that it does not exhibit maximum free Hamming distance ($d_{\text{free}} = 5$, not 6). The most important reason for increasing complexity, as we will see in section 4, is that we can achieve larger values of χ_{min} yielding significant coding gains in the frame error–rate performance.

States	$F = 4$			$F = 8$		
$R = .5$ bits/dimension (2–AM)						
States	d_{H}^{4}	χ_{min}	gen.	d_{H}^{8}	χ_{min}	gen.
4	**3**	**6.35**	**5,7**	4	5.66	5.7
8	**3**	**10.08**	**31,51**	**5**	**4.00**	**11,31**
16	**3**	**13.21**	**32,13**	**5**	**5.28**	**13,33**
32	**3**	**14.54**	**75,57**	**5**	**8.19**	**54,33**
64	**3**	**17.93**	**721,561**	**5**	**10.90**	**111,771**
$R = 1$ bit/dimension (4–AM)						
States	d_{H}^{4}	$\chi_{\text{min}}^{\text{lb}}$	gen.	d_{H}^{8}	$\chi_{\text{min}}^{\text{lb}}$	gen.
4	**3**	**1.6**	**5,7**	3	2.02	5.7
8	**3**	**2.02**	**11,31**	4	1.60	31,51
16	**3**	**2.78**	**23,73**	**5**	**1.27**	**23,71**
32	**3**	**3.33**	**54,73**	**5**	**1.84**	**75,26**
64	**3**	**4.25**	**631,571**	**5**	**2.21**	**741,361**
$R = .5$ bits/dimension(4–AM)						
States	d_{H}^{4}	$\chi_{\text{min}}^{\text{lb}}$	gen.	d_{H}^{8}	$\chi_{\text{min}}^{\text{lb}}$	gen.
4	**4**	**2.58**	**5,7,3,7**	6	2.02	5.7.3.7
8	**4**	**3.76**	**11,31,51,61**	**7**	**1.77**	**11,60,50,61**
16	**4**	**4.60**	**32,72,13,53**	**7**	**2.55**	**32,53,50,33**
32	**4**	**5.55**	**35,75,16,57**	**7**	**3.27**	**14,57,37,54**
64	**4**	**6.04**	**311,171,551,561**	**7**	**3.79**	**541,131,701,750**

Table 1 Rate 1/2 and 1 bit/dimension convolutionally coded 2 and 4-AM modulations

We now turn to a simple example of convolutional code design for non–binary constellations in order to take advantage of the diversity gain offered by constellation expansion. The first important point to take into account is that the diversity level is *independent* of the shape of the constellation and that only its cardinality matters. This is not the case for coded modulation schemes with Euclidean distance as a performance indicator. For illustrative purposes, we therefore consider linear binary convolutional codes with a 4–AM constellation shown in Fig. 3. We assume a *Gray mapping* of adjacent output bits to points in the constellation for rate 1/2 and 1/4 codes (1 and .5 bits/dim). The use binary linear codes simplifies the code search since d_{H}^{F} preserves the linearity of the code (i.e. $d_{\text{H}}^{F}(\mathbf{c}_a, \mathbf{c}_b) = d_{\text{H}}^{F}(0, \mathbf{c}_a \oplus \mathbf{c}_b)$ where each symbol is now composed of 2 bits). We assume further that interleaving over the F blocks is performed on the symbol level. The secondary performance measure in the PEP, χ_{min} depends on the F Euclidean distances between the sub–codewords transmitted in each block, which for 4–AM do not preserve the linearity of the code. As a result of the Gray mapping, we may "fine–tune" the code search by maximizing a lower–bound on χ_{min} which preserves the linearity of the code (i.e. $\chi(\mathbf{c}_a, \mathbf{c}_b) \geq \chi^{\text{lb}}(\mathbf{c}^{(a)}, \mathbf{c}^{(b)}) = \chi(\mathbf{0}, \mathbf{c}_a \oplus \mathbf{c}_b)$. χ^{lb} is a lower bound since under the Gray mapping $d^2(a, b) \geq d^2(0, a \oplus b), \forall a, b$. Some codes using this construction are given in Table 1.

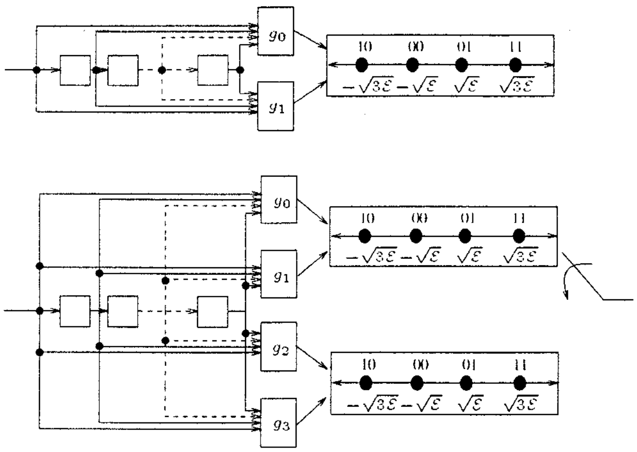

Fig. 3: 4–AM Coding Example

4 Performance Comparison of Various Codes

In order to assess the performance of some of the codes reported in this work, we resort to computer simulations. We have found that a union–bound approach for assessing the performance analytically yields quite unfruitful results for convolutional codes. The reason is that as we approach the maximum diversity F, the number of paths which share a given diversity level increases quickly as we progress through the code's trellis and enumerating them becomes difficult since they cannot be discarded. This is especially true for practical block lengths ($N > 100$). In our simulations, we assumed a block length $N = 100n/\log_2|S|$ coded symbols, where n is the number of output bits of a rate $1/n$ binary convolutional code. We assume a single–path Rayleigh fading channel and soft–decision decoding with perfect channel state information.

The frame and bit–error rate performances for a variety of codes having $F = 4, 8$ at $R = .5$ bits/dim are shown in Figs. 4,5,6,7. We have shown the simulated performance of the 2 GSM codes which have generators (46,26) (full–rate, 16 states, $F = 8$) and (554,764) (half–rate, 64 states, $F = 4$). The $\chi_{\min}$ for these codes are only slightly less than those listed in Table 1 and have comparable performance. We see that increased complexity can yield non–negligible coding gains in the FER performance even though the diversity level is maximum. There is, however, less improvement in the BER performance. The 4AM codes yield significant performance gains, especially in the BER performance, due to the increased diversity level.

5 Conclusion

This work considered coding for block–fading channels with small number of blocks. This channel model has significant practical importance for delay–constrained block–oriented communications, a category in which many mobile radio systems fall. The slow frequency–hopping scheme used in the current GSM mobile radio systems is a prime example. It is reasonable to assume that next generation systems will also use similar, and perhaps more complex techniques.

We discussed the attainable diversity due to coding. We showed that there is a upper–limit to the diversity which depends on the number of blocks, the code rate and the size of the signaling constellation. This relies on what turns out to be a disguised version of the Singleton bound, which indicates the maximum achievable asymptotic diversity for a code of a given rate. It indicates that diversity is asymptotically limited and that it can be increased by constellation expansion. A rather unfortunate result is that for high spectral–efficiency systems, in order to achieve a high asymptotic diversity level, very large constellations are required. We gave examples of convolutional codes for simple AM constellations, which achieve maximum diversity. An important result is that maximum diversity can be achieved with rather simple codes and that, in terms of bit error–rate performance, increased complexity does not yield significant gains. This is not true, however, for the frame–error rate performance, which is often important in both speech and data applications.

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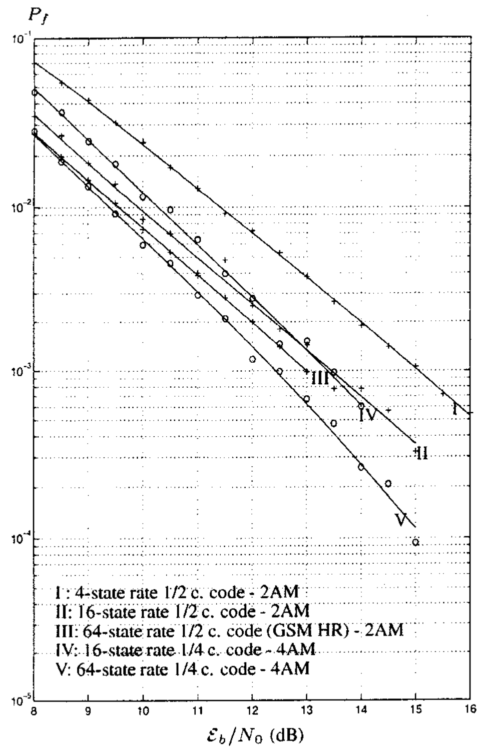
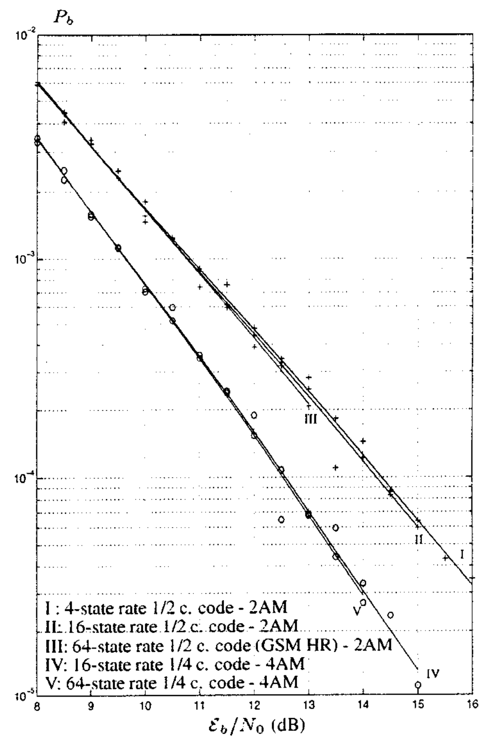


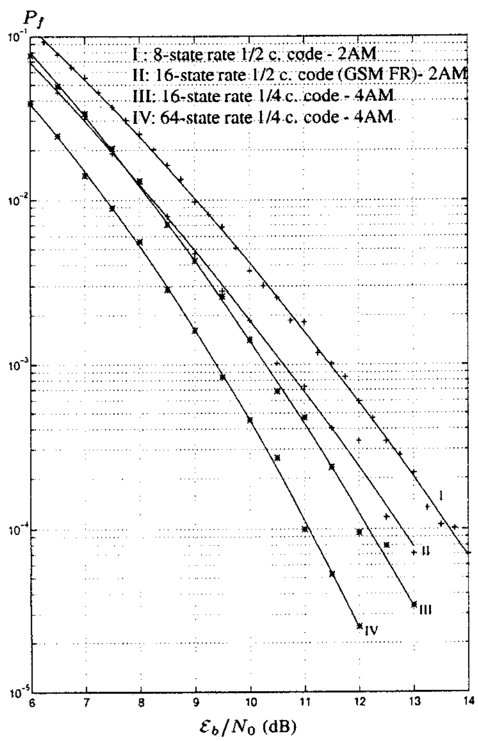
(a) $F = 4$

Fig. 4: Frame Error Probabilities (F=4)



(a) $F = 4$

Fig. 6: Bit Error Probabilities (F=4)



(b) $F = 8$

Fig. 5: Frame Error Probabilities (F=8)

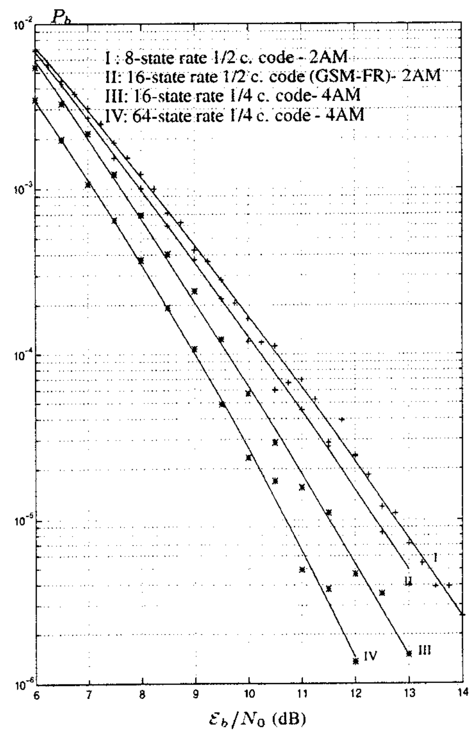


Fig. 7: Bit Error Probabilities (F=8)