# ON MAXIMUM LIKELIHOOD ANGLE OF ARRIVAL ESTIMATION USING ORTHOGONAL PROJECTIONS

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## ABSTRACT

We present a novel and efficient approach for estimating the maximum likelihood (ML) estimates of the angles-of-arrival (AoAs) of multiple sources. The approach is iterative and is based on orthogonal projections in order to optimise the ML cost function, thus the name OPML. As will be shown, the advantage of using an orthogonal basis of the signal manifold would allow solving the ML cost function in an iterative manner. In fact, we propose two algorithms based on OPML, i.e. OPML-1 and OPML-2, which exhibit lower computational complexity and faster convergence than existing ML algorithms. In this paper, we discuss the idea of OPML and its two implementations, followed by simulation results to demonstrate their performance.

***Index Terms*—** Maximum Likelihood, Angle of Arrival, Orthogonal Projections, Alternating Optimization

## 1. INTRODUCTION

The estimation of the angles of arrival, or AoAs, of multiple sources is a well known problem in the context of array signal processing. In fact, this problem emanates in many engineering applications such as navigation, tracking of objects, radar, sonar, and wireless communications [1]. One of the first investigated techniques to deal with the AoA problem was by using Maximum Likelihood [2]. Nonetheless, it didn't receive much attention due to the high computational load of the multivariate nonlinear maximisation problem involved, since it requires a $q$-dimensional search, where $q$ is the number of signals. To cope with this issue, a tradeoff has been done between complexity and performance, hence suboptimal techniques with reduced complexity have dominated the field. The most famous ones are: Minimum Variance Distortionless Response (MVDR) by Capon [3], followed by Multiple Signal Classification (MUSIC) developed in [4] and [5], independently. Also, less complex algorithms were implemented to replace the 1-D search of MUSIC by a polynomial root finding process [6], or a least squares fit [7]. The performance of these algorithms are inferior to the ML technique. In addition, these suboptimal methods can not resolve coherent sources, which is the case of a specular multipath channel. Therefore, if a single user was transmitting a signal in such a channel, then all the above techniques (except for ML) couldn't properly estimate the signal parameters, unless a preprocessing of the data is done, such as spatial smoothing [8].

In the literature, existing techniques for obtaining ML estimates of the AoAs have been proposed. The iterative quadratic ML technique (IQML) was developed [9] to compute the ML estimates of the signal parameters such as AoAs, but the technique is only applicable to uniform linear antenna arrays. Also, the alternating projections method in [10] was used to obtain ML estimates of the AoAs. The expectation maximisation (EM) technique in [11] was used to update the signal parameters simultaneously to maximise the ML cost function. This may lead to slow convergence and a difficulty in maximisation steps due to coupling when smoothness penalties are used. These two problems were a motivation to the space-alternating generalised EM (SAGE) algorithm in [12] where the signal parameters are updated sequentially in small groups. Furthermore, an iterative method was developed in [13] to estimate the AoAs of incoming signals, but it is only applicable to signals with known waveforms. Other recent ML methods devoted to AoA estimation, mostly based on optimization heuristics, can be found in [14–22].

In this paper, we take a different approach. We express the signal manifold in an orthogonal basis, which allows implementing iterative algorithms to update the signal parameters, i.e. the AoAs, depending on the orthogonal basis chosen. In other words, we iteratively update the AoAs using Orthogonal Projections to optimise the ML cost function (OPML). The performance of the algorithm would depend on the choice of the orthogonal basis. For this reason, we have implemented two versions of OPML, i.e. OPML-1 and OPML-2. It turns out that OPML-1 gives the same estimates as the alternating projections method in [10]. Also, we propose a complexity reduction of OPML-1. Furthermore, simulations have shown that OPML-2 converges faster than OPML-1, and in some scenarions OPML-2 requires only 1 iteration to converge.

This paper is organised as follows. First, the general system model of the AoA problem is stated. In Section 3, a review of the deterministic ML estimator of the AoAs is presented. Then, the concept of using Orthogonal Projections for ML estimation (OPML) is introduced in Section 4, where both implementations OPML-1 (with its complexity reduction) and OPML-2 are described. In Section 5, a short discussion on OPML is given, followed by simulation results comparing the performance of OPML-1 and OPML-2 per iteration. We conclude the paper in Section 7.

**Notations:** Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^T$ and $(.)^{\mathrm{H}}$ represent the transpose and the transpose-conjugate operators. The matrix $\mathbf{I}_{\mathrm{N}}$ is the identity matrix of dimensions $N \times N$. The operators "$\mathrm{tr}\{\mathbf{X}\}$" and $\|\mathbf{X}\|$ denote the trace and *Frobenius* norm of a square matrix $\mathbf{X}$.

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## 2. SYSTEM MODEL

Assume a planar arbitrary array of $N$ antennas. Furthermore, consider $q < N$ narrowband sources attacking the array from different angles, i.e. $\mathbf{\Theta} = [\theta_1 \dots \theta_q]$. Collecting $L$ time snapshots and following [23], we can write

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{N} \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{\mathrm{N}\times\mathrm{L}}$ is the data matrix with $l^{th}$ time snapshot, $\mathbf{x}(t_l)$, stacked in the $l^{th}$ column of $\mathbf{X}$. The matrix $\mathbf{S} \in \mathbb{C}^{\mathrm{q}\times\mathrm{L}}$ is the source matrix. The steering matrix, or signal manifold, $\mathbf{A} \in \mathbb{C}^{\mathrm{N}\times\mathrm{q}}$ is composed of $q$ steering vectors, i.e. $\mathbf{A} = [\mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_q)]$. Each vector $\mathbf{a}(\theta_i)$ is the response of the array to a source impinging the array from direction $\theta_i$. The form of $\mathbf{a}(\theta_i)$ depends on the array geometry. The matrix $\mathbf{N} \in \mathbb{C}^{\mathrm{N}\times\mathrm{L}}$ is background noise. The noise is modelled as a white circular complex Gaussian process of zero mean and covariance $\sigma^2\mathbf{I}_{\mathrm{N}}$ and independent from the source signals. Before stating the problem, we admit that the number of sources $q$ is known a priori. The problem of estimating the number of sources is, in fact, a *detection* problem in signal processing. Techniques for estimating $q$ are found in [24–27]. Now, we are ready to address our *estimation* problem: *"Given the available snapshots $\mathbf{X}$ and the number of sources q, estimate the angles of arrival of the incoming signals, i.e. $\mathbf{\Theta}$."*

## 3. THE DETERMINISTIC MAXIMUM LIKELIHOOD ESTIMATOR

This section serves as a review of the *deterministic* ML estimator of the angles of arrival of the transmitting sources. For detailed derivations, the reader is referred to [10].

In a *deterministic* approach, the signal parameters (i.e. $\mathbf{S}$ and $\mathbf{\Theta}$) are modelled as unknown deterministic sequences, i.e. the signal parameters are assumed to be nonrandom and unknown. These quantities are jointly estimated through the criterion:

$$[\hat{\mathbf{\Theta}}^{\mathrm{ML}}, \hat{\mathbf{S}}^{\mathrm{ML}}] = \arg\min_{\mathbf{\Theta},\mathbf{S}} \|\mathbf{X} - \mathbf{A}\mathbf{S}\|^2 \tag{2}$$

The ML estimate of $\mathbf{\Theta}$, obtained by optimising (2) over $\mathbf{S}$ first, is given by

$$\hat{\mathbf{\Theta}}^{\mathrm{ML}} = \arg\max_{\mathbf{\Theta}} \operatorname{tr}\Big\{\mathscr{P}_{\mathbf{A}}\hat{\mathbf{R}}\Big\} \tag{3}$$

where $\mathscr{P}_{\mathbf{A}}$ is the projector onto the signal subspace, i.e. the space spanned by columns of $\mathscr{P}_{\mathbf{A}}$

$$\mathscr{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^{\mathrm{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{H}} \tag{4}$$

The matrix $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^{\mathrm{H}}$ is the sample covariance matrix of the received snapshots. The issue, here, is that the optimisation problem in (3) is computationally exhaustive, i.e. to obtain an ML estimate of $\mathbf{\Theta}$, one should go through a $q$-dimensional search, which is somewhat impossible. In the following section, we propose an approach to solve the problem using orthogonal projections.

## 4. ORTHOGONAL PROJECTIONS FOR ML ESTIMATION (OPML)

In this section, we present the OPML approach for estimating the angles of arrival of the incoming signals. We begin the section by exploiting a property of the projector matrix, i.e. $\mathscr{P}_{\mathbf{A}}$. It is easy to see that

$$\mathscr{P}_{\mathbf{A}}\mathbf{A} = \mathbf{A}(\mathbf{A}^{\mathrm{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{A} \tag{5a}$$

or, equivalently

$$\mathscr{P}_{\mathbf{A}}\mathbf{a}(\theta_i) = \mathbf{a}(\theta_i), \qquad i = 1 \dots q \tag{5b}$$

Equation (5) tells us that $\mathscr{P}_{\mathbf{A}}$ has $q$ eigenvalues equal to 1, with corresponding eigenvectors being the columns of $\mathbf{A}$. Note that the rank of $\mathscr{P}_{\mathbf{A}}$ is $q$, which means that the dimension of the null space of $\mathscr{P}_{\mathbf{A}}$ is $N - q$. The null space of $\mathscr{P}_{\mathbf{A}}$ is called the noise subspace. Since the columns of $\mathbf{A}$ are linearly independent, they could be expressed in an orthonormal basis of dimension $q$. Using the spectral representation theorem, we could write $\mathscr{P}_{\mathbf{A}} = \mathbf{V}\mathbf{V}^{\mathrm{H}}$ where $\mathbf{V} = [\mathbf{v}_1 \dots \mathbf{v}_q]$ and $\mathbf{V}^{\mathrm{H}}\mathbf{V} = \mathbf{I}_{\mathrm{N}}$. Plugging this representation of $\mathscr{P}_{\mathbf{A}}$ in (3), we can say

$$\hat{\mathbf{\Theta}}^{\mathrm{ML}} = \arg\max_{\mathbf{\Theta}} \operatorname{tr}\Big\{\mathbf{V}\mathbf{V}^{\mathrm{H}}\hat{\mathbf{R}}\Big\} = \arg\max_{\mathbf{\Theta}} \sum_{i=1}^{q} \mathbf{v}_i^{\mathrm{H}}\hat{\mathbf{R}}\mathbf{v}_i \tag{6}$$

We have made implicit that $\{\mathbf{v}_i\}_{i=1}^q$ are, indeed, functions of $\mathbf{\Theta}$. Referring to the second equality in (6), one could see that we have decoupled the $q$-dimensional maximisation problem into $q$ positive and quadratic functions in $\mathbf{v}_i$'s. The OPML approach is based on solving (6) by maximising each term of the form $\mathbf{v}_i^{\mathrm{H}}\hat{\mathbf{R}}\mathbf{v}_i$, iteratively. In the two subsections that follow, we propose two different implementations of the OPML approach, OPML-1 and OPML-2. Each approach is based on a different choice of the orthogonal basis $\mathbf{V}$.

### 4.1. First Implementation of OPML (OPML-1)

Since the columns of $\mathbf{A}$ are linearly independent and span the signal subspace, it turns out that we could orthonormalise the column vectors of $\mathbf{A}$ via sequence of matrix operations that can be interpreted as multiplication on the right by upper-triangular matrices [28]. These matrix operations could be done sequentially through *Gram-Schmidt* projections, i.e.

$$\mathbf{v}_k = \frac{(\mathbf{I}_{\mathrm{N}} - \sum_{i=1}^{k-1}\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}})\mathbf{a}(\theta_k)}{\|(\mathbf{I}_{\mathrm{N}} - \sum_{i=1}^{k-1}\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}})\mathbf{a}(\theta_k)\|} \tag{7}$$

starting from $k = 1$ till $q$. OPML-1 aims at maximising (6) using a *Gram-Schmidt* representation in (7), in an iterative fashion. Like any other iterative algorithm, an initialisation is needed to run the main loop of the algorithm. Before presenting OPML-1, we fix a notation that we use throughout the paper: let $\mathbf{Y}_i^{(k)}$ be a matrix of a defined dimension that denotes the value (or estimate) of the quantity $\mathbf{Y}_i$ at iteration $k$ and sub-iteration $i$. For the initialisation phase, we use $k = 0$.

#### *4.1.1. Initialisation of OPML-1*

In the initialisation phase of OPML-1, the estimate of $\theta_1^{(0)}$ is obtained by solving

$$\hat{\theta}_1^{(0)} = \arg\max_{\mathbf{v}_1} \mathbf{v}_1^{\mathrm{H}}\hat{\mathbf{R}}\mathbf{v}_1 = \arg\max_{\theta_1} \frac{\mathbf{a}^{\mathrm{H}}(\theta_1)\hat{\mathbf{R}}\mathbf{a}(\theta_1)}{\|\mathbf{a}(\theta_1)\|^2} \tag{8}$$

Note that $\hat{\theta}_1^{(0)}$ is nothing other than the conventional Bartlett [29] beamformer estimate of $\theta_1$. In an attempt of estimating $\theta_2^{(0)}$, one needs $\mathbf{v}_1^{(0)}$ to proceed. This vector is obtained as

$$\mathbf{v}_1^{(0)} = \frac{\mathbf{a}(\hat{\theta}_1^{(0)})}{\|\mathbf{a}(\hat{\theta}_1^{(0)})\|} \tag{9}$$

In general, one has the estimates $\{\hat{\theta}_j^{(0)}\}_{j<i}$ and the vectors $\{\mathbf{v}_j^{(0)}\}_{j<i}$ in order to get an estimate of $\theta_i^{(0)}$ and the vector $\mathbf{v}_i^{(0)}$, which is done by

$$\hat{\theta}_i^{(0)} = \arg\max_{\mathbf{v}_i} \mathbf{v}_i^H \hat{\mathbf{R}} \mathbf{v}_i = \arg\max_{\theta_i} \frac{\mathbf{a}^H(\theta_i)\mathscr{P}_i^{(0)}\hat{\mathbf{R}}\mathscr{P}_i^{(0)}\mathbf{a}(\theta_i)}{\|\mathscr{P}_i^{(0)}\mathbf{a}(\theta_i)\|^2} \tag{10a}$$

where

$$\mathscr{P}_i^{(0)} = \mathbf{I}_{\mathrm{N}} - \sum_{j=1}^{i-1} \mathbf{v}_j^{(0)}\mathbf{v}_j^{(0)\mathrm{H}} \quad \text{with} \quad \mathbf{v}_i^{(0)} = \frac{\mathscr{P}_i^{(0)}\mathbf{a}(\hat{\theta}_i^{(0)})}{\|\mathscr{P}_i^{(0)}\mathbf{a}(\hat{\theta}_i^{(0)})\|} \tag{10b}$$

#### 4.1.2. Main Loop of OPML-1

After the initialisation phase, one has access to the quantities $\hat{\theta}_i^{(0)}$, $\mathbf{v}_i^{(0)}$, and $\mathscr{P}_i^{(0)}$ for all $i = 1 \dots q$. In what follows, we omit the superscript $(k)$ from $\mathbf{v}_i^{(k)}$, i.e. $\mathbf{v}_i$, because we update all these quantities, jointly, per sub-iteration. Now, in order to estimate $\hat{\theta}_i^{(k)}$, one should apply Gram-Schmidt orthogonalisation to the vectors $\big\{\mathbf{a}(\hat{\theta}_1^{(k)}) \dots \mathbf{a}(\hat{\theta}_{i-1}^{(k)}), \mathbf{a}(\hat{\theta}_{i+1}^{(k-1)}) \dots \mathbf{a}(\hat{\theta}_q^{(k-1)})\big\}$, then treat $\mathbf{a}(\hat{\theta}_i^{(k)})$ as if it were the last vector to be orthogonalised, viz.

$$\hat{\theta}_i^{(k)} = \arg\max_{\theta_i} \mathbf{v}_i^{\mathrm{H}} \hat{\mathbf{R}} \mathbf{v}_i = \arg\max_{\theta_i} \frac{\mathbf{a}^{\mathrm{H}}(\theta_i)\mathscr{P}_i^{(k)}\hat{\mathbf{R}}\mathscr{P}_i^{(k)}\mathbf{a}(\theta_i)}{\|\mathscr{P}_i^{(k)}\mathbf{a}(\theta_i)\|^2} \tag{11a}$$

where

$$\mathscr{P}_i^{(k)} = \mathbf{I}_{\mathrm{N}} - \sum_{\substack{j=1 \\ j\neq i}}^{q} \mathbf{v}_j\mathbf{v}_j^{\mathrm{H}} \tag{11b}$$

and, sequentially compute $\big\{\mathbf{v}_p\big\}_{\substack{p=1 \\ p\neq i}}^{q}$ as

$$\begin{aligned}
\mathbf{v}_p &= \frac{(\mathbf{I}_{\mathrm{N}} - \sum_{j=1}^{p-1} \mathbf{v}_j\mathbf{v}_j^H)\mathbf{a}(\hat{\theta}_p^{(k)})}{\|(\mathbf{I}_{\mathrm{N}} - \sum_{j=1}^{p-1} \mathbf{v}_j\mathbf{v}_j^H)\mathbf{a}(\hat{\theta}_p^{(k)})\|}, \qquad \text{if } p < i \\
\mathbf{v}_p &= \frac{(\mathbf{I}_{\mathrm{N}} - \sum_{\substack{j=1 \\ j\neq i}}^{p-1} \mathbf{v}_j\mathbf{v}_j^H)\mathbf{a}(\hat{\theta}_p^{(k-1)})}{\|(\mathbf{I}_{\mathrm{N}} - \sum_{\substack{j=1 \\ j\neq i}}^{p-1} \mathbf{v}_j\mathbf{v}_j^H)\mathbf{a}(\hat{\theta}_p^{(k-1)})\|}, \quad \text{if } p > i
\end{aligned} \tag{11c}$$

The operations in equation (11) are done sequentially from $i = 1$ till $q$, at iteration $k$. After updating all quantites, one could move to the next iteration ($k \leftarrow k+1$). The algorithm terminates upon $\|\mathbf{\Theta}^{(k+1)} - \mathbf{\Theta}^{(k)}\| < \varepsilon$, with $\mathbf{\Theta}^{(k)} = [\hat{\theta}_1^{(k)} \dots \hat{\theta}_q^{(k)}]^{\mathrm{T}}$ and $\varepsilon$ being a pre-defined threshold. It is important to note the following: one could analytically prove that OPML-1 gives the same estimates per iteration and sub-iteration of $\hat{\theta}_i^{(k)}$ as the alternating projections method in [10]. The proof has been omitted due to lack of space.

#### 4.1.3. Reduced Complexity of OPML-1

In the main loop of the OPML-1 algorithm, we notice that a Gram-Schmidt orthogonalisation for $q-1$ vectors of dimension $N$ is needed in order to estimate the angle of arrival of the $i^{th}$ source at the $k^{th}$ iteration, i.e. $\hat{\theta}_i^{(k)}$. To be more precise, these vectors are $\big\{\mathbf{a}(\hat{\theta}_1^{(k)}) \dots \mathbf{a}(\hat{\theta}_{i-1}^{(k)}), \mathbf{a}(\hat{\theta}_{i+1}^{(k-1)}) \dots \mathbf{a}(\hat{\theta}_q^{(k-1)})\big\}$. The Gram-Schmidt process to $q-1$ vectors of dimension $N$ costs $\mathcal{O}\big(N(q-1)^2\big)$ operations.

At an iteration $k$ and sub-iteration $i$, it is possible to update the orthogonal vectors $\big\{\mathbf{v}_1 \dots \mathbf{v}_{i-1}, \mathbf{v}_{i+1} \dots \mathbf{v}_q\big\}$ by a technique that demands $\mathcal{O}\big(N(q-1)\big)$ operations. The updates are based on *Givens* matrices and the Gram-Schmidt process with reorthogonalisation. We refer the reader to [30] for details on QR updates with real-valued vectors. An extension to the complex case is straightforward.

### 4.2. Second Implementation of OPML (OPML-2)

One could, indeed, choose another orthogonal basis $\mathbf{U} = [\mathbf{u}_1 \dots \mathbf{u}_q]$ (we use $\mathbf{U}$ instead of $\mathbf{V}$ to avoid confusion) that spans the signal subspace to solve (6) in an iterative fashion. Consider the following basis

$$\mathbf{u}_k = \frac{(\mathbf{I}_{\mathrm{N}} - \sum_{\substack{i=1 \\ i\neq k}}^{q} \mathbf{u}_i\mathbf{u}_i^{\mathrm{H}})\mathbf{a}(\theta_k)}{\|(\mathbf{I}_{\mathrm{N}} - \sum_{\substack{i=1 \\ i\neq k}}^{q} \mathbf{u}_i\mathbf{u}_i^{\mathrm{H}})\mathbf{a}(\theta_k)\|}, \quad \forall k = 1 \dots q \tag{12}$$

We claim that the basis in (12) is an orthogonal basis of the columns of $\mathbf{A}$. The proof has been omitted due to lack of space. Note that (12) corresponds to a 2-sided orthogonalisation of the array responses, compared to the 1-sided orthogonalisation in (7). We use the same initialisation for OPML-2 as OPML-1, thus we have the following quantities $\hat{\theta}_i^{(0)}$, $\mathbf{u}_i^{(0)} = \mathbf{v}_i^{(0)}$, and $\mathscr{P}_i^{(0)}$ for all $i = 1 \dots q$. In order to solve (6) using the orthogonal basis in (12), we propose to do the following at iteration $k$ and sub-iteration $i$:

$$\hat{\theta}_i^{(k)} = \arg\max_{\theta_i} \mathbf{u}_i^{\mathrm{H}} \hat{\mathbf{R}} \mathbf{u}_i = \arg\max_{\theta_i} \frac{\mathbf{a}^{\mathrm{H}}(\theta_i)\mathscr{P}_i^{(k)}\hat{\mathbf{R}}\mathscr{P}_i^{(k)}\mathbf{a}(\theta_i)}{\|\mathscr{P}_i^{(k)}\mathbf{a}(\theta_i)\|^2} \tag{13a}$$

where

$$\mathscr{P}_i^{(k)} = \mathbf{I}_{\mathrm{N}} - \sum_{j=1}^{i-1} \mathbf{u}_j^{(k)}\mathbf{u}_j^{(k)\mathrm{H}} - \sum_{j=i+1}^{q} \mathbf{u}_j^{(k-1)}\mathbf{u}_j^{(k-1)\mathrm{H}} \tag{13b}$$

and

$$\mathbf{u}_i^{(k)} = \frac{\mathscr{P}_i^{(k)}\mathbf{a}(\hat{\theta}_i^{(k)})}{\|\mathscr{P}_i^{(k)}\mathbf{a}(\hat{\theta}_i^{(k)})\|} \tag{13c}$$

## 5. DISCUSSION

We see that the concept of OPML is to update $\hat{\theta}_i^{(k)}$ in a successive manner through the orthogonal basis chosen. The orthogonal basis of the signal manifold used for OPML-1 is given in (7), i.e. a *Gram-Schmidt* basis. For OPML-2, the orthogonal basis chosen is expressed in (12). We have noticed that not only does this basis (used for OPML-2) reduce the complexity operations per subiteration, but also converges faster than OPML-1. It is important to state that other implementations of OPML are possible due to the fact that an orthogonal basis of an arbitrary subspace is not unique. In the next section, we present our simulation results, where it is shown that the performance of OPML-2 at iteration 1 is close to that of iterations 3 and 5 of OPML-1.

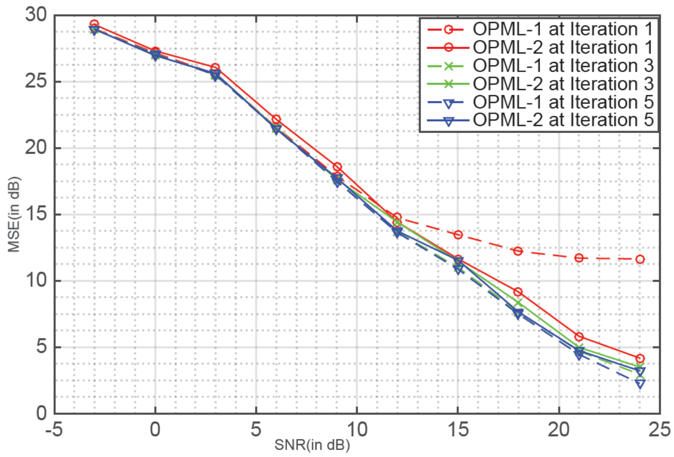


**Fig. 1**: Two sources impinging the array from directions 0° and 20°. The number of snapshots is 10.

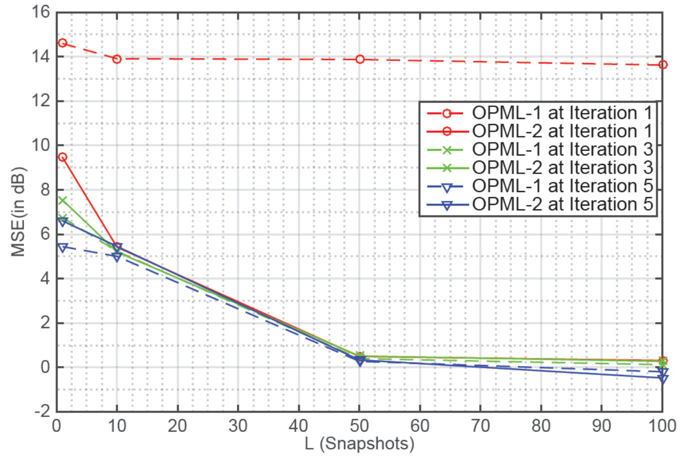


**Fig. 2**: Two sources impinging the array from directions 0° and 20°. The SNR is 20 dB.

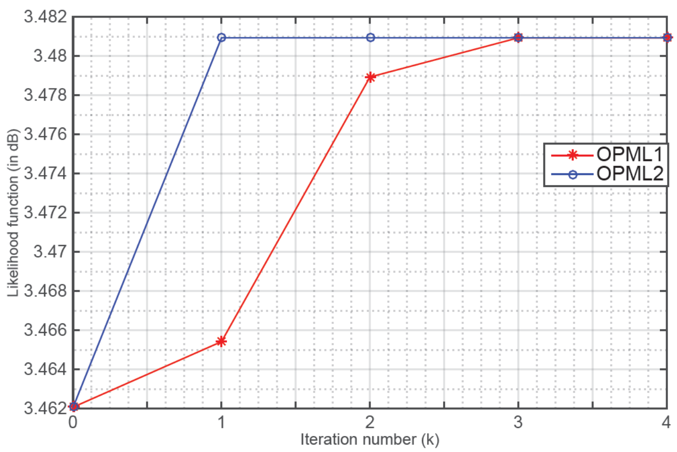


**Fig. 3**: Two sources impinging the array from directions 0° and 20°. The SNR is 20 dB. The number of snapshots is 100.

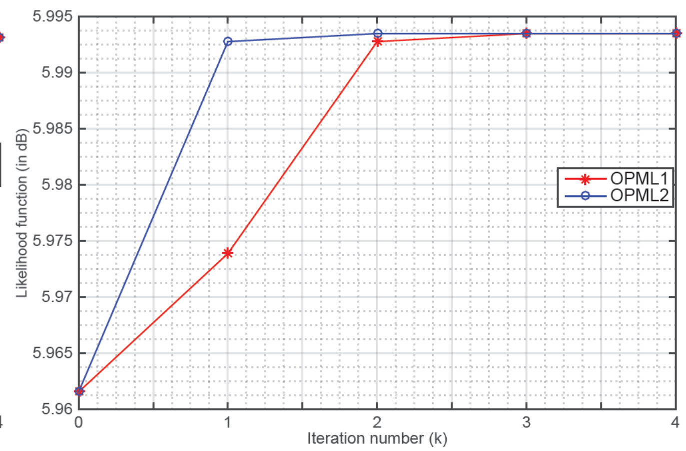


**Fig. 4**: Four sources impinging the array from directions 0°,20°,50°, and 70°. The SNR is 20 dB. The number of snapshots is 100.

## 6. SIMULATION RESULTS

In the first three experiments, the array was linear and uniform with 3 antennas spaced half a wavelength apart. Also, two un-correlated and narrow-band sources impinging the array from directions $\theta_1 = 0^\circ$ and $\theta_2 = 20^\circ$ were fixed, and the noise was additive complex Gaussian with average power $\sigma^2$. The SNR (in dB) is defined as $10 \log \frac{P}{\sigma^2}$, where $P$ is the power of both signals. Finally, 100 Monte-Carlo simulations were done in order to compute the Mean Squared Error (MSE) of the angles of arrival, $\theta_1$ and $\theta_2$.

In the first experiment, i.e. figure 1, we have plotted the MSE of the estimates of $\theta_1$ and $\theta_2$ produced by both OPML-1 and OPML-2 at iterations 1,3, and 5. The number of snapshots is $L = 10$. Recall that iteration 1 is the first iteration, after initialisation. Also, recall that the same initialisation is used for OPML-1 and OPML-2. We notice that there is an MSE difference of about 1 dB between the $1^{st}$ iteration of OPML-2 and the $3^{rd}$ or $5^{th}$ iteration of OPML-1 and OPML-2.

In the second experiment, i.e. figure 2, the MSE has been plotted with respect to the number of snapshots at SNR = 20 dB. We observe the same phenomena as in experiment one, i.e. the $1^{st}$ iteration of OPML-2 performs as good as the $3^{rd}$ or $5^{th}$ iteration of OPML-1 and OPML-2. This means that OPML-2 offers a faster convergence in this scenario.

In the third experiment, we have plotted the likelihood function (equation (3)) as a function of iteration number in figure 3. We have done the same plot in experiment four (see figure 4), but this time 4 sources with 6 antennas were used. The sources imping the array from directions 0°,20°,50°, and 70°. One can see that one iteration of OPML-2 is close to the convergence point.

## 7. CONCLUSION

This paper presents a novel and efficient approach to compute the maximum likelihood estimates of the angles of arrival of the incoming signals. The approach suggests to use an orthogonal basis of the signal manifold and to solve iteratively for the signal parameters, in a successive manner. Furthermore, we have implemented two algorithms based on OPML, i.e. OPML-1 and OPML-2, even though other implementations are possible by representing the signal manifold in another arbitrary orthogonal basis and updating the signal parameters, accordingly. OPML-1 turns out to give the same estimates per iteration as the alternating optimization method. Finally, the second implementation, OPML-2, corresponding to two-sided orthogonalization, converges faster than OPML-1, which is one-sided (Gram-Schmidt) orthogonalization.

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