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Documents de travail GREDEG
GREDEG Working Papers Series

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GREDEG WP No. 2017-24
https://ideas.repec.org/s/gre/wpaper.html

On non-monetary incentives for the provision of public goods*

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GREDEG Working Paper No. 2017–24

Abstract

We propose a non-monetary incentive mechanism to encourage high levels of contribution in public good provision. Based on a generic public good game, we implement a variation that imposes a minimum individual contribution level and offers individuals the choice between respecting it if they decide to contribute, or contributing zero. Restricting the individuals' strategy space in that way can stimulate them toward higher efforts while leaving them the possibility of contributing zero ensures that such efforts remain voluntary. We investigate how to tune the minimum contribution level in order to maximize the total contribution and to reach a stable outcome where no individual has incentive to free-ride. Exploiting the potential nature of the game, we show that one can set the minimum contribution level such that there exists a unique potential maximizer equilibrium in which all the individuals contribute to the public good.

Our work is of particular relevance to the growing field of information economics. Specifically, we provide an application of our model to data analytics projects using information with privacy implications, a domain where individuals (and regulatory provisions) consider as fundamental to be able to exercise control and where monetary compensation has so-far received little traction in practical scenarios.

Keywords: Public goods, Potential games, Non-monetary incentives, Minimum contribution level

JEL Code: C72, H41

*This work was partially funded by the French Government (National Research Agency, ANR) through the "Investments for the Future" Program reference # ANR-11-LABX- 0031-01; and by the Institut Mines-Telecom through the "Futur & Rupture" program. We acknowledge funding from the Alexander von Humboldt foundation. The authors would like to thank Jens Grossklags and Galina Schwartz for helpful comments and suggestions on earlier drafts of this paper.

1 Introduction

1.1 Public goods, voluntary contribution and free-rider problems

In the standard microeconomics literature, public goods are defined as being perfectly *non-rival* in consumption and *non-excludable*. The former implies that one individual's consumption of the good does not reduce the amount available to others. The latter captures the fact that individuals cannot deny each other the opportunity to consume the good, or that the cost of keeping non-payers from enjoying the benefits of the good is prohibitive. Public goods are omnipresent in the economics literature and model a broad range of goods from fresh air to street lightning or national security.

In many public good settings, provision levels are determined in a free market and by the voluntary contributions of the involved individuals. In many important instances, tacit coordination as well as altruism or desire for fairness may enhance efficiency; for example in the context of private donations to charity [Roberts(1984), Young(1982)], campaign funding of political parties or environmental groups. In these cases, experimental analysis has shown the necessity to abandon the classical Nash equilibrium prediction in favor of an ethically-based rule of behavior [Sugden(1982)] in which, even if the contributions to the public goods are perfect substitutes, preferences of people depend also on "having done their bit" [Cornes and Sandler(1984)] or on the social approval that may be consequent to their gestures [Akerlof(1980), Bénabou and Tirole(2006)]. We refer to [Bergstrom et al.(1986)Bergstrom, Blume, and Varian] for an extensive review of works and applications on the topic of the voluntary provision of public goods. However, as observed by [Ledyard(1995)], in many other settings "*one cannot rely on these approaches as a permanent organizing feature without expecting an eventual decline to self-interested behavior*". As a consequence of the non-excludability property, it is well known that the resulting equilibrium outcomes may suffer from (partial) *free riding* problems, where some individuals contribute (almost) zero and nonetheless enjoy the benefits of the public good. Instead, reaching the objective of an efficient and stable outcome typically necessitates some normative interventions or monetary incentives.

1.2 Overcoming the inefficiency of public goods economies

Since the seminal work of [Samuelson(1954)] on public goods, a sizable literature has investigated this problem space and associated improvement approaches, both from a theoretical and an experimental perspective. It is not possible to present here an exhaustive review of these works and we restrict to a very brief overview of some of the main lines of works.

Economic theorists have first proposed various sophisticated incentive compatible mechanisms, based on the general framework for the mechanism design approach to welfare economics of [Hurwicz(1972)], to enhance efficiency in public good problems [Clarke(1971), Groves and Ledyard(1977)]. These mechanisms have been partially criticized for their difficulty of implementation in a real world setting, due to their complexity and their counter-intuitive formulation. Later on, simpler mechanisms have been proposed. In [Varian(1994)], for example, the author proposes a two-stage game in which individuals have the opportunity to subsidize the other individuals' contributions. In [Andreoni and Bergstrom(1996)], the idea is to let the government design a tax-financed subsidies scheme to increase the equilibrium contribution to a public good. A totally different approach is presented in [Ledyard and Palfrey(1994), Ledyard and Palfrey(2002)], where the authors propose to approximate interim efficient mechanisms for the provision of a public good by a referendum in which voters simply vote for or against the provision of the public good. Most of these mechanisms and approaches have been tested experimentally in an increasingly vast literature on behavioral economics and sociology of public good provision, see e.g., [Bohm(1972),Marwell and Ames(1979),Holt and Laury(1997),Zelmer(2003),Lugovskyy et al.(2017)Lugovskyy, Puzzello, Sorensen, Walker, and Williams]. We refer to [Ledyard(1995),Chaudhuri(2011)] for surveys of the experimental literature on the topic.

1.3 Our contribution

In the works mentioned above, the focus is on finding mechanisms to improve the social welfare above the level obtained through voluntary contributions. Complementing this body of research, our work builds on the conceptual public good framework with voluntary contributions and strategic interactions, but with a specific emphasis on the utilization of non-monetary incentives for *improving the total level of contribution* rather than the social welfare. In certain scenarios, this might be more relevant. Indeed, in many public good settings, the public good utility given by the contribution of individuals may be non-excludable not only for free-riders but also for many other individuals who are not participating in the market at the moment because they are not part of it yet, but who would potentially benefit in the future. This is the case, for example, of medical research: contributing to a medical survey may allow doctors to find a cure, which could benefit patients who will get sick in the future for a virtually infinite amount of time. In such a setting, social efficiency is not simply given by the sum of the utilities of the individuals, but it may be represented by an increasing function of the total level of contribution, as this part dominates the private costs of contribution.

Within this context, we investigate the impact of imposing a *minimum individual contribution level*, while leaving individuals the choice to match or exceed the chosen level, or to free-ride by contributing zero. The resulting restriction of the strategy set can stimulate higher contribution levels by individuals. At the same time, leaving them the possibility of choosing a zero level, we ensure that such efforts remain voluntary. We investigate how to tune the minimum contribution level in order to maximize the total contribution and to reach a stable outcome where none of the individuals have incentives to free-ride. It is important to note that our proposal fundamentally differs from works on threshold public good games [van de Kragt et al.(1983)van de Kragt, Orbell, and Dawes,Cadsby and Maynes(1999)] which investigate the impact of a minimum level for the total level of contributions or number of contributors in order for the public good to be provided, rather than a restriction of the strategy set (i.e., the individual contribution level), as is the focus of our work.

Exploiting the potential nature of the proposed game, we show that there exists a minimum contribution level such that, regardless of the possible multitude of Nash equilibria in which some of the individuals free-ride, there exists a unique potential maximizer equilibrium, which has the properties that all individuals contribute to the public good and the total level of contributions is strictly larger than in the original game without minimum contribution level. Potential maximizer Nash equilibria, according to [Monderer and Shapley(1996)], are expected to accurately predict the results obtained through an experimental implementation of the model and, moreover, they are robust to incomplete information following the definition of [Kajii and Morris(1997)]. As such, they represent an important refinement of the equilibrium set. Throughout the paper, we also prove a number of comparative static results that describe how the equilibrium individual and total levels of contribution vary when the number of individuals in the market changes.

1.4 Applicability of the model

Our results provide a widely applicable and easily implementable approach for a policy maker to increase the provision of a public good above voluntary contribution levels, simply by restricting the agents' strategy spaces. The desirability of the approach from the policy maker's perspective lies in the facts that its implementation comes at no additional cost, and that it does not require any coordination or coercion. From the individual contributors' perspective, we argue that the resulting scenario after our modification is as simple to understand as the original one, and that it would likely be perceived as less burdensome in practical scenarios than some more complicated mechanisms (e.g., mechanisms involving payments). Finally, according to well-known findings regarding choice behaviors (e.g., the paradox of choice [Schwartz(2004)]), restricting the individuals' choices might increase their well-being.

The applicability of our proposed model is further validated by the fact that a number of use-cases already indirectly implement it. For instance, taking part in a medical survey is voluntary but, in case of participation, it usually requires devoting a minimum amount of time and releasing a minimum amount of information. Another important case where our model is potentially applicable, in which the total contribution level is socially more important than the aggregated social welfare is environment issues. It is well known that reaching a minimum effort by everyone or by every country in international treaties is fundamental for future generations, and this utility has a much larger weight compared to the individual perceived cost of making the effort of contributing.

1.5 Roadmap

The remainder of this paper is structured as follows. In Section 2, we present a motivating example in the area of data analytics. We develop and describe our generic model in Section 3. We conduct our analysis in detail in Section 4 on a canonical case without minimum contribution level. We investigate the results with a minimum contribution level in Section 5 and we illustrate the potential of our approach. We discuss the results on the specific motivating example in Section 6, and conclude in Section 7. All proofs are relegated to the Appendix.

2 Data analytics as a public good and privacy implications

This paper provides a theoretical analysis of a generic public good game and proposes a new non-monetary incentive mechanism to stimulate individuals toward higher efforts. As described in Section 1.4, it is proposed as a tool for a policy maker wishing to increase the total contribution level that could be applied in a variety of real situations. However, we also describe in more details a specific motivating application in the area of data analytics under privacy concerns, where our proposed mechanism makes particular sense.

In earlier works [Chessa et al.(2015b)Chessa, Grossklags, and Loiseau,Chessa et al.(2015a)Chessa, Grossklags, and Loiseau], we analyzed a game where privacy-conscious users reveal scalar personal data to an analyst who computes the population average. (This is a special case of the model of [Ioannidis and Loiseau(2013)] that introduced a class of linear regression games with privacy costs and the public good nature of the learning precision). Users choose the precision of the personal data revealed (incurring a privacy cost dependent on the chosen precision), which is seen as their contribution to a public good related to the precision of the inferred average. We then showed how, in this specific model, imposing a minimum individual precision level can increase the precision of the inferred average. In this paper, we propose a generalization of such a model to a more generic public good game and we considerably extend the results, hence showing how the idea of setting a minimum contribution level is valid for generic public goods when the aim is to increase the total level of contribution. In Section 6, we present the details of the data analytics problem, show how it is a special case of our more generic model and detail our results on that special case in particular emphasizing the benefits of the minimum contribution level mechanism to improve the level of precision achieved by the data analytics project.

Data analytics projects have a long history. However, in the last decades, the amount of personal information contributed by individuals to digital repositories such as social network sites has grown substantially, and together with it the attention of online services and of researchers. The existence of this data offers unprecedented opportunities for data analytics research in various domains of societal importance, including public health, market-research or political decision-making [Varian(2014)]. The results of these analysis can be considered as a public good which benefits data contributors as well as individuals who are not making their data available. At the same time, the release of personal information carries some privacy costs to the contributors, who may perceive the use of their data as an intrusion of their personal sphere [Altman(1975), Warren and Brandeis(1890)] or as a violation of their dignity [Westin(1970)], or who may fear this data can be abused for unsolicited advertisements, or social and economic discrimination [Acquisti and Fong(2013),Mikians et al.(2013)Mikians, Gyarmati, Erramilli, and Laoutaris]. The game-theoretic model of this paper applies to this trade-off scenario, in line with some literature which has already proposed to treat information as a public good [McCain(1988)]. Our results show how a data analyst can substantially increase the accuracy of her analysis by simply imposing a lower bound on the precision of the data users can reveal, while letting them the choice whether to contribute, and without providing any monetary incentive. This is particularly important as, in this sensible domain of privacy, it has been shown that individuals consider as fundamental to be able to exercise *control* over the release of their personal data [Kass et al.(2003)Kass, Natowicz, Hull, Faden, Plantinga, Gostin, and Slutsman,Damschroder et al.(2007)Damschroder, Pritts, Neblo, Kalarickal, Creswell, and Hayward, Robling et al.(2004)Robling, Hood, Houston, Pill, Fay, and Evans] while monetary compensation has so far received very little traction and it has been shown to meet little acceptance in consumer surveys [Acquisti and Grossklags(2005)].

3 The Model

3.1 The economy

We suppose that our economy consists of a set $N = \{1, \ldots, n\}$ of individuals who may contribute to the provision of a public good. Each individual $i \in N$ has the same unit wealth and contributes to the public good at a (normalized) level $\lambda_i \in [0, 1]$. We denote by $\boldsymbol{\lambda} = [\lambda_i]_{i \in N} \in [0, 1]^n$ the vector of contributions and by $G(N) = \sum_{i \in N} \lambda_i \in [0, n]$ the total level of contribution of the individuals of set N.

In our model, we rely on voluntary contributions for the provision of the public good and we confine our attention to the case in which individuals are assumed to care only about the private cost of the contribution and the utility derived from the total level of contribution. This assumption embeds many of the classical public good models that have received so far the most attention in the economic literature, such as the standard VCM (Voluntary Contribution Mechanism). Our choice enables providing intuitions and comparisons with a large part of the existing results on public goods even if, for applicability reasons, we use concave utility functions instead of the simple VCM linear model[1]. Formally, given her contribution λ_i and the contributions of all the other individuals (for which we use the standard notation $\boldsymbol{\lambda}_{-i}$), i experiences a utility

$$U_i(\lambda_i, \boldsymbol{\lambda}_{-i}) = h(G(N)) - p_i(\lambda_i). \tag{1}$$

The first component, $h(G(N))$, is the *public good utility*, i.e., the homogeneous utility that each individual equally experiences because of the total provision of the public good. We suppose that $h : [0, n] \to \bar{\mathbb{R}}$[2] is twice continuously differentiable, that the individuals experience positive and diminishing marginal utility from the provision of the public good, hence h is strictly increasing and strictly concave. Moreover, we suppose that the utility can be arbitrarily low in absence of contributions, i.e., that $h(0) = -\infty$[3] and that the marginal utility becomes negligible for high contribution levels, i.e., $h'(x) \to 0$ when $x \to +\infty$. The second component, $-p_i$, with $p_i : [0, 1] \to \mathbb{R}_+$, represents the *cost of contribution* of individual i. Such a cost is heterogeneous, i.e., it depends on the type of each individual, but we suppose that all the p_i's are twice continuously differentiable, non-negative, increasing, strictly convex and s.t. $p_i(0) = p_i'(0) = 0$.

Throughout our analysis and when specified, we often make the assumption that the individuals can be ordered in such a way that, for any contribution level $\lambda \in [0, 1]$, an individual choosing λ has higher marginal cost of contribution (and hence higher cost since $p_i(0) = 0$ for all individuals) than the previous individuals if they choose the same contribution level. Formally:

Assumption 3.1 *The costs of contribution are such that $p_1'(\lambda) \leq \cdots \leq p_n'(\lambda)$, for all $\lambda \in [0, 1]$.*

Assumption 3.1 may require some re-ordering from the initial ordering, which comes without loss of generality.

3.2 The public good game

We represent the individuals' strategic interaction as the non-cooperative public good game $\Gamma(N) = \langle N, [0, 1]^n, (U_i)_{i \in N} \rangle$, with set of players N, strategy space $[0, 1]$ for each individual $i \in N$ and utility function U_i given by (1). We analyze the game $\Gamma(N)$ as a *complete information game* between the individuals, i.e., we assume that the set of individuals, the action sets and the utilities are common knowledge. A *Nash equilibrium* in pure strategies (NE) of this game is a strategy profile $\boldsymbol{\lambda}^* \in [0, 1]^n$ satisfying

$$\lambda_i^* \in \arg\max_{\lambda_i \in [0,1]} U_i(\lambda_i, \boldsymbol{\lambda}_{-i}^*), \quad \forall i \in N. \tag{2}$$

Throughout our analysis, we always refer to NE (and the refinements) in pure strategies, even when not explicitly specified. A NE $\boldsymbol{\lambda}^*$ is a *strict Nash equilibrium* (SNE), if, for each $i \in N$ and for each $\lambda_i \in [0, 1]$, $\lambda_i \neq \lambda_i^*$, it holds that

$$U_i(\lambda_i^*, \boldsymbol{\lambda}_{-i}^*) - U_i(\lambda_i, \boldsymbol{\lambda}_{-i}^*) > 0,$$

i.e., each individual strictly decreases her utility by deviating.

[1] Some of the additional assumptions of our specific model are well motivated by some applications, such as the one presented in Section 6.

[2] We denoted by $\bar{\mathbb{R}}$ the extended real number line $\mathbb{R} \cup \{-\infty, +\infty\}$.

[3] Without loss of generality, we can relax this assumption by assuming $h(0) = a$, with a finite but arbitrarily low. Assuming the utility at zero contribution equal to $-\infty$ makes some of the following deductions more straightforward.

4 The Public Good Economy without Incentives

We observe that $\Gamma(N)$ is a potential game [Monderer and Shapley(1996)], with potential function $\Phi : [0,1]^n \to \bar{\mathbb{R}}$, s.t., for each $\boldsymbol{\lambda} \in [0,1]^n$,

$$\Phi(\boldsymbol{\lambda}) = h(G(N)) - \sum_{j \in N} p_j(\lambda_j). \tag{3}$$

In particular, the potential function is concave and the strategy sets are closed intervals of the real line. It follows that the set of Nash equilibria and the set of profiles which are maximizers of the potential function coincide. Exploiting its potential nature, we can perform a complete analysis of the game $\Gamma(N)$ and provide its stable outcomes. In particular, we observe that a Nash equilibrium in pure strategies for such a game exists and it is unique, as stated in Theorem 4.1. As expected, at equilibrium individuals with higher contribution costs select lower contribution levels.

Theorem 4.1 *The game $\Gamma(N)$ has a unique NE $\boldsymbol{\lambda}^*$. This equilibrium is strict and s.t. $\lambda_i^* > 0$ for each $i \in N$. If the costs of contribution satisfy Assumption 3.1, then the equilibrium is s.t., $0 < \lambda_n^* \leq \cdots \leq \lambda_1^*$.*

In most of the public good literature, we observe that only one small subset of the individuals will actually contribute to the provision of the public good at NE. In our model, owing to the assumption that $p_i'(0) = 0$, the equilibrium is such that we do not observe free-riding behaviors. Nevertheless, this comforting result does not prevent the equilibrium contribution from suffering of some partial free-riding behaviors, i.e., the non-zero but limited voluntary contribution of some individuals. As a consequence, when aiming at maximizing the total level of contribution at equilibrium, that we denote by $G^*(N)$, we may observe an inefficiency, compared to the maximal total level of contribution n that the individuals could potentially reach. In the following, and whenever necessary, we use the notation $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*(N)$ and $\lambda_i^* = \lambda_i^*(N)$ for each $i \in N$ to denote that the equilibrium depends on the specific identity of the agents in the set of individuals N (and, in particular, on their contribution costs).

Proposition 4.2 shows how, at equilibrium, the individual contribution and the total level of contribution vary when a new individual enters the game. In particular, we show that the individual contribution becomes lower, as expected and according to the standard public good literature. However, the total level of contribution increases. The result holds without any ordering assumption, in particular, it holds irrespective whether the new individual has higher or lower contribution cost compared to the rest of the population.

Proposition 4.2 *Given the game $\Gamma(N)$, suppose that an additional $(n+1)$-th individual enters the game. Then,*

(i) *the i-th individual's contribution at equilibrium for each $i \in N$ is s.t. $\lambda_i^*(N \cup \{n+1\}) \leq \lambda_i^*(N)$;*

(ii) *the total level of contribution is s.t. $G^*(N \cup \{n+1\}) > G^*(N)$.*

As a direct consequence of Proposition 4.2, we observe that, at equilibrium, the individual utility is strictly bigger when a new individual enters the game.

Corollary 4.3 *Given the game $\Gamma(N)$, suppose that an additional $(n+1)$-th individual enters the game. Then, at equilibrium, $U_i^*(\boldsymbol{\lambda}^*(N \cup \{n+1\})) > U_i^*(\boldsymbol{\lambda}^*(N))$.*

We can summarize the results presented in this section by affirming that our economy, modeled by the public good game $\Gamma(N)$ and when we rely on the individuals' voluntary contribution, is inefficient in term of total level of contribution. However, artificially enlarging the set of potential contributors to the public good is beneficial at this scope. Moreover, also from the individual point of view, it is convenient to welcome as many individuals as possible into the economy. However, improving the equilibrium provision level by enlarging the set of individuals is often not feasible, as some objective constraints could prevent this solution. In Section 5 we will propose and illustrate an alternative way to increase the total contribution level, while maintaining the population of contributors fixed. Before doing that, in the last part of this section, we further detail the analysis of the game $\Gamma(N)$ in the totally homogeneous case, i.e., when the individuals not only have the same public good utility, but also have identical costs of contribution. This allows us to strengthen our results for the proposed case.

4.1 The homogeneous case

Formally, now we assume that the individuals utility is still defined by (1), but it is s.t. $p_i(\cdot) = p(\cdot)$ for each $i \in N$. When the game $\Gamma(N)$ is homogeneous, we denote it by $\Gamma(n)$, i.e., as dependent on the number of individuals and not on their identity.

In the following, we detail Theorem 4.1 and Proposition 4.2 for the totally homogeneous case. The first result, that we present below, simply follows as a corollary of Theorem 4.1.

Corollary 4.4 *Let the game $\Gamma(n)$ be homogeneous, the unique NE $\boldsymbol{\lambda}^*$ is s.t. $\lambda_i^* = \lambda^* > 0$ for each $i \in N$.*

When the game $\Gamma(n)$ is homogeneous, we adopt the notation $\lambda^*(n)$ and $G^*(n)$ to denote the individual and the total equilibrium level of contribution respectively. This helps us emphasize that, under the hypothesis of homogeneity, the equilibrium level does not depend on the specific identity of the set of the individuals N, but only on its cardinality.

The second result, that we present in Proposition 4.5, could be partially shown as a corollary of Proposition 4.2. However, we present it in a more complete version, as the result of a comparative statics analysis of the individual contribution and the total level of contribution at equilibrium, studying how these quantities vary when changing the parameter n. In particular, we show that the individual contribution goes to zero when the number of individuals goes to infinity, and that the total level of contribution goes to infinity.

Proposition 4.5 *Let the game $\Gamma(n)$ be homogeneous, the individual contribution at equilibrium $\lambda^*(n)$ and the total level of contribution at equilibrium $G^*(n)$ satisfy:*

(i) *$\lambda^*(n)$ is a non-increasing function of the number n of individuals, and it is s.t. $\lim_{n\to+\infty} \lambda^*(n) = 0$;*

(ii) *$G^*(n) = n\lambda^*(n)$ is an increasing function of the number n of individuals, and it is s.t. $\lim_{n\to+\infty} G^*(n) = +\infty$.*

To conclude the analysis of this homogeneous case, we observe that, even if the total level of contribution goes to infinity when the number of individuals goes to infinity, it still goes slower than the optimal total level of contribution, as it holds that

$$\frac{n}{G^*(n)} = \frac{n}{n\lambda^*(n)} = \frac{1}{\lambda^*(n)} \longrightarrow +\infty.$$

5 The Public Good Economy with Incentives

5.1 The modified public good game

As we have seen in the previous section, the public good game $\Gamma(N)$, representing our economy and relying on the voluntary contribution of the individuals, has a unique Nash equilibrium. In such a model, when $|N| < +\infty$, i.e., when the number of individuals is finite, the resulting total level of contribution, $G^*(N)$, can be inefficiently low. In the following, we suppose that the equilibrium contribution of $\Gamma(N)$ is such that $\lambda_i^*(N) < 1$ for some $i \in N$, i.e., it is such that the total level of contribution is not maximal.

In this section, we propose a mechanism to incentivize a higher level of contribution by the individuals. The resulting game is a variation of the original model based on a restriction of the individuals' strategy space to $\{0\} \cup [\eta, 1]$ for a given $\eta \in [0, 1]$, that we call the *minimum contribution level*. Restricting the strategy set can stimulate the individuals toward higher efforts; letting them the possibility of choosing a zero level of contribution, we ensure that such efforts remain voluntary. We investigate how to tune the minimum precision level in order to improve the total level of contribution.

To analyze the strategic interaction between the agents in this variation, we define the game $\Gamma(N, \eta) = \left\langle N, \left[\{0\} \cup [\eta, 1]\right]^n, (U_i)_{i \in N} \right\rangle$, where the utility function U_i is still defined by (1), but it is now restricted on the domain $\left[\{0\} \cup [\eta, 1]\right]^n$. Observe that the original game $\Gamma(N)$ is a special case of this modified game $\Gamma(N, \eta)$, when $\eta = 0$. Then, from now on, we suppose that $\eta \in (0, 1]$. As we did for $\Gamma(N)$, we analyze the game $\Gamma(N, \eta)$ as a *complete information game* between the individuals, i.e., we assume that the set of individuals, the action sets (in particular, the minimum contribution level η) and the utilities are common knowledge. A Nash equilibrium (in pure strategy) of the modified game $\Gamma(N, \eta)$ is a strategy profile $\boldsymbol{\lambda}^*(\eta) \in \left[\{0\} \cup [\eta, 1]\right]^n$ satisfying

$$\lambda_i^*(\eta) \in \operatorname*{arg\,max}_{\lambda_i \in \{0\} \cup [\eta, 1]} U_i(\lambda_i, \boldsymbol{\lambda}_{-i}^*(\eta)), \quad \forall i \in N. \tag{4}$$

5.2 The economy with a minimum contribution level

We observe that the modified game $\Gamma(N, \eta)$ is still a potential game, with potential function Φ as in (3), but defined on the restricted domain $\left[\{0\} \cup [\eta, 1]\right]^n$. Differently from $\Gamma(N)$, the strategy sets of the modified game are not closed intervals of the real line and the equilibria may not coincide with the global maxima of the potential function. As a consequence, we can no longer exploit directly its potential nature to perform a complete analysis of the Nash equilibrium structure, as we did in the previous section. However, we observe that, given its strict concavity, the potential function Φ has still a unique global maximum when restricted to the smaller and convex domain $[\eta, 1]^n$. We denote by $\boldsymbol{\lambda}^M(N, \eta) \in [\eta, 1]^n$ such a maximum

$$\boldsymbol{\lambda}^M(N, \eta) = \operatorname*{arg\,max}_{\boldsymbol{\lambda} \in [\eta, 1]^n} \Phi(\boldsymbol{\lambda}), \tag{5}$$

and by $G^M(N, \eta)$ the corresponding total level of contribution. In the following, we show how these quantities still play a fundamental role in the analysis of the Nash equilibrium structure of the game $\Gamma(N, \eta)$.

At first, we observe that introducing a minimum level of contribution, we may incur some free-riding behaviors. Some individuals (in particular, the ones with the highest contribution costs) may find more advantageous to stay out of the game rather than being pushed to contribute more. In Definition 5.1, we introduce a parameter η^* such that, as we see in Proposition 5.2, if we set a minimum precision level in $(0, \eta^*]$, there still exists a NE such that even the individual with the highest cost of contribution does not have an incentive to drop out and to free-ride and such an equilibrium coincides with $\boldsymbol{\lambda}^M(N, \eta)$.

Definition 5.1 *Suppose that Assumption 3.1 is satisfied. We define the parameter η^* as the smallest parameter in $(\lambda_n^*, 1]$[4], if there exists one, s.t.,*

$$h(G^M(N, \eta^*)) - h(G^M(N, \eta^*) - \eta^*) = p_n(\eta^*). \tag{6}$$

If such a parameter does not exist, we set $\eta^* = 1$.

To better understand this definition, we first observe that, when choosing a minimum contribution level $\eta \in (\lambda_n^*, 1]$, vector $\boldsymbol{\lambda}^M(N, \eta)$ is necessarily on the border (as no critical point of Φ was in $(\lambda_n^*, 1]^n$), i.e., it is s.t. the individual with the highest cost contributes exactly $\lambda_n^M(N, \eta) = \eta$. Given that, according to Theorem 4.1, the equilibrium of the original game is strict, intuitively, we define η^* as the smallest contribution level at which, if there still exists an equilibrium in which all the individuals are contributing, such an equilibrium is not strict anymore, i.e., individual n has exactly the same utility by contributing and by free-riding. If such a parameter does not exist, it follows that individual n is always better off by contributing than by free-riding, regardless of the minimum contribution level, and then we set it at maximum, i.e., equal to 1. In that case, it simply holds that

$$h(G^*(N, 1)) - h(G^*(N, 1) - 1) = h(n) - h(n-1) > p_n(1). \tag{7}$$

We now investigate whether, by introducing a minimum precision level, we still have an equilibrium in which also the individual with the highest contribution costs is not free-riding. In Proposition 5.2 we show that, by setting a precision level no bigger than η^*, this is still possible, and moreover such an equilibrium is the unique one with non-zero components.

Proposition 5.2 *Suppose that Assumption 3.1 is satisfied. It holds that:*

(i) *if $|N| = 1$, then for any $\eta \in (0, 1]$, $\Gamma(N, \eta)$ has a unique NE $\lambda^*(N, \eta) = \max\{\lambda^*, \eta\}$, where λ^* is the single entrance NE of $\Gamma(N)$ when only one individual is present;*

(ii) *if $|N| > 1$, then for any $\eta \in (0, \eta^*]$, every NE $\boldsymbol{\lambda}^*$ of $\Gamma(N, \eta)$ is s.t. $\lambda_i^* = \lambda_i^M(S, \eta)$ for each $i \in S$, where $S = \{i \in N | \lambda_i^* \neq 0\}$. In particular, $\boldsymbol{\lambda}^*(N, \eta) := \boldsymbol{\lambda}^M(N, \eta)$ is the unique NE of $\Gamma(N, \eta)$ s.t. $\lambda_i^*(N, \eta) > 0$ for all $i \in N$.*

Proposition 5.2 translates the fact that, if the analyst selects a minimum precision level that is not "too high", i.e., that is not bigger than η^*, there exists an equilibrium where all the individuals are still willing to contribute. This does not exclude the existence of other equilibria where some individuals may contribute zero. Whether the result is still valid or not for value of η bigger than this threshold is still an open question. However, as better stated in the final part of this section, we conjecture that this result will not hold anymore for a minimum contribution level which is larger

[4] We recall that we denote by λ_n^* the equilibrium contribution level in $\Gamma(N)$ of the individual with the highest contribution cost.

than η^*, translating the fact that, in that case, the individuals would feel to be forced to a too important effort and prefer not to take part (e.g., in the survey).

The results of Proposition 5.2 allow us to establish the first main result of this section. In Theorem 5.3 we state that it is possible to strictly increase the total level of contribution at the non-zero equilibrium by imposing a minimum precision level. In particular, such an increase is monotonic in η in the domain $[0, \eta^*]$.

Theorem 5.3 *Suppose that Assumption 3.1 is satisfied. The total level of contribution $G^*(N, \eta)$ at equilibrium $\boldsymbol{\lambda}^*(N, \eta)$ of the modified game $\Gamma(N, \eta)$ is a non-decreasing function of $\eta \in [0, \eta^*]$ and, in particular, it is increasing in $[\lambda_n^*, \eta^*]$.*

In this subsection, we have shown, up to a certain value of the minimum contribution level, the existence of a NE without free-riders in which the increase of the total contribution level is strictly positive and it is monotonic in the choice of the parameter η. However, the uniqueness of such a NE is not assured and the equilibrium prediction could suggest the rise of a suboptimal NE with potentially many free-riders. In the following, we address this problem by refining the NE definition and by showing that, up to a more restrictive value of the minimum contribution level η, $\boldsymbol{\lambda}^*(N, \eta)$ is in fact the only equilibrium which is likely to arise by implementing the corresponding modified game.

5.3 The Nash equilibria refinement

By introducing a minimum precision level, we face the problem of possibly having a multiplicity of Nash equilibria. We have already observed that the modified game $\Gamma(N, \eta)$ is still a potential game, but in which the equilibria may not coincide with the global maxima of its potential function. Even if we cannot exploit anymore directly its potential nature to perform a complete analysis of the Nash equilibrium structure, we observe that, in potential games, the set of global maxima is a subset of the Nash equilibrium set which refines it both in terms of equilibrium prediction and robustness. Indeed, first, in [Monderer and Shapley(1996)] the authors argue that such a refinement concept is expected to accurately predicts the results obtained through an experimental implementation of the model; in particular, they show that this refinement concept accurately predicts the experimental results obtained by [Van Huyck et al.(1990)Van Huyck, Battalio, , and Beil][5]. Second, this refinement concept has been justified theoretically in [Oyama and Tercieux(2009)] because of its robustness to incomplete information following the definition of [Kajii and Morris(1997)]. For these reasons, by implementing a potential game, we can assume that the individuals will converge to a *potential maximizer Nash equilibrium* (PMNE) $\boldsymbol{\lambda}^P(\eta) \in \big[\{0\} \cup [\eta, 1]\big]^n$ satisfying

$$\boldsymbol{\lambda}^P(\eta) \in \operatorname*{arg\,max}_{\boldsymbol{\lambda} \in [\{0\} \cup [\eta, 1]]^n} \Phi(\boldsymbol{\lambda}). \tag{8}$$

In the following we investigate whether, for some choices of the minimum contribution level η, we can ensure the existence of a unique PMNE and whether this solution coincides or not with the non-zero component NE of the previous subsection. In Definition 5.4, we introduce a new parameter $\bar{\eta}$ smaller than or equal to η^* such that, as we see in Proposition 5.5, if the analyst sets a minimum precision level strictly smaller than $\bar{\eta}$, the unique NE s.t. there are no free-riders is also the unique

[5] As we will discuss in the conclusions, an experimental validation of this affirmation on our model will be the main direction for future work on the topic.

PMNE and then it represents the only one that is likely to emerge through an implementation of the model.

Definition 5.4 *Suppose that Assumption 3.1 is satisfied. We define the parameter* $\bar{\eta}$ *as the smallest parameter in* $(\lambda_n^*, \eta^*]$*, if there exists one, s.t.,*

$$\Phi|_{\lambda_n=0}(\boldsymbol{\lambda}^{M0}(N \setminus \{n\}, \bar{\eta})) = \Phi(\boldsymbol{\lambda}^M(N, \bar{\eta})), \tag{9}$$

where $\boldsymbol{\lambda}^{M0}(N \setminus \{n\}, \bar{\eta})$ *is the unique global maximum of the potential function* Φ *defined on the restricted domain* $[\bar{\eta}, 1]^{n-1} \cup \{0\}$ *and* $\boldsymbol{\lambda}^M(N, \bar{\eta})$ *is the unique global maximum of* Φ *on* $[\bar{\eta}, 1]^n$*. If such a parameter does not exist, we impose* $\bar{\eta} = \eta^*$*.*

To better understand this definition, we observe that the parameter $\bar{\eta}$ is defined as the smallest minimum contribution level at which the maximum value of the potential function is the same by letting individual n free to choose whether to contribute or not or by forcing him to contribute zero. Of course, for value of the minimum level of contribution in $[0, \lambda_n^*]$, the maximum value of the potential function is unique and this condition will never be satisfied.

Proposition 5.5 *Suppose that Assumption 3.1 is satisfied. For any* $\eta \in (0, \bar{\eta})$*,* $\boldsymbol{\lambda}^*(N, \eta) = \boldsymbol{\lambda}^M(N, \eta)$ *is the unique PMNE of the modified game* $\Gamma(N, \eta)$*.*

Complementary to the results of Proposition 5.2 and Theorem 5.3, Proposition 5.5 states the second main result of this section. We have shown in Proposition 5.2 how, introducing a minimum precision level which is smaller than η^*, we ensure that an equilibrium such that no individual is free riding still exists, and such an equilibrium is unique, i.e., any other equilibrium is such that some individuals contribute zero. By further bounding the choice of the minimum level of contribution to a parameter smaller than $\bar{\eta}$, we ensure in Proposition 5.5 that such an equilibrium is the unique potential maximizer and, as such, the unique one which is likely to arise implementing the model. Finally, Theorem 5.3 ensures that, by imposing such a minimum precision level, the total level of contribution at the nonzero components NE, i.e., at PMNE, is strictly higher, compared to the one of the original game. We conclude by summing up that choosing a parameter η as close as possible to $\bar{\eta}$ represents, up to our knowledge, the best choice to maximize the total contribution level. Moreover we conjecture, as it is still an open question, that any choice of the parameter bigger than η^* will not produce any equilibrium of the game in which no one is free-riding or with a bigger total level of contribution.

In Corollary 5.6, we illustrate how there could exist other PMNE for choices of the parameter η which are too large, i.e., when trying to force the individuals to some contribution levels which are too high.

Corollary 5.6 *When* $\eta = \eta^*$*, for each* $j \in N$ *s.t.* $p_j'(\cdot) = p_n'(\cdot)$ *there exists a NE* $\boldsymbol{\nu}$ *of* $\Gamma(N, \eta^*)$ *s.t.,* $\nu_i = \lambda_i^*(N, \eta^*) = \lambda_i^*(N \setminus \{n\}, \eta^*)$ *for each* $i \neq j$ *and* $\nu_j = 0$*. Such an equilibrium is such that* $\Phi(\boldsymbol{\nu}) = \Phi(\boldsymbol{\lambda}^*(N, \eta^*))$*.*

From Corollary 5.6 we may observe that when implementing the game $\Gamma(N, \eta^*)$, any strategy vector which corresponds to $\boldsymbol{\lambda}^*(N, \eta^*)$, but s.t. one of the individuals who have the largest contribution cost deviates to zero, is still a NE (in particular, it is still a PMNE if the original contribution vector is a PMNE). This because, by Definition 5.1, η^* is the contribution level s.t. not only her individual utility, but also the potential function does not vary when individual n deviates. However, such a PMNE is inefficient in terms of total level of contribution, as it is s.t. $n-1$ individuals are contributing the same and 1 individual who was contributing strictly more than zero is now free-riding.

5.4 The homogeneous case

Similarly to the previous section for the case without incentives, we now detail the results of Propositions 5.2 and 5.5 and Theorem 5.3 for the homogeneous modified game, that we denote by $\Gamma(n, \eta)$. First, in Definitions 5.7 and 5.8, we introduce a parameter $\eta^*(n)$ and a parameter $\bar{\eta}(n)$, which are the analogous of parameters η^* and $\bar{\eta}$ defined in Definitions 5.1 and 5.4 for the heterogeneous case.

Definition 5.7 *We define the parameter $\eta^*(n)$ as the smallest parameter in $(\lambda^*(n), 1]$, if there exists one, s.t.*

$$h(n\eta^*(n)) - h((n-1)\eta^*(n)) = p(\eta^*(n)). \tag{10}$$

If such a parameter does not exist, we impose $\eta^*(n) = 1$.

Definition 5.8 *We define the parameter $\bar{\eta}(n)$ as the smallest parameter in $(\lambda^*(n), \eta^*(n)]$, if there exists one, s.t.*

$$h((n-1)\lambda^*(n-1)) - (n-1)p(\lambda^*(n-1)) = h(n\bar{\eta}(n)) - np(\bar{\eta}(n)). \tag{11}$$

If such a parameter does not exist, we impose $\bar{\eta}(n) = \eta^*(n)$.

In Theorem 5.9, we provide a more detailed analysis of the NE structure for the homogeneous case. In particular, in this subcase we are able to prove the conjecture we presented at the end of the previous section for the heterogeneous case. Indeed, we show that, when the minimum precision level is too high (bigger than $\eta^*(n)$), at equilibrium some individuals necessarily free-ride.

Theorem 5.9 *Let the game $\Gamma(n)$ be homogeneous, and let $\boldsymbol{\lambda}^*(n)$ be the unique Nash equilibrium of the original game $\Gamma(n)$. It holds:*

(i) *if $n = 1$, then for any $\eta \in (0, 1]$, $\Gamma(n, \eta)$ has a unique Nash equilibrium $\lambda^*(n, \eta) = \max\{\lambda^*(1), \eta\}$;*

(ii) *if $n > 1$, then*

(iia) *for any $\eta \in (0, \eta^*(n))$, $\Gamma(N, \eta)$ has a unique NE $\boldsymbol{\lambda}^*(n, \eta)$ s.t. $\lambda_i^*(n, \eta) > 0$ for each $i \in N$. This equilibrium is s.t., $\lambda_i^*(n, \eta) = \lambda^*(n, \eta)$ for each $i \in N$, with*

$$\lambda^*(n, \eta) = \begin{cases} \lambda^*(n) & \text{if } 0 \le \eta \le \lambda^*(n) \\ \eta & \text{if } \lambda^*(n) < \eta < \eta^*(n); \end{cases} \tag{12}$$

In particular, if $\eta \in (0, \bar{\eta}(n))$, such an equilibrium is the unique PMNE.

(iib) *for $\eta = \eta^*(n)$, $\Gamma(n, \eta^*(n))$ has at least $n+1$ NE: an equilibrium $\boldsymbol{\lambda}^*(n, \eta^*(n))$ s.t. $\lambda_i^*(n, \eta^*(n)) = \eta^*(n)$ for each $i \in N$, and n equilibria $\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^n$ s.t. for each $j = 1 \ldots n$ $\nu_i^j = \eta^*(n)$ for each $i \neq j$ and $\nu_j^j = 0$. In particular, if $\boldsymbol{\lambda}^*(n, \eta^*(n))$ is a PMNE, then all the $\boldsymbol{\nu}^j$ are PMNE for each $j = 1 \ldots n$.*

(iic) *for any $\eta \in (\eta^*(n), 1]$, there does not exist a Nash equilibrium $\bar{\boldsymbol{\lambda}}$ of $\Gamma(n, \eta)$ s.t. $\bar{\lambda}_i > 0$ for each $i \in N$.*

The homogeneous case allows us to understand in more details the equilibrium structure of the modified model in the particular situation in which all the individuals have the same contribution cost. It is also of particular interest when implementing a public good game in which the contribution costs of the individuals are not known precisely, and only an estimation or an average is available. Moreover the homogeneous case, as in the previous section, allows us to perform a comparative statics analysis, investigating what happens when the number of individuals increases. The last result of this section shows the monotonicity of the parameter $\eta^*(n)$, which is a non-increasing function of the parameter n, going to zero for an infinitely large number of individuals.

Corollary 5.10 *The function $\eta^*(n)$ is non-increasing in the number n of individuals, and it is s.t. $\lim_{n\to+\infty} \eta^*(n) = 0$.*

It follows that also the parameter $\bar{\eta}(n)$ goes to zero when the number of individuals goes to infinity. As a consequence, similarly to what we observed at the end of Section 4 without minimum contribution level, also the modified game performs worst than the maximal level of contribution n for an infinite number of individuals, as it holds that

$$\frac{n}{n\lambda^*(n, \bar{\eta}(n))} = \frac{n}{n\bar{\eta}(n)} = \frac{1}{\bar{\eta}(n)} \longrightarrow +\infty.$$

However, as we will show for a particular case in the next section, the improvement can still be very significant.

6 A Data Analytics Project as a Public Good Economy

Our results provide a widely applicable method to increase the provision of a public good above voluntary contributions, simply by restricting the individuals' strategy spaces. This method is attractive by its simplicity and, consequently, its applicability. In this section, we illustrate the model and our results on a modern application around privacy, where the economy is given by a data analytics research project[6].

6.1 The data analytics project

In our personal data economy, a set of individuals $N = \{1, \dots, n\}$ may decide to voluntary contribute (or not) to a data analytics research project, providing some personal data at a given level of precision. We suppose that their personal data are collected and contained in a data repository. In particular, each individual $i \in N$ is associated with a *private variable* $y_i \in \mathbb{R}$, which contains sensitive information. Throughout our analysis, we suppose that there exists $y_M \in \mathbb{R}$, s.t., the private variables are of the form

$$y_i = y_M + \epsilon_i, \quad \forall i \in N, \tag{13}$$

[6] In this application, we adopt a notation which is consistent with the one of the general model presented in the previous sections.

where ϵ_i are i.i.d., zero-mean random variables with finite variance $\sigma^2 < \infty$, which capture the inherent noise. We make no further assumptions on the noise; in particular, we do not assume that it is Gaussian. As a result, such a model applies to a wide range of statistical inference problems, even cases where the distribution of variables is not known.

Parameter y_M represents the *mean* of the private variables y_i, and its knowledge is valuable to the analyst. The analyst wishes to observe the available private variables y_i and to compute their average as an estimation of y_M. In our model, we suppose that the analyst does not know the mean y_M that she wishes to estimate, but she knows the variance σ^2. Such an assumption is justified by the fact that in many statistical analyses observing the variability of an attribute in a population is easier than estimating the mean, both for the analyst and for the population (in [Huberman et al.(2005)Huberman, Adar, and Fine], for example, the authors show how individuals value their age and weight information according to the relative variability).

We suppose that the analyst cannot directly access the private variables; instead she needs to ask the individuals for their consent to be able to retrieve the information. As such, the individuals have full control over their own private variables, and they have the choice to authorize or to deny the analyst's request. In particular, if wishing to contribute, but concerned about privacy, an individual can authorize the access to a perturbed value of the private variable. The *perturbed variable* has the form $\tilde{y}_i = y_i + z_i$, where z_i is a zero-mean random variable with variance σ_i^2 chosen by the individual. We assume that the $\{z_i\}_{i\in N}$ are independent and are also independent of the inherent noise variables $\{\epsilon_i\}_{i\in N}$. In practice, the individual chooses a given *precision* λ_i which is the inverse of the total variance (inherent noise plus artificially added noise) of the perturbed variable $\tilde{y}_i$, i.e.,

$$\lambda_i = 1/(\sigma^2 + \sigma_i^2) \in [0, 1/\sigma^2], \quad \forall i \in N,$$

and which corresponds to the contribution of individual i to the personal data economy. In the choice of the precision level, we have the following two extreme cases:

(i) when $\lambda_i = 0$, individual i has very high privacy concerns. This corresponds to adding noise of infinite variance or, equivalently, this represents the fact that individual i denies access to her data. In our public good model, a zero contribution corresponds to free-riding;

(ii) when $\lambda_i = 1/\sigma^2$, individual i has very low privacy concerns. This corresponds to authorizing access to the private variable y_i without adding any additional noise to the data. In our public good model, a maximal precision corresponds to a maximal level of contribution.

The strategy set $[0, 1/\sigma^2]$ contains all the possible choices for individual i: *denying*, *authorizing*, or any *intermediate level* of precision (which captures a wide range of privacy concerns as documented in behavioral studies [Spiekermann et al.(2001)Spiekermann, Grossklags, and Berendt]). We denote by $\boldsymbol{\lambda} = [\lambda_i]_{i\in N}$ the vector of the precisions.

Once each individual $i \in N$ has made her choice about the level of precision λ_i and, consequently, the perturbed variable $\tilde{y}_i$ has been computed, the analyst has access to both the set of precisions and the set of perturbed variables. Then, the analyst estimates the mean as

$$\hat{y}_M(\boldsymbol{\lambda}) = \frac{\sum_{i\in N} \lambda_i \tilde{y}_i}{\sum_{i\in N} \lambda_i}, \tag{14}$$

where perturbed variables with higher precision (i.e., smaller variance) receive a larger weight. This estimator is the standard *generalized least squares estimator*. It minimizes a weighted square error in which the i-th term is weighted by the precision of the perturbed variable $\tilde{y}_i$. This estimator is unbiased, i.e., $\mathbb{E}[\hat{y}_M] = y_M$, and has variance

$$\sigma_M^2(\boldsymbol{\lambda}) = \mathbb{E}[(\hat{y}_M(\boldsymbol{\lambda}) - y_M)^2] = \frac{1}{\sum_{i \in N} \lambda_i} \in [\sigma^2/n, +\infty]. \tag{15}$$

It is reasonable to assume that the analyst would use this estimator, as it is "good" for several reasons. In particular, it coincides with the *maximum-likelihood estimator* for Gaussian noise and, most importantly, it has minimal variance amongst the linear unbiased estimators for arbitrary noise distributions (this optimality result is known as Aitken theorem [Aitken(1935)]).

In the estimation, we have the following two extreme cases:

(i) when $\lambda_i = 0$ for each $i \in N$, the variance (15) is infinite. This corresponds to the situation in which each individual denies access to her data, and then the analyst cannot estimate y_M. In our public good model, this situation leads to a zero total level of contribution;

(ii) when $\lambda_i = 1/\sigma^2$ for each $i \in N$, the analyst estimates y_M with variance σ^2/n, resulting only from the inherent noise. This corresponds to the situation in which each individual is authorizing access to her data with maximum precision, i.e., no agent is perturbing her private variable. In our public good game, this corresponds to a maximal total level of contribution equal to n/σ^2.

For any level of precision in $[0, 1/\sigma^2]^n$, the estimated variance will be in $[\sigma^2/n, +\infty]$. The set of precision vectors for which the estimator has a finite variance is $[0, 1/\sigma^2]^n \setminus \{(0, \dots, 0)\}$.

6.2 The Estimation Game Γ^E

We describe the interaction between the individuals as follow. We assume that each individual $i \in N$ wishes to minimize a cost function[7] $J_i : [0, 1/\sigma^2]^n \to \bar{\mathbb{R}}_+$, s.t., for each $\boldsymbol{\lambda} \in [0, 1/\sigma^2]^n$,

$$J_i(\lambda, \boldsymbol{\lambda}_{-i}) = c_i(\lambda_i) + f(\boldsymbol{\lambda}). \tag{16}$$

The cost function J_i of individual $i \in N$ comprises two non-negative components. The first component $c_i : [0, 1/\sigma^2] \to \mathbb{R}_+$ represents the privacy attitude of individual i, and we refer to it as the *privacy cost*: it is the (perceived or actual) cost that the individual incurs on account of the privacy violation sustained by revealing the private variable perturbed with a given precision. The second component $f : [0, 1/\sigma^2]^n \to \bar{\mathbb{R}}_+$ is the *estimation cost*, and we assume that it takes the form $f(\boldsymbol{\lambda}) = F(\sigma_M^2(\boldsymbol{\lambda}))$ where $F : [\sigma^2/n, +\infty) \to \mathbb{R}_+$ if the variance is finite, and $+\infty$ otherwise. It represents how well the analyst can estimate the mean y_M and it captures the idea that it is not only in the interest of the analyst, but also of the agents, that the analyst can determine an accurate estimate of the population average y_M. We assume that the privacy costs $c_i : [0, 1/\sigma^2] \to \mathbb{R}_+$, $i \in N$, are twice continuously differentiable, non-negative, non-decreasing, strictly convex and s.t. $c_i(0) = c_i'(0) = 0$, and that function $F : [\sigma^2/n, +\infty) \to \mathbb{R}_+$ is twice continuously differentiable, non-negative, non-decreasing and convex. To describe the strategic interaction between the individuals, we define the *estimation game* $\Gamma^E = \langle N, [0, 1/\sigma^2]^n, (J_i)_{i \in N} \rangle$ with set of agents N, strategy space $[0, 1/\sigma^2]$ for each agent $i \in N$ and cost function J_i given by (16).

[7] We chose to present the estimation game model in its cost formulation for coherence with some previous work [Chessa et al.(2015b)Chessa, Grossklags, and Loiseau, Chessa et al.(2015a)Chessa, Grossklags, and Loiseau]

We observe that the estimation game Γ^E is a particular case of the public good game $\Gamma(N)$ defined in Section 3. In fact, up to a normalization, or assuming $\sigma^2 = 1$, the strategy set of each individual corresponds to the normalized strategy set $[0, 1]$. Moreover, the minimization of the cost function J_i is equivalent to the maximization of a utility function $U_i^E : [0, n]^n \to \bar{\mathbb{R}}$ defined as

$$U_i^E(\lambda_i, \boldsymbol{\lambda}_{-i}) = h^E(\boldsymbol{\lambda}) - p_i^E(\lambda_i), \tag{17}$$

where the public good utility is given by $h^E(\boldsymbol{\lambda}) = -F\left(\frac{1}{\sum_{i\in N}\lambda_i}\right)$ and the cost of contribution by $p_i^E(\lambda_i) = c_i(\lambda_i)$. We may observe that this transformation leads to a negative utility. This is not in contradiction with the general formulation of the public good model proposed in Section 3, but it is, of course, unusual in the standard public good literature. However, we observe that it would be sufficient to do a trivial rescaling and translation of the utility function to write the same problem in a more ordinary form.

6.3 The Estimation Game with Monomial Privacy Costs and Linear Estimation Cost

In this section, we explicit the results of the previous sections and we run some simulations in the special case where the privacy cost is homogeneous and monomial and the estimation cost is linear; i.e., we assume that the cost function in (16) has the form

$$J_i(\lambda_i, \boldsymbol{\lambda}_{-i}) = c\lambda_i^k + \sigma_M^2(\boldsymbol{\lambda}), \tag{18}$$

where $c \in (0, \infty)$ and $k \geq 2$ are constants. In terms of the utility functions defined in Section 3, this corresponds to a cost of contribution $p_i^E(\lambda_i) = c\lambda_i^k$, and a public good utility $h^E(\boldsymbol{\lambda}) = -\frac{1}{\sum_{i\in N}\lambda_i}$. Note that, without loss of generality, in the linear estimation cost, we omit the constant factor (adding a constant to the cost does not modify the game solutions) as well as the slope factor (adding it would give an equivalent game with constant c rescaled). For this special case, we can determine both the equilibrium precision (with and without a minimum precision level) and the optimal minimum precision level in closed form. We can then graphically depict how the quantities vary while moving the model parameters, and explicitly compute the estimation improvement when introducing a minimum precision level.

In the special case of costs given by (18), the equilibrium precision chosen by the agents in the game Γ^E simplifies to:

$$\lambda^*(n) = \begin{cases} \left(\dfrac{1}{ckn^2}\right)^{\frac{1}{k+1}} & \text{if } \left(\dfrac{1}{ckn^2}\right)^{\frac{1}{k+1}} \leq 1/\sigma^2 \\ 1/\sigma^2 & \text{if } \left(\dfrac{1}{ckn^2}\right)^{\frac{1}{k+1}} > 1/\sigma^2. \end{cases} \tag{19}$$

The upper bound of the minimum precision level to guarantee the existence of a non-zero NE is given by

$$\eta^*(n) = \begin{cases} \left(\dfrac{1}{cn(n-1)}\right)^{\frac{1}{k+1}} & \text{if } \left(\dfrac{1}{cn(n-1)}\right)^{\frac{1}{k+1}} \leq 1/\sigma^2 \\ 1/\sigma^2 & \text{if } \left(\dfrac{1}{cn(n-1)}\right)^{\frac{1}{k+1}} > 1/\sigma^2, \end{cases}$$

while the upper bound to guarantee that such an equilibrium is the unique PMNE is given by $\bar{\eta}(n) = \eta^*(n)$. We may conclude that the optimal choice for the data analyst in order to maximize the total level of contribution (which is equivalent to minimizing the variance of the estimation) and to have every individual participating, is to set up a minimum contribution level as close as possible to $\eta^*(n)$.

Writing explicitly the previous two key quantities, we can more easily analyze the properties of our public good game and its modification that we have already discussed in the previous sections for the general case. In particular, we observe that when c increases, i.e., when the individuals are more concerned about privacy, they choose at equilibrium a smaller precision level $\lambda^*(n)$. Further, the minimum precision level $\eta^*(n)$ becomes smaller if the individuals are more sensitive about the protection of their data. Finally, we have $\lambda^*(n) < \eta^*(n)$ for each $n \in \mathbb{N}^*$, and both of these quantities decrease and go to zero when n increases and goes to $+\infty$.

Most interestingly, the closed-form expressions that we have for this special case allow us to analyze the rate of decrease of the variance (equivalent to the rate of increase of the contribution), and to quantify the improvement that can be achieved by imposing a minimum precision level. For n large enough (such that both $\lambda^*(n)$ and $\eta^*(n)$ are strictly smaller than $1/\sigma^2$), the variance at equilibrium level $\lambda^*(n)$ of game Γ^E is given by

$$\sigma_M^2(\boldsymbol{\lambda}^*(n)) = \frac{1}{n\left(\frac{1}{ckn^2}\right)^{\frac{1}{k+1}}},$$

while by setting up a minimum precision level η as close as possible to $\eta^*(n)$, we can reach a variance given by

$$\sigma_M^2(\boldsymbol{\lambda}^*(n,\eta)) = \frac{1}{n\left(\frac{1}{cn(n-1)}\right)^{\frac{1}{k+1}}} + \epsilon,$$

with ϵ arbitrarily small.

Both appear to have the same rate of decrease in $n^{\frac{-k+1}{k+1}}$ which is smaller than n^{-1} but becomes closer to n^{-1} as k tends to infinity. Intuitively, as the privacy cost becomes closer to a step function, the equilibrium precision level becomes less dependent on the number of agents so that we get closer to the case of averaging iid random variables of fixed variance. Consequently, for n large enough, the improvement is given by a factor:

$$\frac{\sigma_M^2(\boldsymbol{\lambda}^*(n))}{\sigma_M^2(\boldsymbol{\lambda}^*(n,\eta))} = \left(\frac{kn}{n-1}\right)^{\frac{1}{k+1}} > 1, \tag{20}$$

which asymptotically becomes constant:

$$\frac{\sigma_M^2(\boldsymbol{\lambda}^*(n))}{\sigma_M^2(\boldsymbol{\lambda}^*(n,\eta^*(n)))} \sim_{n\to\infty} k^{\frac{1}{k+1}}. \tag{21}$$

Interestingly, we notice that this ratio of variances (characterizing the improvement when setting the optimal minimum precision level) depends on k, but not on c. (This holds even before the asymptotic regime, as long as n is large enough such that both $\lambda^*(n)$ and $\eta^*(n)$ are strictly smaller than $1/\sigma^2$.)

Figure 1 illustrates the asymptotic improvement ratio (21) for different values of k. We observe that it is bounded, it goes to 1 for large k's and it is in the range of $25 - 30\%$ improvement for

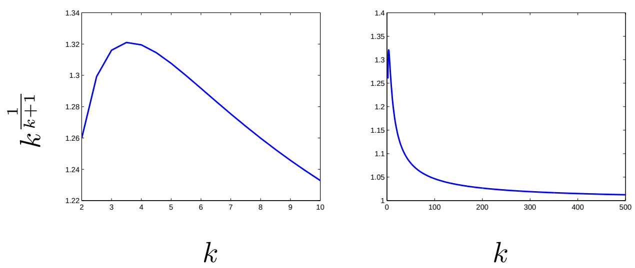

Figure 1: Asymptotic improvement of the estimation choosing the optimum precision level $\eta^*(n)$ for values of $k = 2, \ldots, 10$ and for values of $k = 2, \ldots, 500$.

values of k around $2 - 10$. Given that the ratio (20) converges towards its asymptote from above, this asymptotic improvement represents a lower bound of the improvement the analyst can achieve by implementing our mechanism with any finite number n of agents.

7 Concluding Remarks

In this paper, we propose a mechanism to improve the total contribution level in public good provision. Our objective of improving the total contribution level contrasts with previous works in the public good area that focus on improving social efficiency. Our mechanism simply works by imposing a minimum individual contribution level while allowing to free-ride if they prefer not to respect it. Then our theoretical results predict an important rationale: using this mechanism to push individuals to contribute more than their voluntary contribution level can indeed work and increase the total contribution but it is necessary to carefully set up the minimum contribution level to avoid creating free-riding.

Our theoretical analysis explains the efficacy of some existing protocols that are already adopted and require a minimum effort when asking some individuals to contribute to a public good. Such an idea could be implemented in many other public good settings in which some other approaches, such as providing monetary incentives, have failed in enhancing participation. In future work, we plan to validate and complement our theoretical results through a behavioral study.

A Proof of Theorem 4.1

Game $\Gamma(N)$ is a potential game with a concave potential function Φ (defined as in (3)) and where the strategy sets are closed intervals of the real line. It follows that a strategy profile is a Nash equilibrium if and only if it maximizes the potential function. As the potential function Φ is strictly concave on the convex set on which it is defined, it has a unique global maximum $\boldsymbol{\lambda}^* \in [0,1]^n$ which coincides with the unique Nash equilibrium of $\Gamma(N)$ and such an equilibrium is strict. Moreover, owing to the assumption that $h(0) = -\infty$, this equilibrium is s.t., $\boldsymbol{\lambda}^* \neq (0, \ldots, 0)$. In particular, the equilibrium $\boldsymbol{\lambda}^*$ is such that for each $i \in N$, λ_i^* satisfies the following KKT conditions

$$\begin{cases} h'(G^*(N)) - p_i'(\lambda_i^*) + \psi_i^* - \phi_i^* = 0 \\ \psi_i^* \lambda_i^* = 0 \quad \phi_i^*(\lambda_i^* - 1) = 0, \quad \psi_i^*, \phi_i^* \geq 0, \end{cases} \tag{22}$$

where by $G^*(N)$ we denote the total contribution level at equilibrium. Observe that, as a consequence of the assumption that for each $i \in N$, $p_i'(0) = 0$, it follows that $\lambda_i^* > 0$ for each $i \in N$. Indeed, if we suppose that there exists $i \in N$ s.t. $\lambda_i^* = 0$, the ith-equation of the KKT conditions cannot be satisfied, as

$$h'(G^*(N)) + \psi_i^* > 0,$$

because $\psi_i^* \geq 0$ and $h' > 0$ as h is strictly increasing.

If $\boldsymbol{\lambda}^*$ is a Nash equilibrium with aggregate level of contribution $G^*(N)$, then for all $i \in N$, λ_i^* is the unique solution in x of

$$p_i'(x) = h'(G^*(N)), \tag{23}$$

if the solution is smaller than or equal to 1, and 1 otherwise.

As at equilibrium the right term $h'(G^*(N))$ is the same for each $i \in N$, it immediately follows that, if the p_i's are s.t. $p_1'(\lambda) \leq \ldots \leq p_n'(\lambda)$, for each $\lambda \in [0,1]$, then $\lambda_n^* \leq \ldots \leq \lambda_1^*$.

B Proof of Proposition 4.2

(i) We suppose by contradiction that there exists $i \in N$ s.t. $\lambda_i^*(N) < \lambda_i^*(N \cup \{n+1\}) \leq 1$. Because of the strict convexity of p_i,

$$p_i'(\lambda_i^*(N)) < p_i'(\lambda_i^*(N \cup \{n+1\})),$$

and, from equation (23), and as $\lambda_i^*(N) < 1$,

$$h'(G^*(N)) = p_i'(\lambda_i^*(N)) < p_i'(\lambda_i^*(N \cup \{n+1\})) \leq h'(G^*(N \cup \{n+1\})).$$

From the first and from the last term, and because of the concavity of h, it follows that

$$G^*(N) > G^*(N \cup \{n+1\}). \tag{24}$$

If the total contribution without individual $n+1$ is strictly bigger than after her entrance, this implies that there exists at least one individual $j \in N$ which verifies the opposite inequality compared to i, i.e., such that $1 \geq \lambda_j^*(N) > \lambda_j^*(N \cup \{n+1\})$. Following the same reasoning than before, we conclude that $G^*(N) < G^*(N \cup \{n+1\})$ and this contradicts (24).

(ii) From (i), we know that $\lambda_i^*(N \cup \{n+1\}) \leq \lambda_i^*(N)$ for each $i \in N$. In particular, if the equality holds for each $i \in N$, the thesis follows trivially, as $G^*(N \cup \{n+1\}) = G^*(N) + \lambda_{n+1}^*(N \cup \{n+1\})$ and from Theorem 4.1 we know that at equilibrium individual $n+1$ provides a non-zero contribution. On the contrary, if there exists $i \in N$ s.t. $\lambda_i^*(N \cup \{n+1\}) < \lambda_i^*(N)$, then we can still conclude, following the same reasoning than in (i).

C Proof of Corollary 4.3

It is sufficient to observe that, at equilibrium,

$$\begin{aligned} U_i^*(N \cup \{n+1\}) &= h(G^*(N \cup \{n+1\})) - p_i(\lambda_i^*(N \cup \{n+1\})) \\ &> h(G^*(N)) - p_i(\lambda_i^*(N)) \\ &= U_i^*(N) \end{aligned}$$

where the inequality follows because of Proposition 4.2 and because of the strict concavity and convexity of function h and p_i respectively.

D Proof of Corollary 4.4

When the game Γ is homogeneous, the potential function Φ in (3) is a symmetric function on a symmetric domain. As a consequence, the unique maximum is also symmetric, i.e., $\lambda_i^* = \lambda^*$ for each $i \in N$.

E Proof of Proposition 4.5

(i) Firstly, from Equation (23), remembering we are now in a homogeneous case, we observe that $\lambda^* > 0$ is the unique solution of the following fixed point problem

$$\lambda = g(n, \lambda), \tag{25}$$

where function $g : \mathbb{N}_+ \times (0,1] \to [0, +\infty]$ is defined for each $\lambda \in (0,1]$ and for each $n \in \mathbb{N}_+$ as

$$g(n, \lambda) = \min\left\{(p')^{-1}(h'(n\lambda)), 1\right\}. \tag{26}$$

We consider the problem with the parameter n defined on the real interval $[1, +\infty]$. For each $n \in [1, +\infty]$, g is continuous in λ. Moreover, function g is monotonic non-increasing in n. Indeed,

$$\frac{\partial g}{\partial n} = \frac{\lambda}{p''((p')^{-1}(h'(n\lambda)))} h''(n\lambda) < 0$$

for each λ satisfying Equation (23), and g is identically 1 otherwise. Applying Corollary 1 of [Milgrom and Roberts(1994)], the unique fixed point $\lambda^*(n) > 0$ is non-increasing in n and, consequently, $\lim_{n \to +\infty} \lambda^*(n) \geq 0$ is well defined.

Secondly, we observe that, for each $\lambda > 0$,

$$\lim_{n \to +\infty} g(n, \lambda) = 0$$

pointwise. Indeed, $\lim_{x\to+\infty} h'(x) = 0$, $p'(0) = 0$ and p' is strictly monotonic.

If we suppose by contradiction that $\lim_{n\to+\infty} \lambda^*(n) = a > 0$, then, from Equation (25), it follows that

$$0 < \lim_{n\to+\infty} \lambda^*(n) = \lim_{n\to+\infty} g(n, \lambda^*(n)) = \lim_{n\to+\infty} g(n, a) = 0.$$

(ii) By Proposition 4.2-(ii), we know that $G^*(n+1) > G^*(n)$ for each $n \in N$. We suppose by contradiction that $\lim_{n\to+\infty} G^*(n) < +\infty$. It follows that $\lim_{n\to+\infty} h'(n\lambda^*(n)) > 0$ and then, by (26), that the solution to the fixed point problem is s.t. $\lim_{n\to+\infty} \lambda^*(n) > 0$, and this contradicts the second statement of Proposition 4.5-(i).

F Proof of Proposition 5.2

We observe that the modified game $\Gamma(N, \eta)$ is still a potential game, with potential function Φ as in (3), but defined on the restricted domain $\big[\{0\} \cup [\eta, 1]\big]^n$. Differently from $\Gamma(N)$, the strategy sets of the modified game are not closed intervals of the real line and the equilibria may not coincide with the global maxima of the potential function.

(i) When $|N| = 1$, the potential function and the utility function of the only individual coincide. Then, it still holds that a strategy profile is a Nash equilibrium if and only if it is a global maximum of Φ. If $\eta \leq \lambda^*$, the unique global maximum of Φ is still λ^*. If $\eta > \lambda^*$, owing to the assumption that $h(0) = -\infty$, the unique global maximum is η.

(ii) Now, let $|N| > 1$. For a better reading of the following of the proof, we present it organized in 6 steps.

Step 1 First,we observe that, as the unique NE of the original game $\Gamma(N)$ $\boldsymbol{\lambda}^*$ is strict, in particular it holds that for each $i \in N$,

$$h(G^*(N)) - p_i(\lambda_i^*) > h(G^*(N) - \lambda_i^*),$$

i.e., by deviating to zero any individual strictly decreases her utility.

Step 2 Given a subset of the individuals $S \subseteq N$, with $s = |S|$, and for each $\eta \in [0, 1]$, we define a new modified game $\Gamma'(S, \eta) = \langle S, [\eta, 1]^s, (U_i)_{i\in S}\rangle$, where the utility function U_i is defined by (1) for each individual $i \in S$, but it is now restricted on the domain $[\eta, 1]$. $\Gamma'(S, \eta)$ is still a potential game, with potential function Φ as in (3), defined on the restricted domain $\{0\}^{n-s} \cup [\eta, 1]^{s}$[8]. As for the original game $\Gamma(N)$ and the modified game $\Gamma(N, \eta)$, we observe that also the game $\Gamma'(S, \eta)$ is a potential game with a concave potential function and where, similarly to $\Gamma(N)$ but differently from $\Gamma(N, \eta)$, the strategy sets are closed intervals of the real line. It follows that a strategy profile is a Nash equilibrium if and only if it maximizes the potential function. Moreover, the potential function is strictly concave on the convex set on which it is defined, then it has a unique global maximum $\boldsymbol{\lambda}^M(S, \eta) \in [\eta, 1]^s$, which coincides with the unique Nash equilibrium of $\Gamma'(S, \eta)$. In particular, for each $\eta \in (0, 1]$, $\boldsymbol{\lambda}^M(N, \eta)$ is the unique Nash equilibrium of $\Gamma'(N, \eta)$.

[8]For simplicity, we make an abuse of notation by assuming that the zero components correspond to the first $n - s$ individuals.

Step 3 We define η^* as in (6). We show that $\boldsymbol{\lambda}^M(N,\eta^*)$ is a Nash equilibrium of $\Gamma(N,\eta^*)$. At first, observe that no individual has incentives to deviate to a quantity in $[\eta^*,1]$, because $\boldsymbol{\lambda}^M(N,\eta^*)$ is a Nash equilibrium of the game $\Gamma'(N,\eta)$. It remains to be shown that no individual has incentives to deviate to zero. Individual n does not have incentives by (6) or by (7). For any other individual $i \neq n$, s.t. $\lambda_i^M(N,\eta^*) = \eta^*$, if agent n, who is the most privacy concerned, does not have incentives to deviate from η^*, that is still valid for i. For any other agent $i \neq n$, s.t. $\lambda_i^M(N,\eta^*) > \eta^*$, as i does not have incentives to deviate to η^*, then, because of the concavity of the utility function, she cannot have incentives to deviate to 0.

Step 4 We observe that for each $\eta \in [0,\eta^*)$, $\boldsymbol{\lambda}^M(N,\eta)$ is a Nash equilibrium of $\Gamma(N,\eta)$. This follows trivially from Theorem 4.1 when $\eta \in (0,\lambda_n^*]$. Moreover, when $\eta \in (\lambda_n^*,\eta^*)$, we can repeat the same reasoning of Step 3, with the only difference that now individual n (and any other individual contributing η) is always strictly better off by contributing rather than free-riding, and then the inequality is always strict.

Step 5 We observe that for any $\eta \in (0,\eta^*]$, $\boldsymbol{\lambda}^M(N,\eta)$ is the unique Nash equilibrium of $\Gamma(N,\eta)$ s.t. each individual has a non-zero contribution, i.e., s.t. $\lambda_i^M(N,\eta) > 0$ for each $i \in N$. To show that, it is sufficient to observe that an equilibrium of $\Gamma(N,\eta)$ s.t. each individual has a non-zero contribution, is also an equilibrium of the game $\Gamma'(N,\eta)$ (as if an individual does not have incentives to deviate in $\{0\} \cup [\eta,1]$, she does not have incentives to deviate in the restricted strategy set $[\eta,1]$ either), and the equilibrium of $\Gamma'(N,\eta)$ is unique.

Step 6 We show that any other NE $\bar{\boldsymbol{\lambda}}$ of $\Gamma(N,\eta)$ is s.t. $\bar{\lambda}_i = \lambda_i^M(S,\eta)$ for each $i \in S$, where $S = \{i \in N | \bar{\lambda}_i \neq 0\}$. Let $\eta \in (\lambda_n^*,\eta^*]$ and let $\bar{\boldsymbol{\lambda}} = \bar{\boldsymbol{\lambda}}(N)$ be a Nash equilibrium of $\Gamma(N,\eta)$ s.t. $\bar{\lambda}_i(N) = 0$ for at least one $i \in N$. We define S as the set of individuals who contribute non-zero at this equilibrium. As, by definition of NE, none of the individuals has incentives to deviate, and, in particular, none of the individuals in S has incentives to deviate in $[\eta,1]$, it follows that $\bar{\boldsymbol{\lambda}}(S)$, i.e., the vector of the non-zero elements of the vector $\bar{\boldsymbol{\lambda}}(N)$, is a Nash equilibrium of the game $\Gamma'(S,\eta)$ defined in step 2 of this proof. As we have seen, this equilibrium is unique and it coincides with the unique Nash equilibrium $\boldsymbol{\lambda}^M(S,\eta)$ of the game $\Gamma(S,\eta)$ s.t. no individual is free-riding. Trivially, we can conclude that $\boldsymbol{\lambda}^*(N,\eta)$ is the unique NE of $\Gamma(N,\eta)$ s.t. $\lambda_i^*(N,\eta) > 0$ for all $i \in N$.

G Proof of Theorem 5.3

Remember that we assumed that the equilibrium contribution $\boldsymbol{\lambda}^*(N)$ of $\Gamma(N)$ is such that $\lambda_n^*(N) < 1$, i.e., it is such that the total level of contribution is not optimal.

When $\eta \in [0,\lambda_n^*(N)]$, $\boldsymbol{\lambda}^*(N,\eta) = \boldsymbol{\lambda}^*(N)$ and, consequently, G^* is constant, i.e., $G^*(N,\eta) = G^*(N)$.

When $\eta \in (\lambda_n^*(N),\eta^*]$, individual n is contributing at equilibrium $\lambda_n^*(N,\eta) = \eta > \lambda_n^*(N)$. We show now that for $\eta^* \geq \eta_2 > \eta_1 \geq \lambda_n^*$, $G^*(N,\eta_2) > G^*(N,\eta_1)$. Assume, by contradiction, that $G^*(N,\eta_2) \leq G^*(N,\eta_1)$. It follows that

$$h'(G^*(N,\eta_2)) \geq h'(G^*(N,\eta_1)), \tag{27}$$

because of the concavity of the public good utility function h. Moreover, as the total level of contribution with η_2 is smaller than or equal to the level of contribution with η^1, but we know that individual n has a strictly larger level of contribution, it follows that there exists $i \in N$ who contributes strictly less, i.e., s.t.,

$$\eta_2 \leq \lambda_i^*(N, \eta_2) < \lambda_i^*(N, \eta_1) \leq 1. \tag{28}$$

By equation (28), because of the convexity of the cost of contribution function p_i and from the KKT conditions for the potential function of the modified game $\Gamma^*(N, \eta)$, it follows that

$$h'(G^*(N, \eta_2)) \leq p_i'(\lambda_i^*(N, \eta_2)) < p_i'(\lambda_i^*(N, \eta_1)) \leq h'(G^*(N, \eta_1))$$

and this contradicts equation (27).

H Proof of Proposition 5.5

Let $\bar{\eta}$ be defined as in Definition 5.4, $\eta \in (\lambda_n^*, \bar{\eta})$, and $\boldsymbol{\lambda}^*(N, \eta) = \boldsymbol{\lambda}^M(N, \eta)$ be the unique non-zero components Nash equilibrium of the modified game $\Gamma(N, \eta)$, which is the unique maximizer of Φ on $[\eta, 1]^n$. As $\eta < \bar{\eta}$, it follows that

$$\Phi(\boldsymbol{\lambda}^M(N, \eta)) > \Phi|_{\lambda_n = 0}(\boldsymbol{\lambda}^{M0}(N \setminus \{n\}), \eta). \tag{29}$$

As $\boldsymbol{\lambda}^{M0}(N \setminus \{n\})$ is the potential maximizer of function Φ restricted on the domain $[\bar{\eta}, 1]^{n-1} \cup \{0\}$, it also holds that

$$\Phi|_{\lambda_n = 0}(\boldsymbol{\lambda}^{M0}(N \setminus \{n\}), \eta) \geq \Phi|_{\lambda_i = 0, \forall i \in N \setminus S}(\boldsymbol{\lambda}^{M0}(S, \eta)). \tag{30}$$

From (29) and (30), it follows that $\boldsymbol{\lambda}^*(N, \eta) = \boldsymbol{\lambda}^M(N, \eta)$ is the unique PMNE of the modified game $\Gamma(N, \eta)$.

I Proof of Corollary 5.6

First, we observe that it holds

$$h(G^*(N, \eta^*)) - p_j(\lambda_j^*(N, \eta^*)) \geq h(G^*(N, \eta^*) - \lambda_j^*(N, \eta^*)) \quad \forall j \in N \tag{31}$$

because, as we proved in Proposition 5.2, $\boldsymbol{\lambda}^*(N, \eta^*)$ is a NE of $\Gamma(N, \eta^*)$, and then no individual has incentives to deviate to zero. Now, we suppose by contradiction that the vector $\boldsymbol{\nu}$ s.t., $\nu_i = \lambda_i^*(N, \eta^*) = \lambda_i^*(N \setminus \{n\}, \eta^*)$ for each $i \neq n$ and $\nu_n = 0$ is not a NE of $\Gamma(N, \eta^*)$. This means that there exists an individual $j \in N \setminus \{n\}$ for whom it is convenient to deviate, i.e., s.t., it holds that

$$h(G^*(N, \eta^*) - \eta^*) - p_j(\lambda_j^*(N, \eta^*)) < h(G^*(N, \eta^*) - \eta^* - \lambda_j^*(N, \eta^*)). \tag{32}$$

From Equations (31) and (32), it follows that

$$\begin{aligned} h(G^*(N, \eta^*)) - h(G^*(N, \eta^*) - \lambda_j^*(N, \eta^*)) &\geq p_j(\lambda_j^*(N, \eta^*)) \\ &> h(G^*(N, \eta^*) - \eta^*) - h(G^*(N, \eta^*) - \eta^* - \lambda_j^*(N, \eta^*)) \end{aligned}$$

and this cannot hold because of the concavity of h. It follows that $\boldsymbol{\nu}$ is a NE of $\Gamma(N, \eta^*)$.

Second, we observe that $\boldsymbol{\nu}$ provides the same value of the potential function than $\boldsymbol{\lambda}^*(N, \eta^*)$. Then, if the second is a potential maximizer, the first one is as well.

J Proof of Theorem 5.9

(i) The trivial case when $n = 1$ is unchanged from the analogous case of Theorem 5.2-(i).

(ii) Now, let $n > 1$. For each $\eta \in (0, 1]$, let $\boldsymbol{\lambda}^M(n, \eta)$ be the unique global maximum of Φ on $[\eta, 1]^n$. When the individuals are homogeneous, the potential function Φ is a symmetric function that we are maximizing on a symmetric domain. As a consequence, the maximum is s.t., $\lambda_i^M(n, \eta) = \lambda^M(n, \eta)$ for each $i \in N$, with

$$\lambda^M(n, \eta) = \begin{cases} \lambda^*(n) & \text{if } 0 < \eta \leq \lambda^*(n) \\ \eta & \text{if } \lambda^*(n) < \eta \leq 1. \end{cases} \tag{33}$$

(iia) As a particular case of Proposition 5.2-(ii), for each $\eta \in (0, \eta^*(n)]$, where $\eta^*(n)$ is defined as in Definition 5.7, $\boldsymbol{\lambda}^*(n, \eta) = \boldsymbol{\lambda}^M(n, \eta)$ is the unique NE of the modified game $\Gamma(n, \eta)$ s.t. no individual is free riding. As a particular case of Proposition 5.5, for each $\eta \in (0, \bar{\eta}(n)]$, where $\bar{\eta}(n)$ is defined as in Definition 5.8, $\boldsymbol{\lambda}^*(n, \eta)$ is the unique PMNE of the modified game $\Gamma(n, \eta)$.

(iib) This result follows as the homogeneous case of the statement of Corollary 5.6.

(iic) For each $\eta \in (\eta^*(n), 1]$, we show that there does not exist a NE $\bar{\boldsymbol{\lambda}}$ of $\Gamma(n, \eta)$ s.t. $\bar{\lambda}_i > 0$ for each $i \in N$. We assume $\eta^*(n) < 1$. First we observe that, with a reasoning similar to the one in the proof of Proposition 5.2, F-Step 5, whenever we have a non-zero components NE of $\Gamma(n, \eta)$, this has to be a NE of the corresponding game Γ', an then a maximum of the potential function Φ restricted on the domain $[\eta, 1]^n$. It follows that, because of the symmetry, the only possible candidate non-zero components NE is the vector $\bar{\boldsymbol{\lambda}} = \boldsymbol{\eta} = (\eta, \ldots, \eta)$. Second, we observe that by definition, $\eta^*(n)$ is the smallest solution of the following fixed point problem

$$p(\eta) = h(n\eta) - h((n-1)\eta) \tag{34}$$

or, equivalently, of

$$\frac{p(\eta)}{\eta} = \frac{h(n\eta) - h((n-1)\eta)}{\eta} \tag{35}$$

where n is a fixed parameter. Then, we show that $\eta^*(n)$ is, in fact, the only solution of (35). Indeed, because of the concavity of h, we have that

$$\frac{h(y) - h(x)}{y - x}$$

is a decreasing function both in y and in x and then, in particular, the right term of (35) is decreasing in η. Moreover, because of the convexity of p, we have that

$$\frac{p(y) - p(x)}{y - x}$$

is an increasing function both in y and in x and then, in particular, the left term of (35) is increasing in η (when $y = \eta$, $x = 0$ and, consequently, $p(0) = 0$). It follows that the fixed point problem in (35) has at most one solution.

Third, we know by Theorem 4.1 that $\boldsymbol{\lambda}^*(n)$ is a SNE of $\Gamma(n)$, i.e., that $p(\lambda^*(n)) < h(n\lambda^*(n)) - h((n-1)\lambda^*(n))$. Moreover, we have just observed that $\eta^*(n)$ verifies the equality by definition whenever we have that $\eta^*(n) < 1$. As the solution of the previous fixed point problem, if it exists, is unique, it follows that

$$p(\eta) > h(n\eta) - h((n-1)\eta) \tag{36}$$

for each $\eta \in (\eta^*(n), 1]$. Equation (36) translates the fact that, when the individuals adopt the strategy $\boldsymbol{\eta} = (\eta, \ldots, \eta)$, with $\eta \in (\eta^*(n), 1]$, each individual in N is strictly better of by deviating to zero, and then the only candidate non-zero components NE cannot be a NE.

K Proof of Corollary 5.10

We have already observe that $\eta^*(n)$ is the smallest solution of the fixed point problem in (35). In particular, we have observed that the left term is constant in n and increasing in η, while the right term is decreasing both in n and in η. Applying Corollary 1 of [Milgrom and Roberts(1994)], the smallest fixed point $\eta^*(n) > 0$ is non-increasing in n and, consequently, $\lim_{n\to+\infty} \eta^*(n) \geq 0$ is well defined. Indeed, $\lim_{x\to+\infty} h(nx) - h((n-1)x) = 0$, $p(0) = 0$ and p is strictly monotonic.

If we suppose by contradiction that $\lim_{n\to+\infty} \eta^*(n) = a > 0$, then, from Equation (34), it follows that

$$\begin{aligned}
0 &< p(a) \\
&= \lim_{n\to+\infty} p(\eta^*(n)) \\
&= \lim_{n\to+\infty} h(n\eta^*(n)) - h((n-1)\eta^*(n)) \\
&= \lim_{n\to+\infty} h(na) - h((n-1)a) \\
&= 0.
\end{aligned}$$

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