ON MUTUAL COUPLING FOR ULAS: ESTIMATING AOAS IN THE PRESENCE OF MORE COUPLING PARAMETERS

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ABSTRACT

The problem of Angle-of-Arrival estimation of multiple sources in the presence of mutual coupling is addressed. The presence of unknown mutual coupling between antenna array elements is known to degrade the performance of direction-finding algorithms. We first present a result explaining why some traditional methods, that estimate Angles-of-Arrival (AoAs) of multiple sources in the presence of mutual coupling, suffer from an identifiability issue, when the number of coupling parameters exceeds a certain level. Then, we present a first method that estimates AoAs of sources when more coupling parameters are present, namely when the number of coupling parameters exceeds that certain level. Finally, we propose a refinement of the proposed algorithm, which could further enhance the AoA estimates. Simulation results have demonstrated the potential of the proposed method and its refined version, for different scenarios, as it enjoys better performance than existing methods. A better description of the paper could be found in the Conclusions section.

Index Terms— Angle-of-Arrival, Estimation, Mutual Coupling, MUSIC

1. INTRODUCTION

Mutual coupling between antennas is a popular problem in array signal processing. This phenomenon arises when antennas are close to each other [1], and thus the current developed in an antenna element depends on its own excitation and on the contributions from adjacent antennas. As a consequence, an ideal model is no longer valid, and therefore the performance of the high resolution algorithms that perform Angle-of-Arrival (AoA) estimation, such as MUSIC [2], ESPRIT [3], etc., deteriorate significantly. It is also worth mentioning other phenomena that perturb an ideal model, when not taken into account, such as different gain/phases [4] across antennas, synchronization and jitter effect [5], local scattering [6], etc.

Methods that aim on solving the mutual coupling problem are sometimes referred to as calibration methods, which are of two types: Offline and Online. In an offline calibration approach, one estimates the mutual coupling parameters using known locations, such as the techniques in [7–9]. In contrast, online calibration consists of jointly estimating the coupling and AoA parameters. In this paper, our main focus is on the latter.

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In the literature, several techniques deal with the online calibration problem. A RAnk-REduction estimator, known as RARE, was first proposed in [10] in the context of partly calibrated arrays. The same idea was used for totally uncalibrated Uniform Circular Arrays (UCA) in [11, 12] and Uniform Linear Arrays (ULA) in [13, 14]. This method makes use of the MUSIC algorithm to estimate AoAs in the presence of mutual coupling via rank reduction of an appropriate matrix. The method in [14] is a Recursive-RARE (R-RARE), which was shown to achieve better performance than the traditional RARE. A similar rank-reduction approach was adopted in [15]. Moreover, the algorithm in [16] is based on minimum eigenvalues instead. In addition, the method in [17] formulates the problem through a quadratic minimisation problem. The paper in [18] is an iterative method that assumes a diagonal source covariance matrix, i.e. the sources have no correlation. This is not always true. In addition, they treat the mutual coupling matrix and its conjugate-transpose as independent matrices. This, in turn, might lead to some sub-optimality.

In this paper, we present a result that shows why all the above algorithms do not function properly when the number of coupling parameters exceed a certain number. More precisely, let N and p denote the number of antennas and coupling parameters, respectively. The result shows that the above methods (except for [18]) will yield fake peaks whenever $p > \frac{N}{2}$. Furthermore, we propose a method that is capable of estimating the AoAs of multiple sources, even when $p > \frac{N}{2}$. In addition, we propose another method that could further refine the estimates of the first method. Simulation results demonstrate the potential of the proposed methods when compared to state-of-the-art methods.

Notations: Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^{\mathrm{T}}$, $(.)^*$ and $(.)^{\mathrm{H}}$ represent the transpose, conjugate and the transpose-conjugate operators. The matrix $\boldsymbol{I}$ is the identity matrix with suitable dimensions. For any matrix $\boldsymbol{B}$, the $(i,j)^{th}$ entry of $\boldsymbol{B}$ is represented as $(\boldsymbol{B})_{i,j}$. The vector $\boldsymbol{e}_k$ is the k^{th} column of $\boldsymbol{I}$. The vector $\mathbb{1}_k$ is a $k \times 1$ vector of all-ones. For any matrix $\boldsymbol{B}$, the operator $\|\boldsymbol{B}\|$ denotes the *Frobenius* norm. The statement $X \implies Y$ means that "if statement X is true, then Y is true."

2. SYSTEM MODEL

Consider q narrowband sources impinging a Uniform Linear Array (ULA), composed of $N > q$ antennas. The angles are denoted as $\boldsymbol{\Theta} = [\theta_1 \dots \theta_q]$. Given L time snapshots, we can write

$$\boldsymbol{X} = \widetilde{\boldsymbol{A}}(\boldsymbol{\Theta})\boldsymbol{S} + \boldsymbol{W} \tag{1}$$

where $\boldsymbol{X} \in \mathbb{C}^{N \times L}$ is the data matrix with l^{th} time snapshot, $\boldsymbol{x}(t_l)$, stacked in the l^{th} column of $\boldsymbol{X}$. The matrix $\boldsymbol{S} \in \mathbb{C}^{q\times L}$ is the source matrix. Similar to $\boldsymbol{X}$, matrix $\boldsymbol{S}$ contains the l^{th} transmitted source vector $\boldsymbol{s}(t_l)$ in its l^{th} column. The matrix $\boldsymbol{W} \in \mathbb{C}^{N\times L}$ is background noise. Moreover, the steering matrix $\widetilde{\boldsymbol{A}}(\boldsymbol{\Theta}) \in \mathbb{C}^{N\times q}$ is composed of q steering vectors, i.e. $\widetilde{\boldsymbol{A}}(\boldsymbol{\Theta}) = [\widetilde{\boldsymbol{a}}(\theta_1)\ldots\widetilde{\boldsymbol{a}}(\theta_q)]$. Each vector $\widetilde{\boldsymbol{a}}(\theta_i)$ is the response of the array to a source impinging from direction θ_i. Note that we shall use the notations $\widetilde{\boldsymbol{a}}(\theta)$ and $\boldsymbol{a}(\theta)$ to denote an array response in the presence and absence of mutual coupling, respectively, where

$$\boldsymbol{a}(\theta) = [1, z_\theta, \ldots z_\theta^{N-1}]^{\mathrm{T}} \tag{2}$$

with $z_\theta = e^{-j2\pi\frac{d}{\lambda}\sin(\theta)}$, d is the inter-element spacing and λ is the signal's wavelength. The response $\widetilde{\boldsymbol{a}}(\theta)$ is usually modelled as

$$\widetilde{\boldsymbol{a}}(\theta) = \boldsymbol{\mathcal{T}}_p(\boldsymbol{m})\boldsymbol{a}(\theta) \tag{3}$$

where $\boldsymbol{\mathcal{T}}_p(\boldsymbol{m}) \in \mathbb{C}^{N\times N}$ is a banded symmetric Toeplitz matrix defined as follows

$$\big(\boldsymbol{\mathcal{T}}_p(\boldsymbol{m})\big)_{i,j} = \begin{cases} m_{|i-j|} & \text{if } |i-j| < p \\ 0 & \text{else} \end{cases} \tag{4}$$

Note that the matrix $\boldsymbol{\mathcal{T}}_p(\boldsymbol{m})$ is independent from the AoAs. The model in equations (3) and (4) suggest that antennas that are placed at least p inter-element spacings apart do not interfere, i.e. $m_i = 0$ for all $i \geq p$. This is due to the fact that the mutual coupling is inversely proportional to the distance between antennas.

Throughout the paper, we assume the following:

- **A1**: $\widetilde{\boldsymbol{A}}(\boldsymbol{\Theta})$ is full column rank.
- **A2**: The transmitted signals $\boldsymbol{s}(t_l)$ are fixed within a snapshot. The signals are allowed to be highly, but not fully, correlated.
- **A3**: The number of sources is known.
- **A4**: The vector $\boldsymbol{w}(t_l)$ is Gaussian noise with zero mean and covariance $\sigma^2\boldsymbol{I}$ and independent from the sources.

For assumption **A3**, algorithms exist for estimating the number of sources, such as Minimum Description Length [19], Modified MDL [20], Benjamin Hochberg procedure [21], and so forth. We are now ready to address our problem: *"Given $\boldsymbol{X}$, p and q, estimate the AoAs $\boldsymbol{\Theta}$ in the presence of mutual coupling $\boldsymbol{\mathcal{T}}_p(\boldsymbol{m})$."*

3. PROPOSED ALGORITHM

This section makes use of the MUSIC algorithm in order to estimate the angles of arrival $\boldsymbol{\Theta}$ in the presence of mutual coupling. We start by exploiting the structure of the received signal covariance matrix. Under assumption **A4**, we have

$$\boldsymbol{R_{xx}} = \mathrm{E}\{\boldsymbol{x}(\mathrm{t})\boldsymbol{x}^{\mathrm{H}}(\mathrm{t})\} = \widetilde{\boldsymbol{A}}(\boldsymbol{\Theta})\boldsymbol{R_{ss}}\widetilde{\boldsymbol{A}}^{\mathrm{H}}(\boldsymbol{\Theta}) + \sigma^2\boldsymbol{I} \tag{5}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots \boldsymbol{u}_N$ be the normalized eigenvectors of $\boldsymbol{R_{xx}}$, where $\boldsymbol{u}_k$ corresponds to the k^{th} largest eigenvalue. It is well known that under assumptions **A1** and **A2**, the following holds:

$$\widetilde{\boldsymbol{a}}^{\mathrm{H}}(\theta_i)\boldsymbol{U_n}\boldsymbol{U_n^{\mathrm{H}}}\widetilde{\boldsymbol{a}}(\theta_i) = 0, \qquad \text{for all } i = 1\ldots q. \tag{6}$$

where $\boldsymbol{U_n} = [\boldsymbol{u}_{q+1}\ldots\boldsymbol{u}_N]$ is the *noise subspace*. However, in practice, one has access to the sample covariance matrix, i.e. $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}} = \frac{1}{L}\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}$. As before, let $\widehat{\boldsymbol{u}}_k$ correspond to the k^{th} largest eigenvalue of $\widehat{\boldsymbol{R}}_{\boldsymbol{xx}}$. Finally, the MUSIC algorithm estimates $\boldsymbol{\Theta}$ as follows

$$\{\widehat{\theta}_i\}_{i=1}^q = \arg\max_\theta \frac{1}{\widetilde{\boldsymbol{a}}^{\mathrm{H}}(\theta)\widehat{\boldsymbol{U}}_{\boldsymbol{n}}\widehat{\boldsymbol{U}}_{\boldsymbol{n}}^{\mathrm{H}}\widetilde{\boldsymbol{a}}(\theta)} \tag{7}$$

However, applying MUSIC directly to the problem in hand is not possible because the functional form of $\widetilde{\boldsymbol{a}}(\theta)$ is unknown. To proceed, we find the following useful:

Theorem 1: *Let $\boldsymbol{\alpha} = [\alpha_0, \alpha_1 \ldots \alpha_{p-1}]^T$ and $\boldsymbol{a} \in \mathbb{C}^{N\times 1}$. Define the corresponding matrix $\boldsymbol{\mathcal{T}}_p(\boldsymbol{\alpha})$, then*

$$\boldsymbol{\mathcal{T}}_p(\boldsymbol{\alpha})\boldsymbol{a} = \boldsymbol{B}_p\boldsymbol{\alpha} \tag{8}$$

where

$$\boldsymbol{B}_p = \left[\ \boldsymbol{a} \mid \boldsymbol{S}_1\boldsymbol{a} \mid \ldots \mid \boldsymbol{S}_{p-1}\boldsymbol{a}\ \right] \tag{9}$$

where $\boldsymbol{S}_k \in \mathbb{C}^{N\times N}$ is an all-zero matrix except at the k^{th} sub- and super-diagonals, which are set to 1.

Proof. See [17,23]. □

Using this theorem, we can say $\widetilde{\boldsymbol{a}}(\theta) = \boldsymbol{\mathcal{T}}_p(\boldsymbol{m})\boldsymbol{a}(\theta) = \boldsymbol{B}(\theta)\boldsymbol{m}$, where $\boldsymbol{B}(\theta)$ is defined as in equation (9). Note that equation (6) could be re-written as

$$\boldsymbol{z}^{\mathrm{H}}\boldsymbol{K}(\theta)\boldsymbol{z} = 0 \Longrightarrow \{\theta \in \boldsymbol{\Theta} \text{ and } \boldsymbol{z} = \boldsymbol{m}\} \tag{10a}$$

where

$$\boldsymbol{K}(\theta) = \boldsymbol{B}^{\mathbf{H}}(\theta)\boldsymbol{U_n}\boldsymbol{U_n^{\mathbf{H}}}\boldsymbol{B}(\theta) \tag{10b}$$

Therefore, one way to formulate the problem of estimating the AoAs in the presence of mutual coupling is to

$$\min_{\boldsymbol{z},\theta} \boldsymbol{z}^{\mathbf{H}}\widehat{\boldsymbol{K}}(\theta)\boldsymbol{z} \tag{11}$$

where $\widehat{\boldsymbol{K}}(\theta) = \boldsymbol{B}^{\mathbf{H}}(\theta)\widehat{\boldsymbol{U}}_{\boldsymbol{n}}\widehat{\boldsymbol{U}}_{\boldsymbol{n}}^{\mathbf{H}}\boldsymbol{B}(\theta)$. A suitable constraint on $\boldsymbol{z}$ is needed on the above problem due to the following consequence

Consequence 1: *For ULA type configurations, define the following sets:*

$$\boldsymbol{\Theta_+} = \Big\{sin^{-1}(\frac{k\lambda}{Nd}), \quad k = -\frac{N}{2}\ldots\frac{N}{2}\Big\} \tag{12}$$

$$\boldsymbol{\Theta_-} = \Big\{sin^{-1}(\frac{(k+\frac{1}{2})\lambda}{Nd}), \quad k = -\frac{N}{2}\ldots\frac{N}{2}\Big\} \tag{13}$$

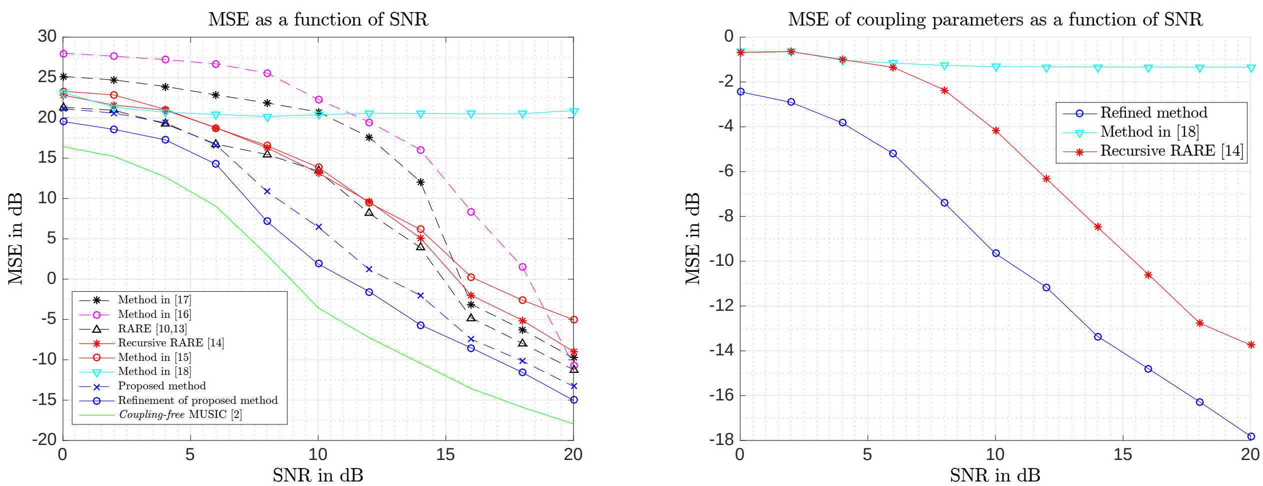
$$\boldsymbol{\Theta_\pm} = \Big\{\boldsymbol{\Theta_+} \cup \boldsymbol{\Theta_-}\Big\} \tag{14}$$

The matrix $\boldsymbol{B}(\theta)$ has the following characteristics:

- *If $p < \frac{N+2}{2}$, the matrix $\boldsymbol{B}(\theta)$ is full column rank.*
- *When $p \geq \frac{N+2}{2}$, we distinguish the following cases:*
  - *If N is even and $\theta \in \boldsymbol{\Theta_+}$, then* $\mathrm{rank}(\boldsymbol{B}(\theta)) = \frac{N}{2}$.
  - *If N is even and $\theta \in \boldsymbol{\Theta_-}$, then* $\mathrm{rank}(\boldsymbol{B}(\theta)) = \frac{N}{2} + 1$.
  - *If N is odd and $\theta \in \boldsymbol{\Theta_\pm}$, then* $\mathrm{rank}(\boldsymbol{B}(\theta)) = \frac{N+1}{2}$.
  - *Else $\boldsymbol{B}(\theta)$ is full column rank.*

Proof. See [22]. □

This means that when $p \geq \frac{N+2}{2}$, the matrix $\boldsymbol{B}(\theta)$, and consequently $\widehat{\boldsymbol{K}}(\theta)$, will admit a null-space, when $\theta \in \boldsymbol{\Theta_\pm}$. Hence, the algorithms proposed in [10] and [13-17] do not operate properly when $p > \frac{N}{2}$. For instance, the RARE [10,13] estimator, which is based on minimising the determinant of $\widehat{\boldsymbol{K}}(\theta)$ will yield fake peaks whenever $\theta \in \boldsymbol{\Theta_\pm}$. A similar argument is held for the methods in [14-17]. A clear explanation is given in [22]. To circumvent this issue, by minimising the cost function in equation (11) first with respect to $\boldsymbol{z}$,



(a) MSE in dB of AoA estimates as a function of SNR.

(b) Error on Coupling Parameters as a function of SNR.

Fig. 1: Experiment 1

the constraint should, in one way or another, prevent $\boldsymbol{z}$ to fall in the null-space of $\boldsymbol{B}(\theta)$. Hence, we propose the following

$$\begin{cases} \underset{\boldsymbol{z},\theta}{\text{minimise}} & \boldsymbol{z}^{\mathbf{H}}\widehat{\boldsymbol{K}}(\theta)\boldsymbol{z} \\ \text{subject to} & \boldsymbol{e}_{\mathbf{1}}^{\mathbf{H}}\boldsymbol{B}(\theta)\boldsymbol{z} = 1 \end{cases} \tag{15}$$

The constraint prevents the vector $\boldsymbol{z}$ to fall in the null-space of $\boldsymbol{B}(\theta)$. The solution to the above optimisation problem is given as [22]

$$\{\widehat{\theta}_k\}_{k=1}^{q} = \arg\max_{\theta}\left[\boldsymbol{a}_p^{\mathrm{T}}(\theta)\widehat{\boldsymbol{K}}^{-1}(\theta)\boldsymbol{a}_p^{*}(\theta)\right] \tag{16}$$

where $\boldsymbol{a}_p(\theta)$ is a $p\times 1$ vector defined as in equation (2). This algorithm could estimate $\boldsymbol{\Theta}$ even though $p > \frac{N}{2}$ given that $p+q\leq N$. For a better explanation, the reader is referred to [22].

4. REFINING THE AOA ESTIMATES BY ALTERNATING MINIMISATION

The problem in (15) is suboptimal in a MUSIC point of view. This is so because the objective function that is being optimised does not take into account that the vector of coupling parameters $\boldsymbol{m}$ is common to all AoAs, $\boldsymbol{\Theta}$. Mathematically, it is true that

$$\left\|\begin{bmatrix} \boldsymbol{U}_{\boldsymbol{n}}^{\mathbf{H}}\boldsymbol{B}(\theta_1) \\ \vdots \\ \boldsymbol{U}_{\boldsymbol{n}}^{\mathbf{H}}\boldsymbol{B}(\theta_q)\end{bmatrix}\boldsymbol{z}\right\|^2 = 0 \Longrightarrow \boldsymbol{m} = \boldsymbol{z} \tag{17}$$

Therefore, we seek to minimise the norm term in the above equation as follows

$$\begin{cases} \underset{\boldsymbol{z},\bar{\theta}_1\dots\bar{\theta}_q}{\text{minimise}} & \boldsymbol{z}^{\mathrm{H}}\widehat{\boldsymbol{S}}(\bar{\boldsymbol{\Theta}})\boldsymbol{z} \\ \text{subject to} & \left(\sum_{k=1}^{q}\boldsymbol{e}_{\mathbf{1}}^{\mathbf{H}}\boldsymbol{B}(\bar{\theta}_k)\right)\boldsymbol{m} = 1 \end{cases} \tag{18}$$

where

$$\widehat{\boldsymbol{S}}(\bar{\boldsymbol{\Theta}}) = \sum_{j=1}^{q}\widehat{\boldsymbol{K}}(\bar{\theta}_j) \tag{19}$$

To avoid confusion, we have used the notation $\bar{\theta}_k$ to indicate a variable of the problem, which is distinguished from the true value θ_k. The constraint here is a generalisation of the constraint in problem (15) in a sense that it prevents the cost function to be zero, when the AoA variables $\bar{\boldsymbol{\Theta}}$ are "simultaneously" in the set $\boldsymbol{\Theta}_{\pm}$, i.e. when $\bar{\theta}_1\in\boldsymbol{\Theta}_{\pm}\dots\bar{\theta}_q\in\boldsymbol{\Theta}_{\pm}$. Minimising first with respect to $\boldsymbol{z}$ gives [22]

$$\widehat{\boldsymbol{\Theta}} = \arg\max_{\bar{\boldsymbol{\Theta}}}\left\{\mathbb{1}_q{}^{\mathrm{T}}\boldsymbol{A}_p^{\mathrm{T}}(\bar{\boldsymbol{\Theta}})\widehat{\boldsymbol{S}}^{-1}(\bar{\boldsymbol{\Theta}})\boldsymbol{A}_p^{*}(\bar{\boldsymbol{\Theta}})\mathbb{1}_q\right\} \tag{20}$$

with $\boldsymbol{A}_p(\bar{\boldsymbol{\Theta}}) = [\boldsymbol{a}_p(\bar{\theta}_1)\dots\boldsymbol{a}_p(\bar{\theta}_q)]$. The cost function in equation (20) involves a $q-$dimensional search in AoA parameters. We, hereby, propose q "$1-$dimensional" searches done by alternating minimisations: At an iteration i, the following AoAs are estimated from previous iterations:

$$\widehat{\boldsymbol{\Theta}}_{\bar{i}} = [\widehat{\theta}_1\dots\widehat{\theta}_{i-1}] \tag{21}$$

Estimate $\widehat{\theta}_i$ as

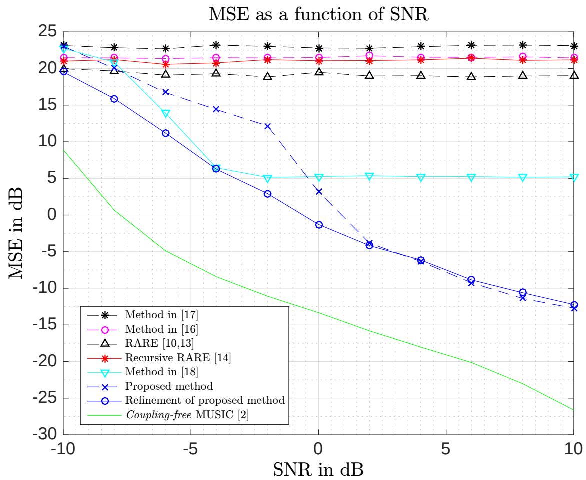
$$\widehat{\theta}_i = \arg\max_{\theta}\left\{\mathbb{1}_i{}^{\mathrm{T}}\boldsymbol{A}_p^{\mathrm{T}}(\widehat{\boldsymbol{\Theta}}_{\bar{i}},\theta)\widehat{\boldsymbol{S}}^{-1}(\widehat{\boldsymbol{\Theta}}_{\bar{i}},\theta)\boldsymbol{A}_p^{*}(\widehat{\boldsymbol{\Theta}}_{\bar{i}},\theta)\mathbb{1}_i\right\} \tag{22}$$

by picking $\widehat{\theta}_i\notin\widehat{\boldsymbol{\Theta}}_{\bar{i}}$ because values in $\widehat{\boldsymbol{\Theta}}_{\bar{i}}$ also maximise the above cost function. After estimating the vector of AoAs, $\widehat{\boldsymbol{\Theta}}$, one could also estimate the vector of coupling parameters via

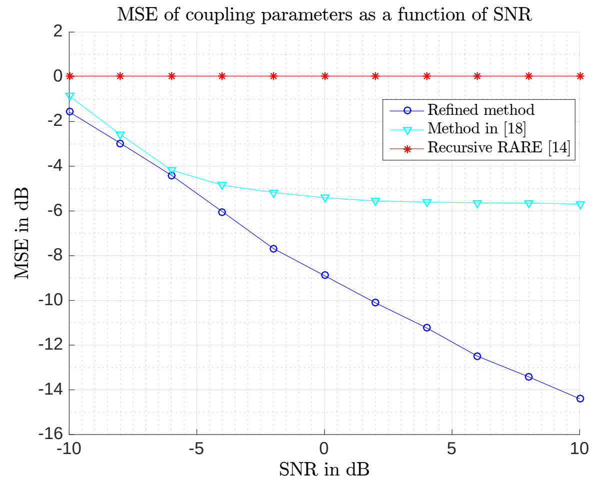
$$\widehat{\boldsymbol{m}} = \widehat{\boldsymbol{S}}^{-1}(\widehat{\boldsymbol{\Theta}})\boldsymbol{A}_p^{*}(\widehat{\boldsymbol{\Theta}})\mathbb{1}_q \tag{23}$$

Note that $\widehat{\boldsymbol{m}}$ is estimated up to an unknown constant, where we normalise $\widehat{\boldsymbol{m}}$ such that its first element is 1.

Remark: It should be noted that this approach is "optimal" in a MUSIC point of view. This is so because the approach forces the same coupling parameters for all the true AoAs $\boldsymbol{\Theta}$. However, we are faced with a multi-dimensional problem, which is equation (20). Nevertheless, we propose an alternating minimisation method to optimise the cost function. It is easily verified that a first iteration of the proposed alternating minimisation approach corresponds to the method in equation (16), hence *refining the AoA estimates*. In addition, the constraint in problem (18) avoids the false peaks that fall in $\boldsymbol{\Theta}_{\pm}$.



(a) MSE in dB of AoA estimates as a function of SNR.



(b) Error on Coupling Parameters as a function of SNR.

Fig. 2: Experiment 2

5. SIMULATION RESULTS

We have conducted two experiments to compare the MSE of estimated parameters with other methods. In all what follows, the experiments are conducted with 500 Monte-Carlo simulations. At a given SNR, let $\widehat{\theta}_k^{(j)}$ be the k^{th} estimate of θ_k at the j^{th} Monte-Carlo simulation. Similarly, let $\widehat{\boldsymbol{m}}^{(j)}$ be the estimate of $\boldsymbol{m}$ at the j^{th} Monte-Carlo simulation. Then, we define the MSE of AoA parameters is given as follows:

$$\text{MSE}_{\text{AoA}} = \frac{1}{500q} \sum_{j=1}^{500} \sum_{k=1}^{q} \left(\theta_k - \widehat{\theta}_k^{(j)}\right)^2 \tag{24}$$

Similarly, the MSE of the coupling parameters are computed as follows

$$\text{MSE}_{\boldsymbol{m}} = \frac{1}{500} \sum_{j=1}^{500} \frac{\left\|\boldsymbol{m} - \widehat{\boldsymbol{m}}^{(j)}\right\|}{\left\|\boldsymbol{m}\right\|} \tag{25}$$

As a testbench, we compare the MSE of several algorithms to the MUSIC algorithm that doesn't include mutual coupling, which we refer to as *Coupling-free* MUSIC.

In Experiment 1 (Fig. 1), we fix the following parameters: $N = 8$, $L = 10^3$, $p = 3$ with

$$\boldsymbol{m} = \left[1;\ 0.2 + 0.46j;\ 0.33 + 0.04j\right]^{\mathrm{T}} \tag{26}$$

Moreover, $q = 2$ Gaussian and correlated sources impinge the array at $\theta_1 = 5^\circ$ and $\theta_2 = 20^\circ$. The source covariance matrix is given as follows

$$\boldsymbol{R_{ss}} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \tag{27}$$

where the correlation coefficient is set to $|\rho| = 0.8$. According to Fig. 1a, where the figure depicts the MSE of the AoAs as a function of SNR, we see that the proposed algorithm (Section 3), along with its refined method (Section 4) outperform the RARE method [10,13], Recursive RARE [14], and the methods [15-18]. Observe that the method in [18] does not perform well at all. This is so because the source covariance matrix is not diagonal, since the sources are correlated. Furthermore, in Fig. 1b, where we have plotted the error on coupling parameters as a function of SNR for the algorithms that could estimate coupling parameters. We compare the performance of the proposed method for estimating coupling parameters (equation (23)), which is the last step in the refined method to those in [14] and [18]. The figure shows the potential of the proposed method compared to [14] and [18].

In Experiment 2 (Fig. 2), we fix the same parameters as in Experiment 1, except for $p = \frac{N+2}{2} = 5$ with

$$\boldsymbol{m} = \left[1;\ 0.2+0.46j;\ 0.33+0.04j;\ 0.12+0.01j;\ 0.01+0.03\right]^{\mathrm{T}} \tag{28}$$

Also, the sources now are un-correlated ($\rho = 0$). According to Fig. 2a, we see that all algorithms except for [18] and the proposed ones do not operate properly. This is so since p was chosen to be $\frac{N+2}{2}$. Therefore, according to ***Consequence 1***, the matrix $\boldsymbol{B}(\theta)$, and consequently $\widehat{\boldsymbol{K}}(\theta)$ admits a null-space whenever $\theta \in \boldsymbol{\Theta_+}$, and therefore the mentioned methods will always choose peaks corresponding to angles in $\theta \in \boldsymbol{\Theta_+}$. This will, indeed, affect the estimation of coupling parameters, as one can see in Fig. 2b.

6. CONCLUSIONS

There are several new results in this paper that should be highlighted: We have first presented ***Consequence 1***, which describes the spectral behaviour of an important matrix, namely $\boldsymbol{B}(\theta)$. This consequence explains the reason why other mentioned algorithms suffer from *"non-identifiability"* (i.e. when $p > \frac{N}{2}$). As a result, the consequence has led to the method in Section 3, which forms an optimisation problem to minimize a suitable cost function with a suitable constraint, to avoid false peaks. Furthermore, in Section 4, we propose another cost function that forces the same coupling parameters for all the true AoAs. The downside of this approach (in Section 4) is that the problem will be a multi-dimensional problem in AoAs (in contrast to the first approach). However, as proposed, we could solve this via alternating minimisations. We called this approach "refining the AoA estimates", because the first iteration of the alternating minimisation approach is, in fact, the proposed algorithm in Section 3.

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