Bits and Flops in Modern Communications: Analyzing Complexity as the Missing Piece of the Wireless-Communication Puzzle

Petros Elia (EURECOM) and Joakim Jaldén (KTH)

Challenges involved in telecommunications

Practical communications seek:

- very reliable communications
- of large quantities of data, at ultra-high rates (channel information)
- with strict delay limitations (no obsolete occupancy information)
- under any channel conditions (distributed users)
- with dynamically changing volumes of information (variable number of primary and secondary users)
- between arbitrary numbers of small and highly independent users (different network providers)
- that cooperate and complete for resources
- with little knowledge of the environment
- small power supplies
- **. . . at affordable computational cost**

Outage limited versus ergodic setting

Ergodic setting:

- long-term transmissions that see the full fading process
- long delays and high mobility
- code over channel fading to combat fading
- designs for the *average case* motivated by the law of large numbers

Outage limited setting:

- short term transmission that only see a snapshot of the fading
- delay constraints and limited mobility (in relation to data rate)
- code for successful transmission over many fading realizations
- designs for a *probabilistic worst case* (channels not in outage)

THIS TUTORIAL TARGETS RATE-RELIABILITY-COMPLEXITY TRADEOFFS IN THE OUTAGE LIMITED SETTING

General multi-dimensional communications

- Consider *outage limited* general MIMO communications

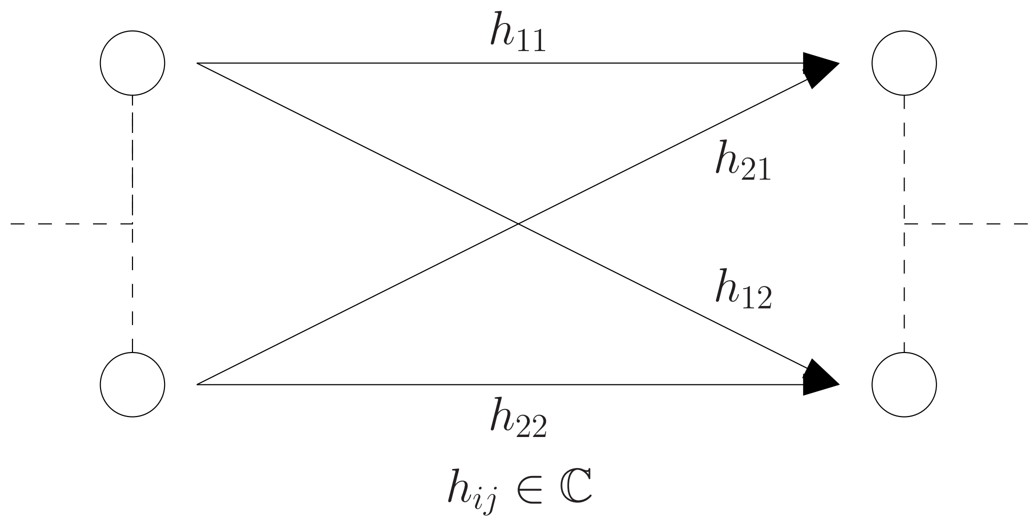
$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

- MIMO, MIMO-OFDM, MIMO-MAC, MIMO-ARQ, COOPERATIVE, HYBRID...

- MIMO: Performance $\uparrow \quad \longleftrightarrow \quad$ transceiver computational complexity $\downarrow$

In this general setting, we present joint performance-complexity limits

Example: the point-to-point quasi-static MIMO channel



Multiple antenna transmission between cooperating antenna arrays

Example: the point-to-point quasi-static MIMO channel

- n_{T}-transmit n_{R}-receive antenna quasi-static (flat-fading) MIMO channel

$$\mathbf{y}_t^c = \sqrt{\rho}\,\boldsymbol{H}^c \boldsymbol{x}_t^c + \boldsymbol{w}_t^c\,, \quad t = 1, \ldots, T$$

  ⋆ $\boldsymbol{H}^c \in \mathbb{C}^{n_{\mathrm{R}} \times n_{\mathrm{T}}}$, $\boldsymbol{x}_t^c \in \mathbb{C}^{n_{\mathrm{T}}}$, $\mathbf{y}_t^c \in \mathbb{C}^{n_{\mathrm{R}}}$, and $\boldsymbol{w}_t^c \in \mathbb{C}^{n_{\mathrm{R}}}$

- equivalent matrix (STBC) form

$$\boldsymbol{Y}^c = \sqrt{\rho}\,\boldsymbol{H}^c \boldsymbol{X}^c + \boldsymbol{W}^c$$

  ⋆ where $\boldsymbol{X}^c = [\boldsymbol{x}_1^c, \ldots, \boldsymbol{x}_T^c]$ and $\boldsymbol{W}^c = [\boldsymbol{w}_1^c, \ldots, \boldsymbol{w}_T^c]$

The general multi-dimensional linear channel model

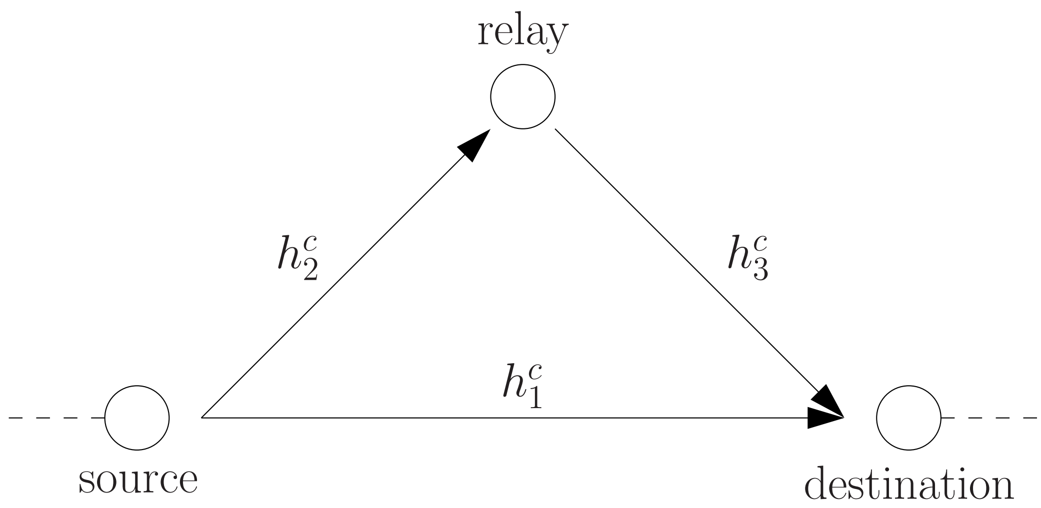
$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

$$\begin{aligned} \boldsymbol{x} &= [\boldsymbol{x}_1^{\mathrm{T}}, \ldots, \boldsymbol{x}_T^{\mathrm{T}}]^{\mathrm{T}} & \qquad \boldsymbol{x}_t^{\mathrm{T}} &= [\Re(\boldsymbol{x}_t^c)^{\mathrm{T}}\,,\ \Im(\boldsymbol{x}_t^c)^{\mathrm{T}}] \\ \boldsymbol{w} &= [\boldsymbol{w}_1^{\mathrm{T}}, \ldots, \boldsymbol{w}_T^{\mathrm{T}}]^{\mathrm{T}} & \qquad \boldsymbol{w}_t^{\mathrm{T}} &= [\Re(\boldsymbol{w}_t^c)^{\mathrm{T}}\,,\ \Im(\boldsymbol{w}_t^c)^{\mathrm{T}}]\,, \end{aligned}$$

and

$$\boldsymbol{H} = \sqrt{\rho}\,\boldsymbol{I} \otimes \begin{bmatrix} \Re(\boldsymbol{H}^c) & -\Im(\boldsymbol{H}^c) \\ \Im(\boldsymbol{H}^c) & \Re(\boldsymbol{H}^c) \end{bmatrix} \in \mathbb{R}^n$$

Example: Amplify and forward



$$\mathbf{y}_t^c = \begin{bmatrix} \sqrt{\rho} h_1^c & 0 \\ \rho b h_2^c h_3^c & \sqrt{\rho} h_1^c \end{bmatrix} \boldsymbol{x}_t^c + \begin{bmatrix} 0 \\ \sqrt{\rho} b h_3^c \end{bmatrix} w_t^c + \boldsymbol{v}_t^c \,, \qquad |b|^2 = \frac{1}{\rho |h_2^c|^2 + 1}$$

$$\downarrow$$

$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

The general MIMO channel

- Many (all?) general MIMO and co-operative scenarios with a centralized decoder fit to the model $\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$
  - ⋆ specific scenario mandates specific fading model for $\boldsymbol{H}$
  - ⋆ specific scenario mandates relevant constraints for $\boldsymbol{x}$
- Co-operative scenarios with decentralized decoders (e.g., dynamic decode and forward relaying) still may use the $\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$ model between transmitter, receiver pairs
  - ⋆ joint exposition of MIMO and co-operative communications
- Results on decoder technologies carry over to multi-user settings (with individual rates)

We use MIMO in the sense of coding over multi-dimensional signal spaces, not necessarily involving multiple co-located antennas

Thematic outline of tutorial

- Rate-Reliability-Complexity aspects of transceiver design
- Analysis focus on the high SNR limit ($\rho \to \infty$)
- Part I:
  - ⋆ outage limited communications
  - ⋆ decoding lattice codes and the available receiver algorithms
  - ⋆ Finding lim-optimal transceivers with subexponential complexity
  - ⋆ Complexity measures
- Part II:
  - ⋆ Covering the gap to optimal performance
  - ⋆ Performance - vs - Complexity tradeoff
  - ⋆ Fundamental rate-reliability-complexity limits
  - ⋆ Feedback
  - ⋆ Applications

Communicating over the general MIMO channel

$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

- Transmitted codeword from codebook $\mathcal{X}$: $\boldsymbol{x} \in \mathcal{X}$
- Rate R (assuming transmission over T time-slots)

$$R = \frac{1}{T}\log_2 |\mathcal{X}|$$

- Decoder (receiver) produce $\hat{\boldsymbol{x}} = \varphi(\mathbf{y}, \boldsymbol{H})$ at some computational cost
- Reliability measured by (average) probability of error

$$P_e = \mathrm{P}\left(\boldsymbol{x} \neq \hat{\boldsymbol{x}}\right) = \mathbb{E}_{\boldsymbol{H}}\left\{\mathrm{P}\left(\boldsymbol{x} \neq \hat{\boldsymbol{x}} | \boldsymbol{H}\right)\right\}$$

How do we measure computational cost?

The rate-reliability-complexity trade-off

- Several competing measures of computational cost
  - ⋆ Floating point operations (flops)
  - ⋆ Number of iterations
  - ⋆ Hardware utilization (processing units, parallelizability, etc)
- Given a maximum allowed computational cost (e.g., number of flops) C, we can achieve certain rates R and reliabilities P_e at a given SNR ρ
  - ⋆ What rate-reliability-complexity triplets $(R \uparrow, P_e \downarrow, C \downarrow)$ are achievable by given classes of codes and decoders?

These are very challenging questions!

How to create new decoding algorithms through complexity regulating policies $\mathcal{P}$

- Consider a detection algorithm (Algorithm A) with
  - Probability of error (reliability) $P_{e,\mathrm{A}}$
  - Required number of flops (for a given input) $F_\mathrm{A} = F_\mathrm{A}(\boldsymbol{H}, \mathbf{y})$
    - worst case complexity $\sup_{\boldsymbol{H},\mathbf{y}} F_\mathrm{A}(\boldsymbol{H}, \mathbf{y})$
- Consider another algorithm (Algorithm B) that use Algorithm A but terminates and calls a decoding error if $F_\mathrm{A}(\boldsymbol{H}, \mathbf{y}) \geq C$ for some C
  - Probability of error $P_{e,\mathrm{B}} \leq P_{e,\mathrm{A}} + \mathrm{P}\left(F_\mathrm{A}(\boldsymbol{H}, \mathbf{y}) \geq C\right)$
  - Algorithm B will always use less than C flops
- Imagine that there is a C such that
  - $\mathrm{P}\left(F_\mathrm{A}(\boldsymbol{H}, \mathbf{y}) \geq C\right) \ll P_{e,\mathrm{A}}$
  - $C \ll \sup_{\boldsymbol{H},\mathbf{y}} F_\mathrm{A}(\boldsymbol{H}, \mathbf{y})$

The is a lot to gain in terms of complexity, with a very small loss in reliability

The big question

Can we in tractably way characterize the set of achievable triplets (R, P_e, C) for any reasonably complex algorithms and codes?

…surprisingly, the answer is a partial yes,
if we rely on, e.g., high SNR asymptotics

The road ahead...

In order to conduct a reasonable rate-reliability-complexity study we need:

1. Flexible and parameterized codes (lattice codes)
2. A representative set of decoding algorithms (lattice, or sphere, decoders)

and a tractable mathematical framework (high SNR large deviations, DMT)

Degrees of freedom proxies for rate and reliability

- Proxies are often used to simplify computations and gain insight
  - ⋆ Reliability (probability of error) $\rightarrow$ diversity
  - ⋆ Rate of communication $\rightarrow$ multiplexing gain

We can do the same for complexity,
using reference algorithmic implementations

Exploiting degrees of freedom

- Diversification of resources: Utilize all the channel dimensions
- Example: SISO coherent BPSK v.s. QPSK

$$y = hx + w \quad y, h, x, w \in \mathbb{C}$$

where $h \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ and

$$x \in \{a(1 + \imath), a(1 - \imath), a(-1 + \imath), a(-1 - \imath)\}.$$

  - ⋆ bits in I and Q directions are independently detected (noise independent in directions)

- Basically same probability of error

$$P_{e,\text{BPSK}} \approx \frac{1}{4\rho} \quad \text{v.s.} \quad P_{e,\text{QPSK}} \approx \frac{1}{2\rho}$$

but double rate

- Overall lesson: seek to increase space dimensions and then use signals that give diversity and efficiently utilize all dimensions

Degrees of freedom and the multiplexing gain

- The capacity of the (non-fading) AWGN channel at SNR ρ is

$$C = \log_2(1 + \rho) \approx \log_2 \rho$$

  at high SNR

- The ergodic capacity of the $n_{\mathrm{T}} \times n_{\mathrm{R}}$ MIMO channel is

$$C \approx \min(n_{\mathrm{T}}, n_{\mathrm{R}}) \log_2 \rho$$

  at high SNR

- A given transceiver design has a *multiplexing gain* of r if

$$R \approx r \log_2 \rho$$

- r is equivalent to the degrees of freedom (used by the code)

EXAMPLE:

- Single-input multiple-output (SIMO) channel (L receive antennas)
- One (BPSK) symbol $x = \pm a$ transmitted

$$\mathbf{y} = \mathbf{h}x + \boldsymbol{w}, \quad \mathbf{y} \in \mathbb{C}^L, \quad \rho = \frac{a^2}{2}$$

where $\boldsymbol{h} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{I}_L)$

- Detection

$$\frac{\mathbf{h}^\dagger}{\|\mathbf{h}\|}\mathbf{y} = \|\mathbf{h}\|x + \frac{\mathbf{h}^\dagger}{\|\mathbf{h}\|}\boldsymbol{w}$$

- Overall error

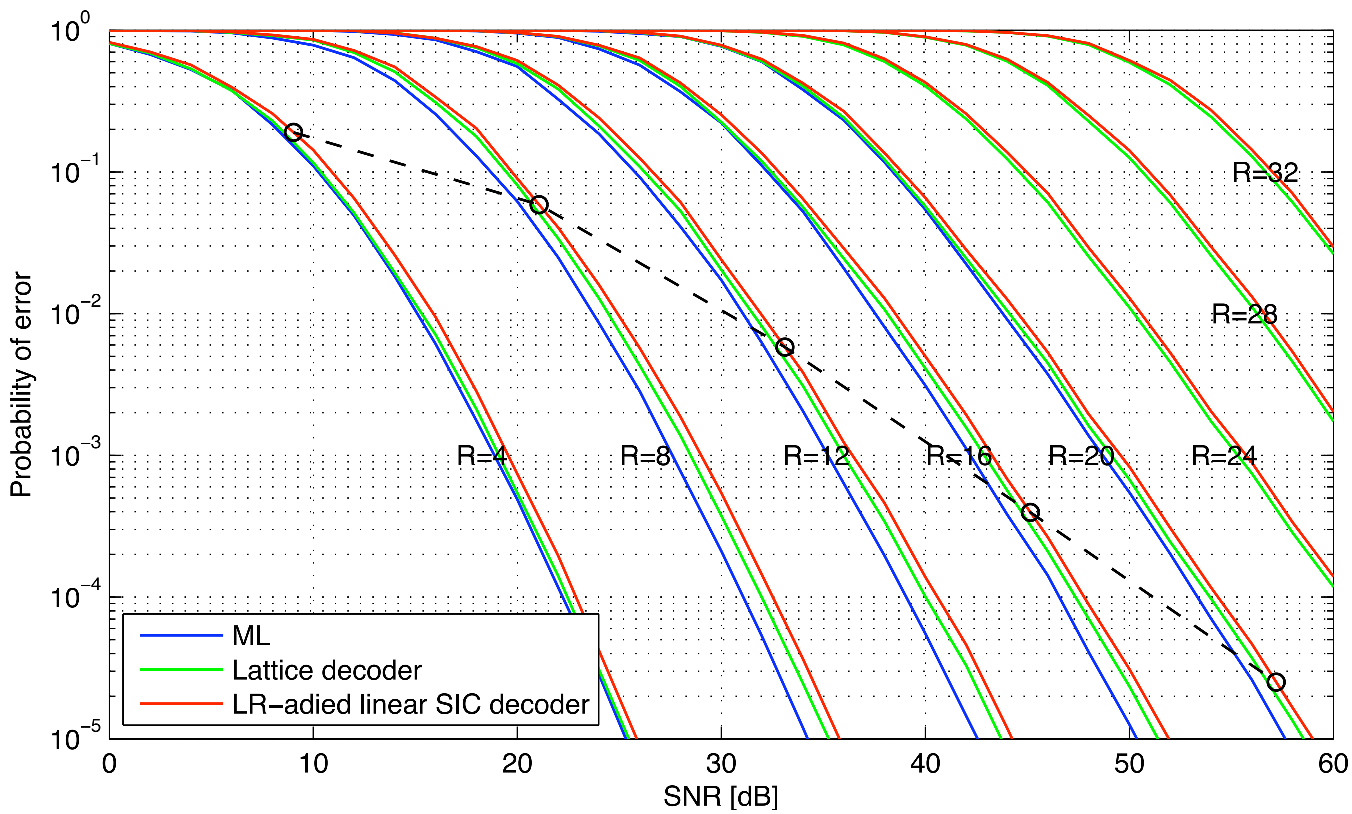
$$P_e \;=\; \mathbb{E}_{\boldsymbol{h}}\{P_{e|\boldsymbol{h}}\} = \int_0^\infty Q(\sqrt{2\|\boldsymbol{h}\|^2\rho}) f_{\|h\|^2}(\|\boldsymbol{h}\|^2) d\|\boldsymbol{h}\|^2 \approx \binom{2L-1}{L} (\frac{1}{4\rho})^L$$

- A transceiver has diversity d is the probability of error P_e satisfies

$$P_e \propto \frac{1}{\rho^d}$$

Numerical example (of rate and reliability)

The titled QAM lattice design over a $n_\mathrm{T} \times n_\mathrm{R} = 2 \times 2$ point-to-point i.i.d. Rayleigh fading MIMO channel



(Rate R in bits per channel use)

Diversity-Multiplexing Gain Tradeoff

IN THE CONTEXT OF:

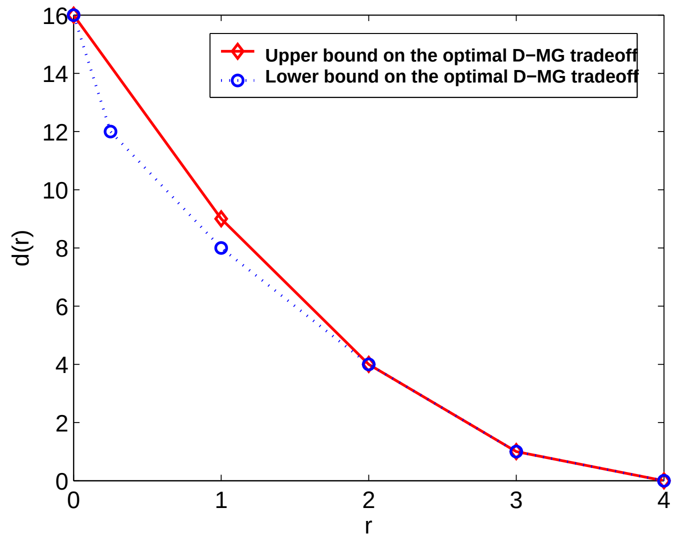
- Space-time schemes
- Ever increasing SNR
- Coding over just one channel realization

THE TWO PARAMETERS: DIVERSITY AND MULTIPLEXING GAIN

$$d \;=\; -\lim_{SNR\to\infty}\frac{\log(P_e)}{\log(SNR)} \qquad r = \lim_{\mathrm{SNR}\to\infty}\frac{R(\mathrm{SNR})}{\log \mathrm{SNR}}$$

- d: rate of decrease of P_e at some distance from the ergodic capacity
  - ⋆ need to step back from ergodic capacity for reliable communications
- r: how close you are to the ergodic capacity
  - ⋆ how many parallel channels you are utilizing for rate

Diversity-Multiplexing Gain Tradeoff (Zheng - Tse)



For a fixed integer multiplexing gain r, and $T \geq n_t + n_r - 1$, the maximum achievable diversity gain[1] $d(r)$ *over the* $n_{\mathrm{R}} \times n_{\mathrm{T}}$ *point-to-point MIMO channel is governed by*

$$d(r) \;=\; (n_{\mathrm{T}} - r)(n_{\mathrm{R}} - r)$$

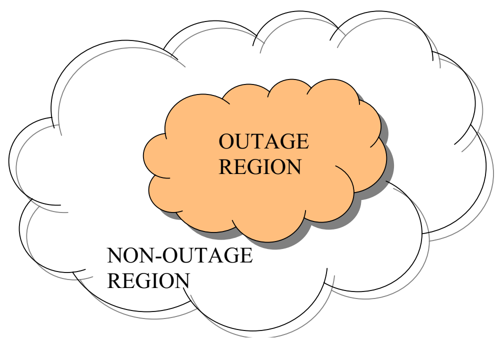
- Straight-line interpolation for non-integral values of r
- $T < n_t + n_r - 1$ gives upper and lower bounds on maximum $d(r)$

[1] L. Zheng and D. N. C. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels," *Trans. IT*, May 2003.

D-MG Tradeoff and Outage

- Outage region: The mutual information of the channel does not support the channel data rate.

$$\{\boldsymbol{H} \ : \ I(\boldsymbol{x}_t; \mathbf{y}_t \mid \boldsymbol{H}) < R\}$$



$$\begin{aligned} P_{\text{out}}(R) &= \mathbb{E}_{\boldsymbol{H}}\left(I(\boldsymbol{x}_t; \mathbf{y}_t \mid \boldsymbol{H}) < R\right) \\ &\doteq P\left[\log\det\left(I + \text{SNR}\boldsymbol{H}\boldsymbol{H}^{\dagger}\right) < R\right] = \text{SNR}^{-d_{\text{out}}(r)} \end{aligned}$$

- No matter what code you use you will have high probability of error

$$d_{\text{out}}(r) \ = \ d(r) \ = \ (n_t - r)(n_r - r)$$

- Corresponding outage based DMT characterizations now available for many co-operative scenarios

Coding-Decoding challenge (Rayleigh fading channel)

- "There exist some random Gaussian codes that meet the outage region"[2]
- "There exist some random lattice codes that meet the outage region"[3]
- "Currently no explicit non-random code is optimal" (dated statement)
- "Up until now DMT optimality required complex ML decoders"
- Result sparked interest and what was called "The worldwide race towards the DMT frontier"
- Is optimality possible? What encoders and decoders can achieve it?

Similar challenges in co-operative scenarios

[2] L. Zheng and D. N. C. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels," *Trans. IT*, May 2003.

[3] H. El Gamal,G. Caire, and M. O. Damen,"Lattice coding and decoding achieve the optimal diversity-multiplexing tradeoff of MIMO channels" *Trans. IT*, June 2004.

- Lattice design: Set of codes
  - ⋆ builds on work by de Buda, Poltyrev, Forney et al., Urbanke-Rimoldi, Erez-Zamir, El Gamal-Caire-Damen, and many others
- Start with a lattice

$$\Lambda_0 \triangleq \{\boldsymbol{G}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$

- Create a variably dense lattice

$$\Lambda \triangleq \phi\Lambda_0 \,, \quad \phi \triangleq \rho^{-\frac{rT}{n}}$$

- $\mathcal{R} \subset \mathbb{R}^n$ is a compact convex shaping region that picks out codewords
- Select codewords from a limited region:

$$\mathcal{X} = \Lambda \cap \mathcal{R} = \Lambda \cap \mathcal{R}$$

$$|\mathcal{X}| = \rho^{rT}, \ \mathrm{E}\left\{\|\boldsymbol{x}\|^2\right\} \leq T$$

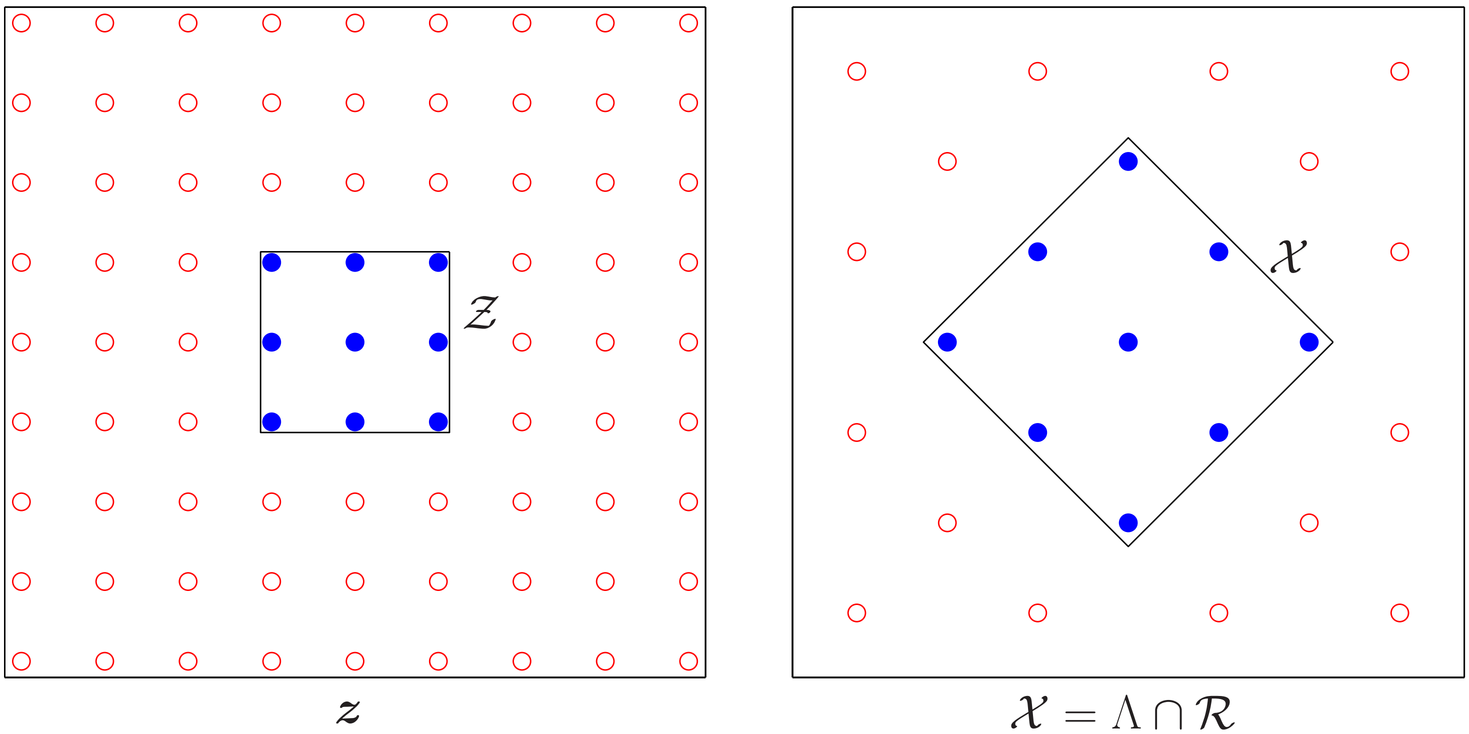
- Creates a composite code-channel MIMO relation

$$\mathbf{y} = \underbrace{\phi\boldsymbol{H}\boldsymbol{G}}_{\boldsymbol{M}}\boldsymbol{z} + \boldsymbol{w}$$

- $\mathbf{y}$ is a perturbed lattice point form the random lattice $\boldsymbol{H}\Lambda = \boldsymbol{M}\mathbb{Z}^n$

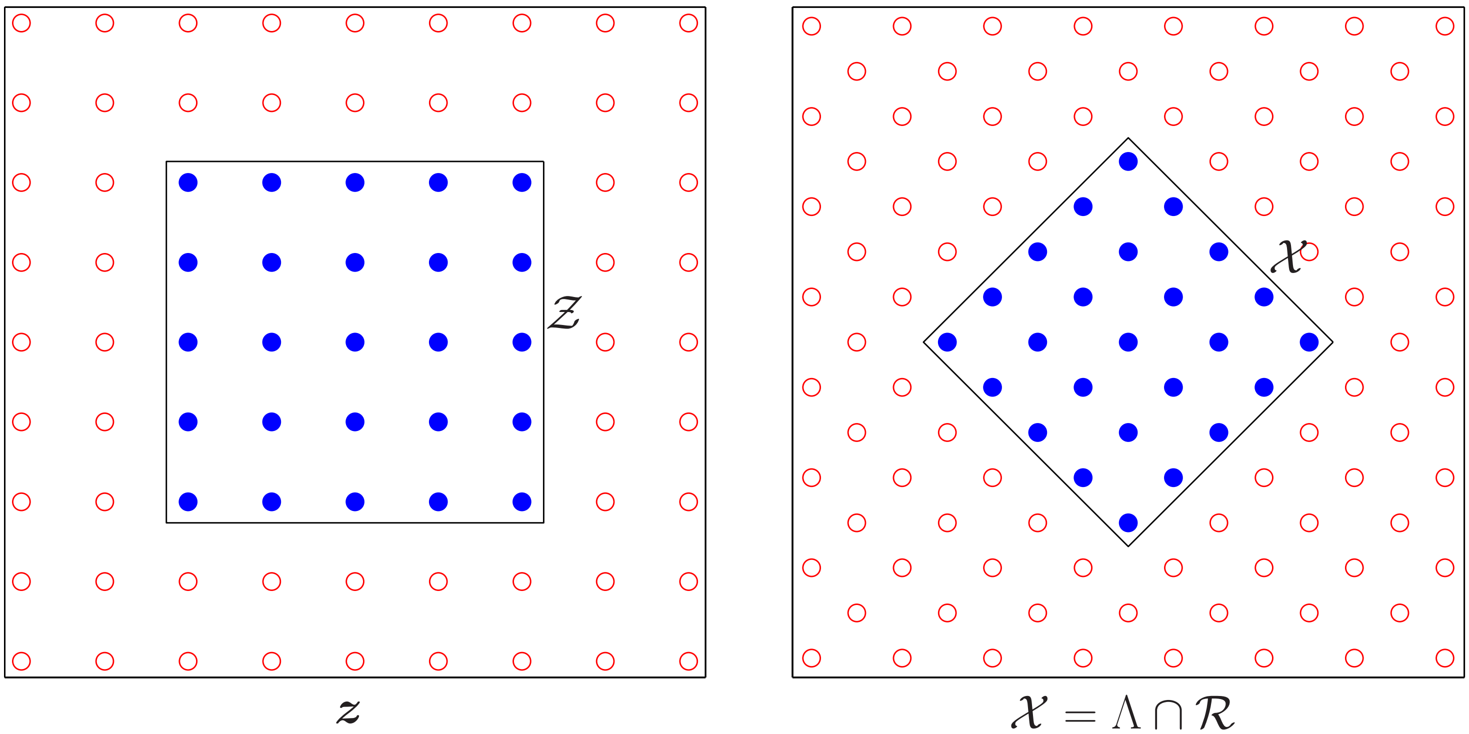
Rate (or multiplexing gain) controlled by lattice density

Lattice designs: Illustration



$$\Lambda = \{\phi \boldsymbol{G} \boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$

$$\mathcal{Z} = (\phi \boldsymbol{G})^{-1} \mathcal{X} \subset \mathbb{Z}^n$$



$$\Lambda = \{\phi \boldsymbol{G}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$

$$\mathcal{Z} = (\phi \boldsymbol{G})^{-1}\mathcal{X} \subset \mathbb{Z}^n$$

FAST FORWARD TO ... FULL DIVERSITY AND FULL DEGREES OF FREEDOM

- Linear dispersion codes[4]

$$X = f_1 A_1 + f_2 A_2 + f_3 A_3 + f_4 A_4, \quad A_i \in \mathbb{C}^{2\times 2}$$

- Threaded algebraic constructions TAST[5]

$$X = \begin{bmatrix} \sigma_1(f_1, f_2) & \sigma_1(f_3, f_4) \\ \sigma_2(f_3, f_4) & \sigma_2(f_1, f_2) \end{bmatrix}$$

  - ⋆ full rate benefits but no coding gain guarantees for increasing rate
  - ⋆ no diversity guarantees for increasing rate

- Cyclic Division Algebra (CDA) Codes [6]

[4] Hassibi-Hochwald
[5] El Gamal-Hammons
[6] Sethuraman et al. ,Belfiore-Rekaya, Kiran-Rajan

Solution: DMT optimal explicit constructions

FIRST DMT OPTIMAL EXPLICIT CONSTRUCTION:
CYCLIC DIVISION ALGEBRAS

- Unified DMT optimal code design and construction criteria[7]
  - codes explicitly constructed for all dimensions
  - CDA-based codes (drawing from work of [8])
  - employ *a single and identifiable* lattice generator matrix
  - codes guarantee continuous DMT optimality for all fading statistics

APPROXIMATE UNIVERSALITY[9]

- Approximate universality crucial for code design in cooperative communications and several other MIMO scenarios

[7] Elia et al. 2006
[8] Sethuraman et al.,Belfiore-Rekaya,Kiran-Rajan
[9] Tavildar and Viswanath 2006

The magic of cyclic division algebras

Division algebra D

holds n^2QAM elements

$x = \sum_{j=0}^{n-1} l_j \gamma^{\frac{j}{n}} \in D,$

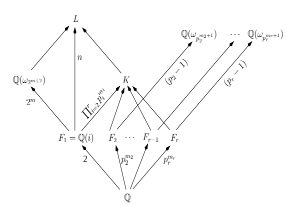
$$\begin{vmatrix} l_0 & \gamma\sigma(l_{n-1}) & \gamma\sigma^2(l_{n-2}) & \cdots & \gamma\sigma^{n-1}(l_1) \\ l_1 & \sigma(l_0) & \gamma\sigma^2(l_{n-1}) & \cdots & \gamma\sigma^{n-1}(l_2) \\ \vdots & & & & \vdots \\ l_{n-2} & \sigma(l_{n-3}) & \sigma(l_{n-4}) & \cdots & \gamma\sigma^{n-2}(l_{n-1}) \\ l_{n-1} & \sigma(l_{n-2}) & \sigma(l_{n-3}) & \cdots & \sigma^{n-1}(l_0) \end{vmatrix}$$

Maximal field L

$l_j = \sum_{i=0}^{n-1} f_{j,i}\theta^i \in L$

Lattice for
transmit constellation

transmit constellation



n^2 QAM information symbols

Base field F

$F = \mathbb{Q}(\imath)$

$$\begin{vmatrix} f_{0,0} & f_{0,1} & \cdots & f_{0,n-1} \\ f_{1,0} & f_{1,1} & \cdots & f_{1,n-1} \\ \vdots & & & \vdots \\ f_{n-1,0} & f_{n-1,1} & \cdots & f_{n-1,n-1} \end{vmatrix}$$

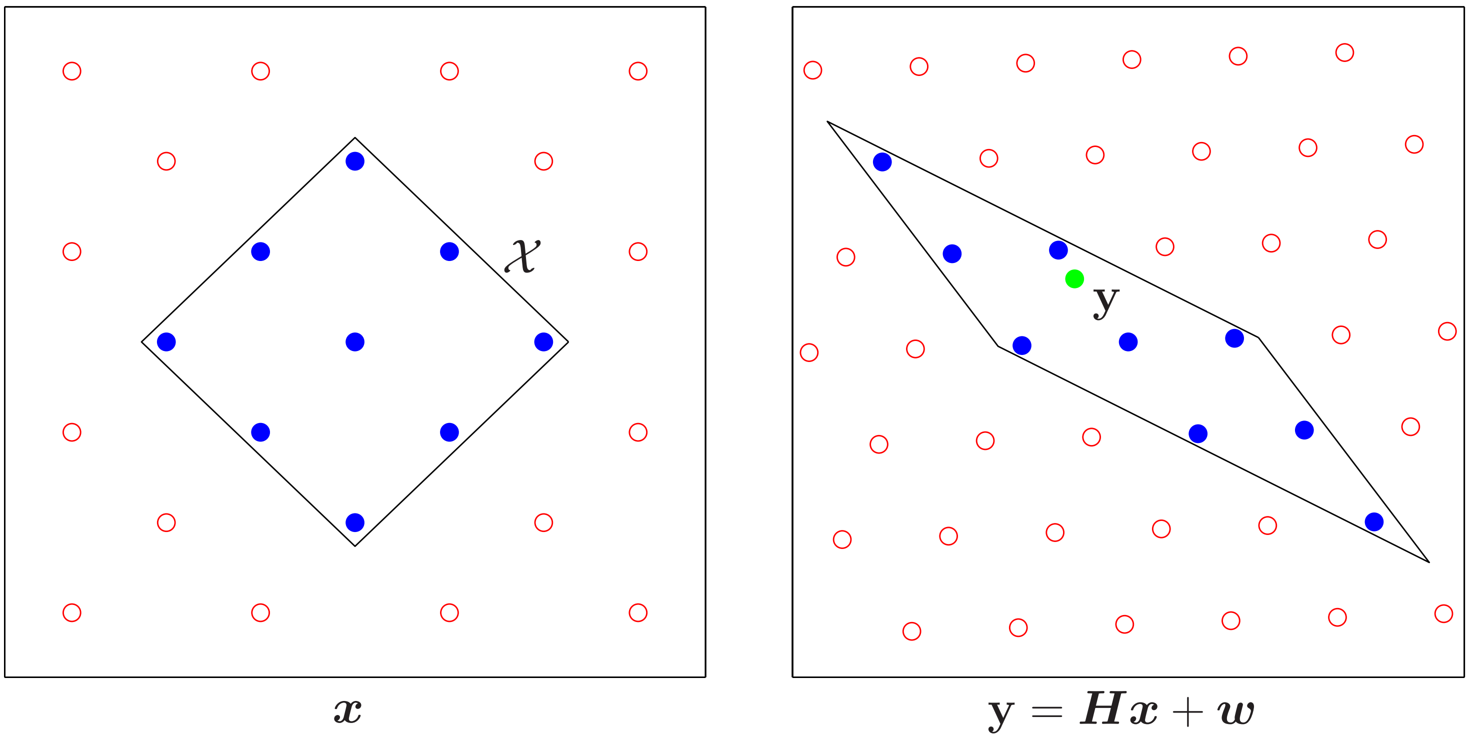
Versatility of approximately universal designs

- Dense & enumerable constellations but distant codematrices
- Distances increase optimally, in increasing time, space, and $-r$
- Code-channel distances manipulated to meet information theoretic limits
  - ⋆ complements of algebraic structure
  - ⋆ even in extreme, puncture-like channels

Powerful properties opened ways to solving puzzles

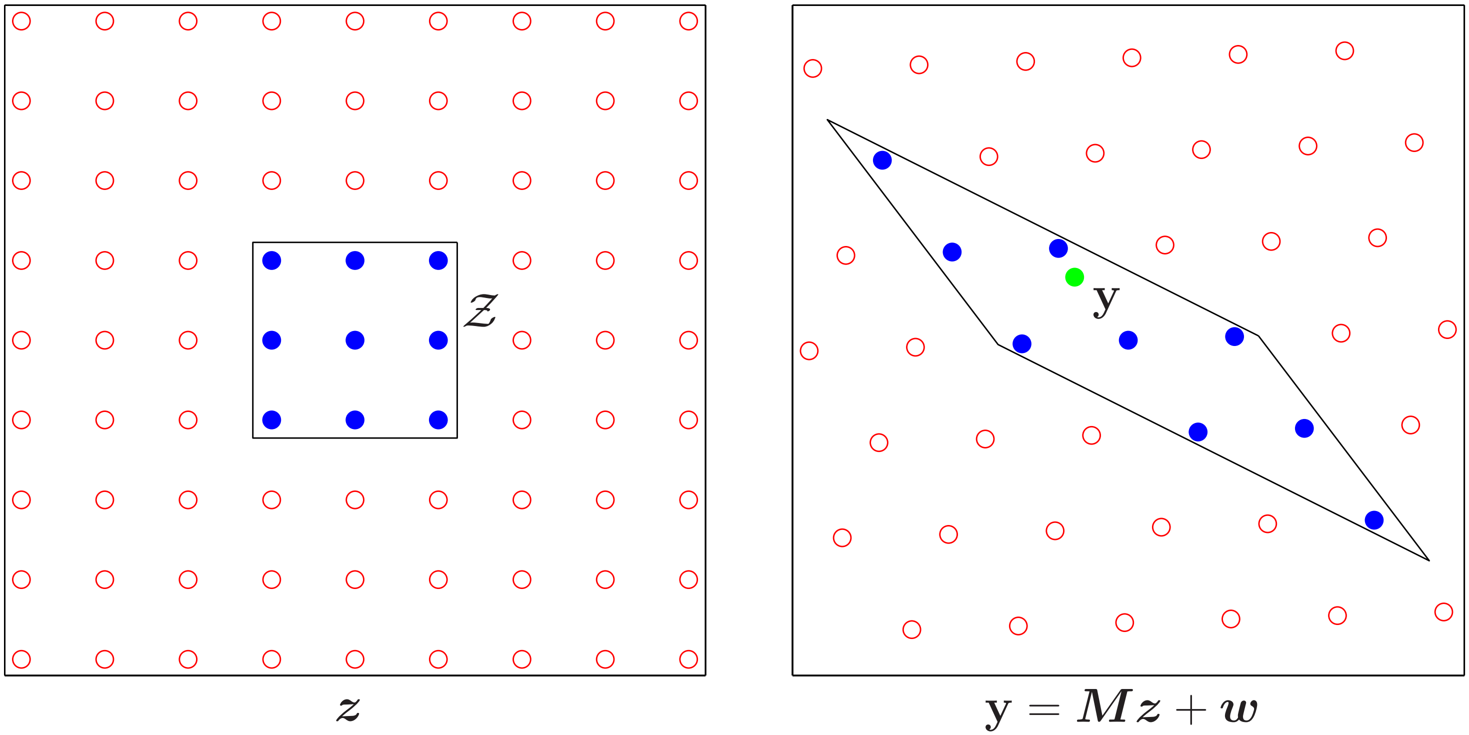
- DMT optimality achieved for several MIMO scenarios, in most general setting
  - ⋆ ([Elia et.al.],[Tavildar-Viswanath],[Elia-Kumar],[Yang-Belfiore])
    ([K.R. Kumar-Caire],[Lu-Hollanti])

The ML decoder: Illustration



The receiver sees a skewed codebook in noise
The ML objective is to find closest codeword hypothesis $\boldsymbol{H}\hat{\boldsymbol{x}}$ to $\mathbf{y}$

The ML decoder: Illustration



The receiver sees a skewed codebook in noise

The ML objective is to find closest codeword hypothesis $\boldsymbol{M}\hat{\boldsymbol{z}}$ to $\mathbf{y}$

MAXIMUM LIKELIHOOD (ML) DECODING

- The ML decoder solves a closest vector problem (CVP) in $\mathcal{X}$

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2$$

- Equivalent formulation in terms of code-channel lattice

$$\hat{\boldsymbol{z}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{z}} \in \mathcal{Z}} \|\mathbf{y} - \boldsymbol{M}\hat{\boldsymbol{z}}\|^2$$

LATTICE DECODING

- The (naive) lattice decoder solves a closest vector problem (CVP) in $\mathbb{Z}^n$

$$\hat{\boldsymbol{x}}_{\mathrm{NLD}} = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{Z}^n} \|\mathbf{y} - \boldsymbol{M}\hat{\boldsymbol{z}}\|^2$$

POTENTIAL PROBLEM: CVPS ARE IN GENERAL NP-HARD
(EVEN WITH FREE PRE-PROCESSING OF $\boldsymbol{M}$)

Need for generally efficient decoding procedures

- Unfortunately, most high performance lattice codes were previously known to perform provably well only in the presence of an ML decoder
  - ⋆ decoding complexity has remained a fundamental limitation in obtaining provably good error probability performance in a computationally efficient manner
  - ⋆ the limitation, roughly speaking, originates from the fact that optimal codes must in general be drawn from lattices whose dimension 'matches' the inherently high dimension of $\boldsymbol{H}$
  - ⋆ on top of that, in all but rare cases, the diversity requirements force code-channel lattices that cannot be decomposed into substantially 'smaller' and simpler component lattices, without severely sacrificing rate gains

Complexity is the missing piece of the pussle

The curse of dimensionality

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2, \qquad \boldsymbol{H} \in \mathbb{R}^{n \times n}$$

- This high dimensionality, in conjunction with the high spectral efficiency, introduce prohibitive ML decoding complexity
- The resulting complexity bottleneck brought to the fore the need for efficient decoding algorithms, some of which we review here

Channel	n
$m \times m$ MIMO	$2m^2$
$m \times m$, L-tone MIMO-OFDM	$2m^2L$
$m \times m$, m-round MIMO-ARQ	$2m^2$
$m \times m$, L-round MIMO-ARQ (AU)	$2m^2L$
m-relay OAF	$2m$
2-relay OSDF, NSDF ($r = 2$)	$32, 162$
m-relay NAF	$8(m-1)$ $8(m-1)^2$
m-relay DDF, L-slots, $m > 2$	$2m^2L$

How do we measure complexity on the DMT scale?

- Linear MIMO channel model

$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

- Maximum likelihood decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2$$

  - ⋆ full search consider $|\mathcal{X}| \doteq \rho^{rT}$ codeword hypothesis

- Zero forcing decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ZF}} = Q_{\Lambda}\left[\boldsymbol{H}^{\dagger}\mathbf{y}\right]$$

  - ⋆ complexity is independent of SNR ρ

Definition: Let C be the complexity (e.g., flops) of a particular decoder structure. We then say that this decoder structure has a *complexity exponent* of c if $C \doteq \rho^c$, i.e.,

$$\limsup_{\rho \to \infty} \frac{\log C}{\log \rho} = c$$

where $0 \leq c \leq rT$ for any reasonable decoder structure

- What (high SNR) rate-reliability-complexity triplets

$$(r \uparrow, d \uparrow, c \downarrow)$$

  are achievable?

Decoding techniques

In order to provide a meaningful discussion of the decoding complexity of DMT optimal codes we now consider several pertinent techniques used in state-of-the-art decoders in the outage limited setting

Linear decoders (receivers)
Sphere decoders (universal lattice decoders)
Lattice reduction techniques

Linear receivers - Zero Forcing

Substantial interest in ZF and MMSE linear receivers, due to the simplicity of implementation

$$\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$$

- Interference caused by a generally non-orthogonal $\boldsymbol{M}$ suppressed by multiplying $\mathbf{y}$ by the pseudo-inverse $\boldsymbol{M}^{\ddagger} \triangleq (\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M})^{-1}\boldsymbol{M}^{\mathrm{T}}$
- Decision by minimum distance (rounding) quantization:

$$\tilde{\boldsymbol{z}}_{\mathrm{ZF}} = \boldsymbol{M}^{\ddagger}\mathbf{y} \underset{\text{quantization}}{\longrightarrow} \hat{\boldsymbol{z}}_{\mathrm{ZF}}$$

- Limitation: ill-conditioned channel matrices cause considerable noise amplification
  - ⋆ on the order of the diagonal entries of $(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M})^{-1}$.

- Interference partially suppressed by a linear MMSE filter

$$\tilde{\boldsymbol{z}}_{\mathrm{MMSE}} = \underbrace{(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M} + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{M}^{\mathrm{T}}}_{\text{L-MMSE filter}}\mathbf{y} \underset{\text{quantization}}{\longrightarrow} \hat{\boldsymbol{z}}_{\mathrm{MMSE}}$$

- MMSE based linear receivers address noise amplification issue
- Can be seen as ZF receivers over an extended system model[10]

$$\tilde{\boldsymbol{y}} \triangleq \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} = \tilde{\boldsymbol{M}}\mathbf{x} + \begin{bmatrix} \boldsymbol{w} \\ \mathbf{0} \end{bmatrix},$$

  - ⋆ w.r.t. better conditioned channel matrix

$$\tilde{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{H} \\ \sigma\boldsymbol{I} \end{bmatrix}, \quad \text{s.t.} \quad \tilde{\boldsymbol{M}}^{\mathrm{T}}\tilde{\boldsymbol{M}} = \boldsymbol{M}^{\mathrm{T}}\boldsymbol{M} + \sigma^2\boldsymbol{I}, \tag{1}$$

[10] D. Wubben, R. Bohnke, V. Kuhn and K.-D. Kammeyer , “Near-maximum-likelihood detection of MIMO systems using MMSE-based lattice reduction”, *ICC*, June 2004

The DMT of linear receivers

- The complexity of linear receivers is well understood (a function of n alone)
  - ⋆ complexity exponent $c = 0$
- For ill-conditioned channel matrices, both ZF and MMSE decoders are generally suboptimal[11]

$$d_{\mathrm{LIN}}(r) \triangleq (n_r - n_t + 1)(1 - \frac{r}{n_t}).$$

  - ⋆ assuming $n_{\mathrm{T}} \times n_{\mathrm{R}}$ ($n_{\mathrm{R}} \geq n_{\mathrm{T}}$) point-to-point quasi-static MIMO channel, i.i.d. Rayleigh fading, and random Gaussian codes
- This is substantially suboptimal as compared to ML

$$d_{\mathrm{ML}}(r) \triangleq (n_{\mathrm{T}} - r)(n_{\mathrm{R}} - r)$$

  for $r = 0, 1, \cdots, \min(n_{\mathrm{T}}, n_{\mathrm{R}})$
- Any triplet (r, d, c) in

$$\left\{(r, d, c) \,|\, d \leq d_{\mathrm{LIN}}(r)\,, c \geq 0\,\right\}$$

  is achievable with linear decoders

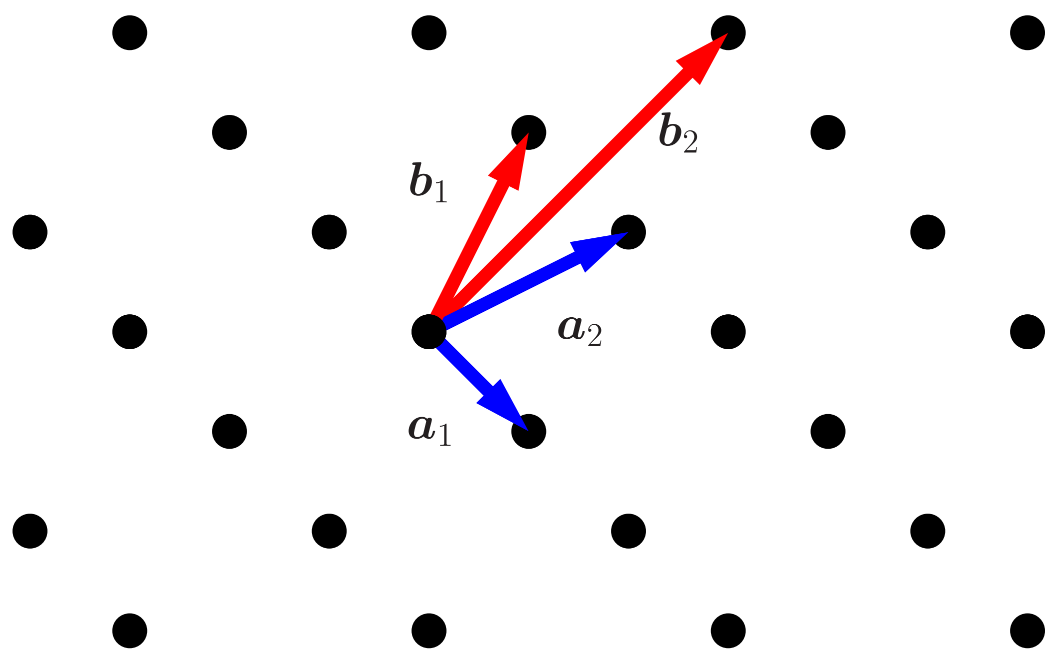
[11] K. R. Kumar, G. Caire, and A. L. Moustakas, “Asymptotic Performance of Linear Receivers in MIMO Fading Channels”, *Trans IT*, Oct. 2009

Lattice generator matrices are not unique:

$$\{\boldsymbol{A}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\} = \{\boldsymbol{B}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\}$$

whenever $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{U}$ for $\boldsymbol{U}$ such that $\mathbb{Z}^n = \boldsymbol{U}\mathbb{Z}^n$

- such $\boldsymbol{U}$ is called unimodular, and satisfies $\boldsymbol{U} \in \mathbb{Z}^{n \times n}$ and $|\det(\boldsymbol{U})| = 1$



Lattice reduction

- Lattice reduction (LR) refers to the task of – given an arbitrary lattice basis $\boldsymbol{B}$ – finding a better basis, e.g.,
  - ⋆ nearly orthogonal basis vectors (columns of $\boldsymbol{B}$)
  - ⋆ short basis vectors
- Different LR criteria and algorithms (for finding $\boldsymbol{U}$)[12]:
  - ⋆ Minkowski reductions[13] and Korkine-Zolotareff reductions[14]
    - ∗ seeks basis with shortest vectors
    - ∗ NP-hard to compute (computationally infeasible)
  - ⋆ LLL reduction[15]
    - ∗ seeks short lattice basis vectors
    - ∗ predominant in the MIMO detection literature (complexity)

---

[12] D. Wübben, D. Seethaler, J. Jaldén, and Gerald Matz, "A Survey of Lattice Reduction Techniques with Applications to Wireless Communications", *SP Mag*, May 2011

[13] H. Minkowski, "Ueber die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen," *Journal für die reine und angewandte Mathematik*, 1891.

[14] A. Korkine and G. Zolotareff, "Sur les formes quadratiques," *Mathematische Annalen*, 1873.

[15] A. K. Lenstra, H. W. Lenstra, and L. Lovász, "Factoring Polynomials with Rational Coefficients," *Mathematische Annalen*, 1982.

Lattice reduction aided linear decoders

- The noise amplification in ZF (applied to $\boldsymbol{M} = \phi \boldsymbol{H}\boldsymbol{G}$) is proportional to diagonal elements of $(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M})^{-1}$
  - ⋆ when applied to $\boldsymbol{M}\boldsymbol{U}$ it is proportional to $(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{U})^{-1}$
- Applying the ZF (or MMSE) decoder in a lattice reduced basis can significantly improve the probability of error performance[16] [17]
- **Note:** The application of this technique assumes lattice decoding
- LLL based LR-aided ZF archives maximal receive diversity in fixed rate V-BLAST scanerio[18]
- LR-aided ZF detection is however not a DMT optimal decoding in the general setting[19]

---

[16] H. Yao and G. W. Wornell, "Lattice-Reduction-Aided Detectors for MIMO Communication Systems," in *Proc. GLOBECOM*, Nov. 2002.

[17] C. Windpassinger and R. F. H. Fischer, "Low-Complexity Near-Maximum-Likelihood Detection and Precoding for MIMO Systems using Lattice Reduction," in *Proc. ITW*, Mar. 2003.

[18] M. Taherzadeh, A. Mobasher, and A. K. Khandani, "LLL Reduction Achieves the Receive Diversity in MIMO Decdoing," *Trans IT*, Dec. 2007.

[19] M. Taherzadeh and A. K. Khandani, "On the limitations of the naive lattice decoding," in *Trans. IT*, Oct. 2010.

Lattice decoding and the MMSE pre-processing

- Naive lattice decoding (i.e., ignoring $\mathcal{R}$) is not generally DMT optimal (not even for the point-to-point MIMO i.i.d. Rayleigh fading channel)
- However, there exist[20] lattice codes that, when decoded using lattice decoding, achieve optimal DMT performance over the point-to-point MIMO i.i.d. Rayleigh fading channel
  - ⋆ an ensemble of random lattice codes
  - ⋆ an MMSE pre-processing step
  - ⋆ and an optimal lattice translate
- The magic is in the MMSE preprocessing!
- Opens up the potential for simultaneously DMT optimal and computationally efficient decoders

[20] H. El Gamal, G. Caire, and M. O. Damen, "Lattice coding and decoding achieve the optimal diversity-multiplexing tradeoff of MIMO channels" *Trans. IT*, June 2004.

- Explicit non-ML transceivers achieving DMT optimality with[21] [22] :
WORST-CASE COMPLEXITY THAT IS AT MOST LINEAR IN RATE!
  - ⋆ for all channel models/fading statistics
  - ⋆ for MIMO, MIMO-OFDM, ISI, multiple-access, cooperative-networks...

- DMT optimality of MMSE lattice-reduction (LR)-aided LINEAR decoders
  - ⋆ most interestingly:

OPTIMALITY OF DECODERS HOLDS IRRESPECTIVE OF THE PARTICULAR LATTICE-CODE APPLIED!

- Any triplet (r, d, c) in

$$\left\{(r, d, c) \,|\, d \leq d_{\text{out}}(r)\,, c \geq 0\,\right\}$$

is achievable with lattice based codes and decoders

[21] J. Jaldén and P. Elia, "DMT Optimality of LR-Aided Linear Decoders for a General Class of Channels, Lattice Designs, and System Models", *Trans. IT*, Oct. 2010.

[22] P. Elia and J. Jaldén, "DMT Optimality of LR-Aided Linear Decoders for a General Class of MIMO-MAC Lattice Designs," *ITW 2010*.

Sphere decoding

- Sphere decoders and its variants are arguably the most well known ML (and near ML) decoder structures
- Complexity is higher than linear detectors, and generally *random*
  - ⋆ ...but how high?

Sphere decoding philosophy

- Maximum likelihood (minimum error probability) decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}} \left\| \mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}} \right\|^2, \qquad \mathcal{X} = \Lambda \cap \mathcal{R}$$

- ML decoder requires the solution to a problem that is generally NP-hard
- The sphere decoder can solve the ML detection problem exactly by enumerating all codewords $\hat{\boldsymbol{x}} \in \mathcal{X}$ that satisfy

$$\left\| \mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}} \right\|^2 \leq \xi^2$$

  - ⋆ codewords in a hyper-sphere centered at the received signal $\mathbf{y}$
- It is considerably more efficient than a full search

A short sphere decoder history

- Based on a paper in the math literature from 85[23]
- First used in communications in early 90s[24] [25]
- Popularized in the late 90s and early 00s[26]
  - ⋆ also where it go the name sphere decoding
- Several semi-tutorial papers now available [27] [28] [29]

---

[23] U. Fincke and M. Pohst, “Improved Methods for Calculating Vectors of Short Length in a Lattice, Including a Complexity Analysis”, *Mathematics of Computation*, Apr. 1985

[24] W. H. Mow., “Maximum Likelihood Sequence Estimation from the Lattice Viewpoint”, *Trans. IT*, Sep. 1994

[25] E. Viterbo, E. Biglieri., “A universal decoding algorithm for lattice codes”. Proc. GRETSI, Juanles- Pins, France, Sep. 1993

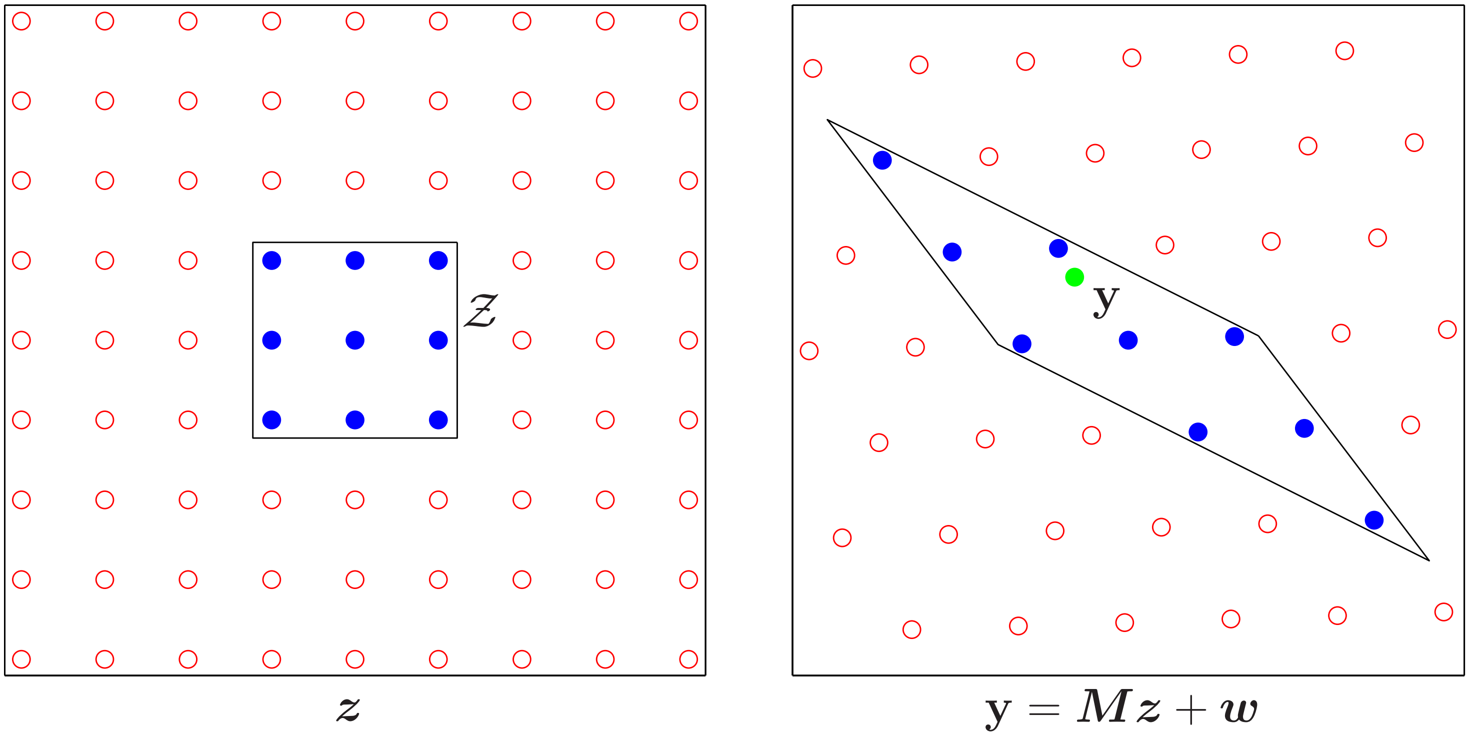
[26] E. Viterbo, J. Boutros. “A universal lattice code decoder for fading channels”, *Trans. IT*, July 1999

[27] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, “Closest point search in lattices”, *Trans. IT*, Aug. 2002

[28] M. O. Damen, H. El Gamal, and G. Caire, “On maximum-likelihood detection and the search for the closest lattice point”, *Trans. IT*, Oct. 2003

[29] A. D. Murugan, H. El Gamal, M. O. Damen, and G. Caire, “A unified framework for tree search decoding: rediscovering the sequential decoder”, *Trans. IT*, Mar. 2006

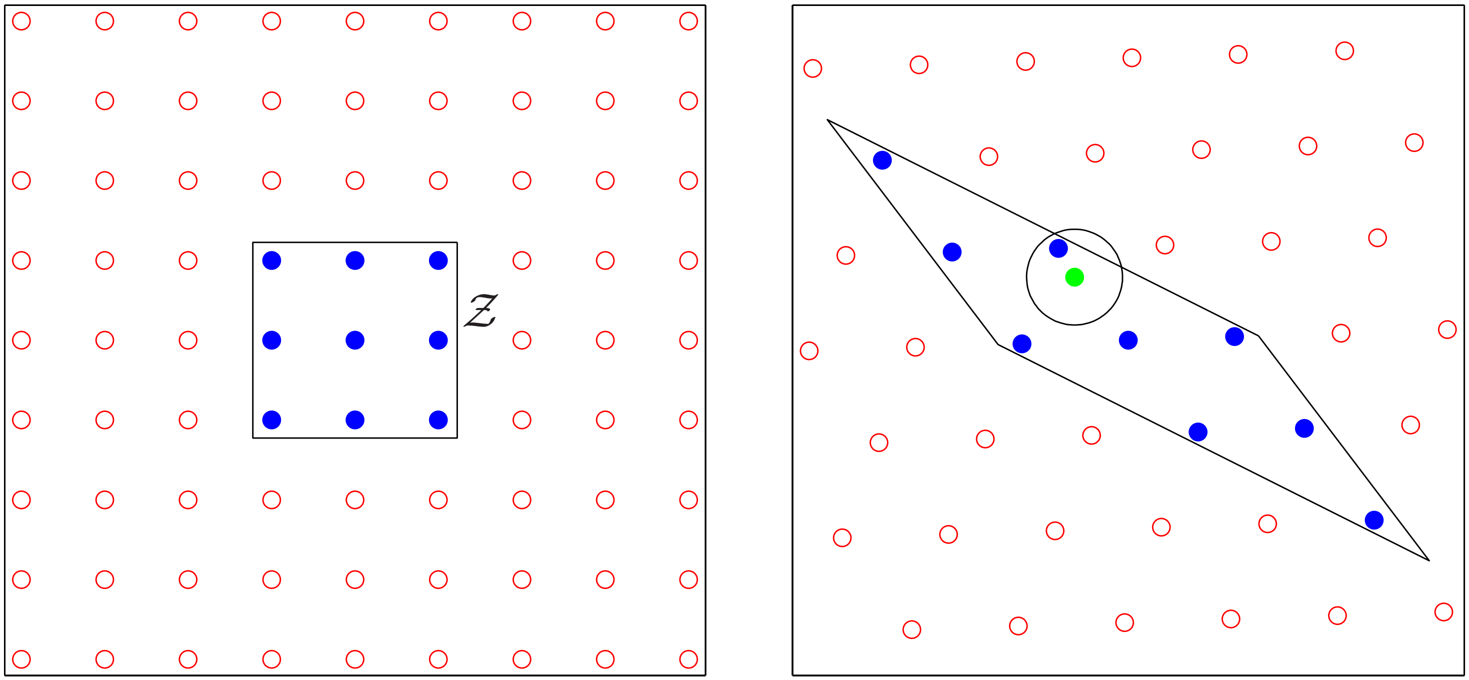
Sphere decoder example



The receiver sees a skewed codebook in noise

The object is to find closest codeword hypothesis $\boldsymbol{M}\hat{\boldsymbol{z}}$ to $\mathbf{y}$

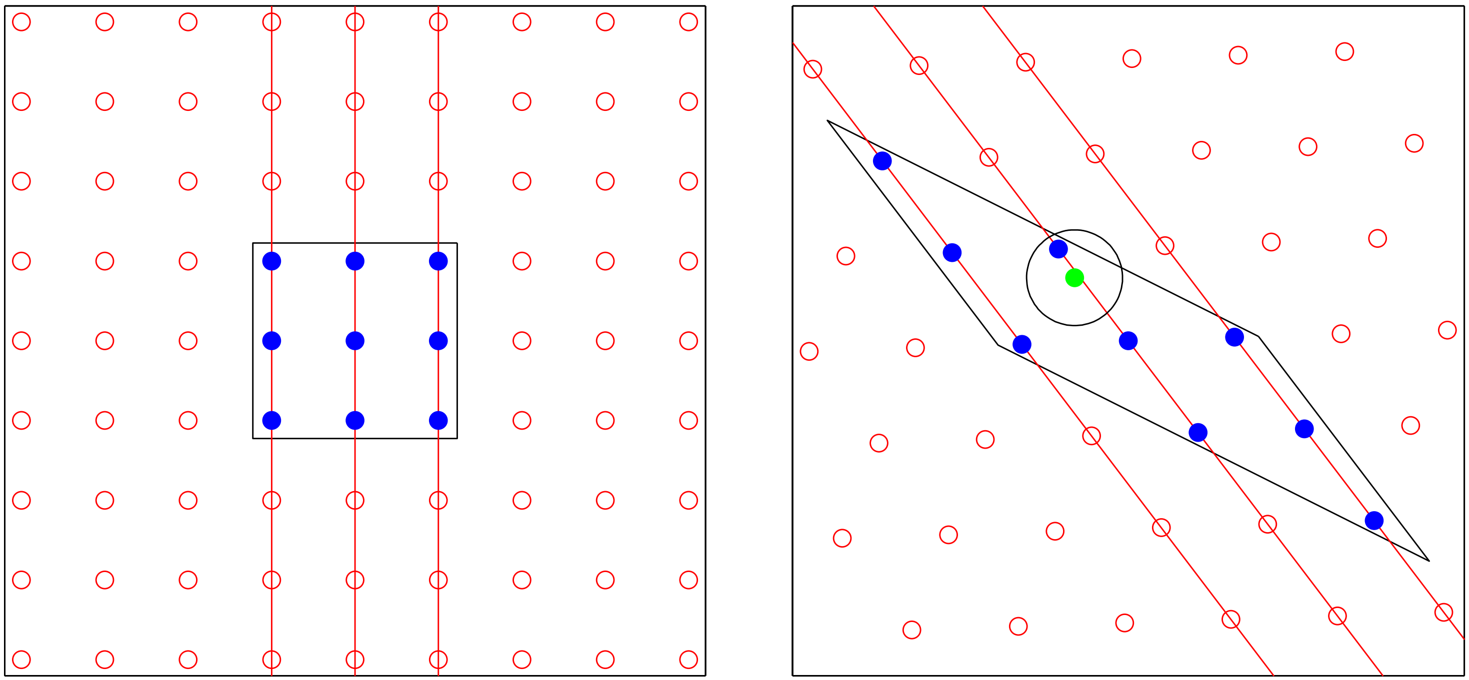
Sphere decoder example



$\boldsymbol{z}$ $\qquad\qquad$ $\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

Sphere decoder (SD) searches for hypotheses in a sphere centered at $\mathbf{y}$
Need a clever codeword enumeration procedure

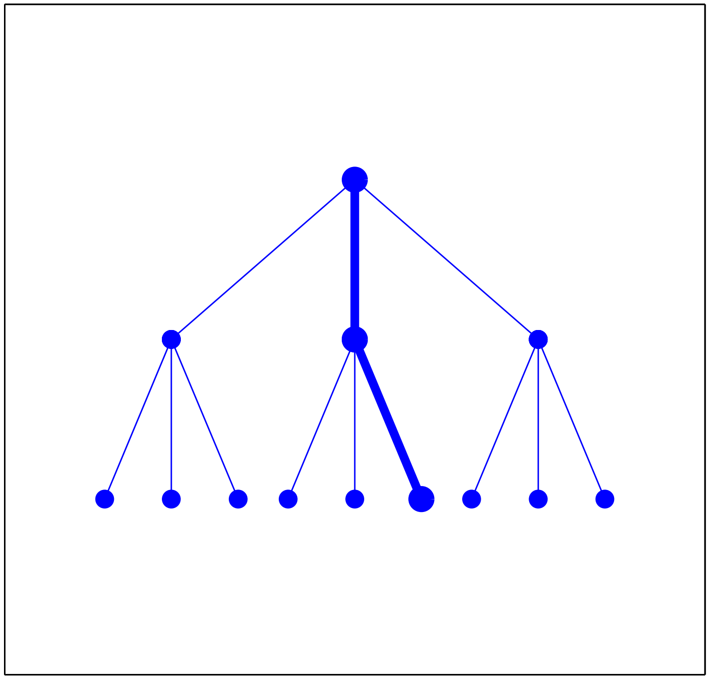
Sphere decoder example



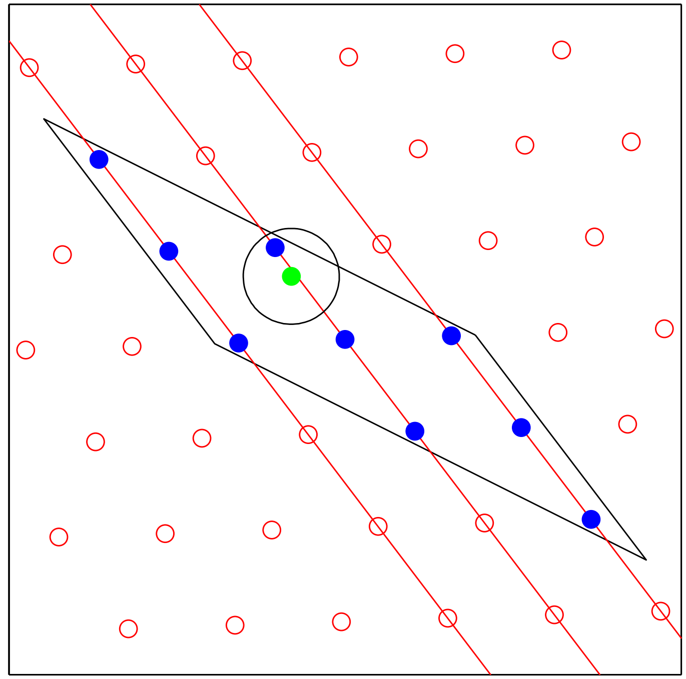
$\boldsymbol{z}$ $\qquad\qquad$ $\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

SD identifies point in the sphere by enumerating lattice layers

Sphere decoder example



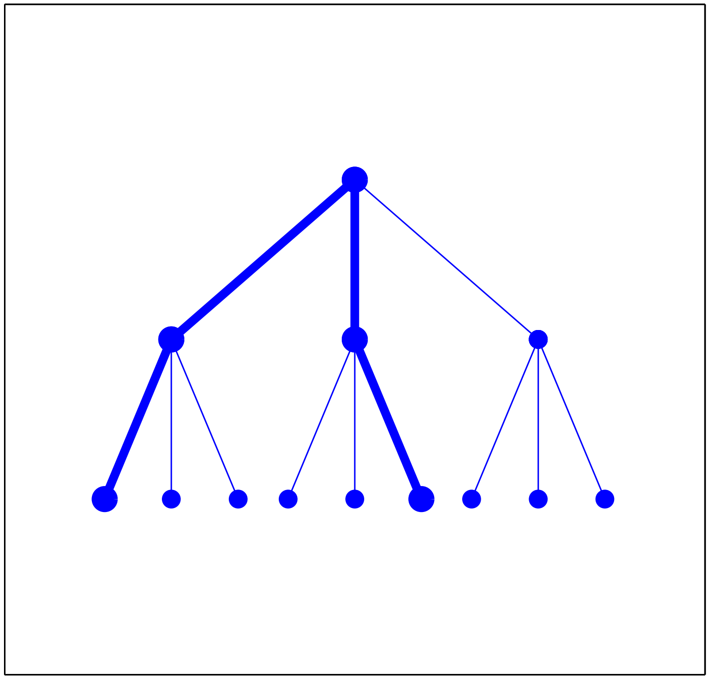
$\mathcal{T}$



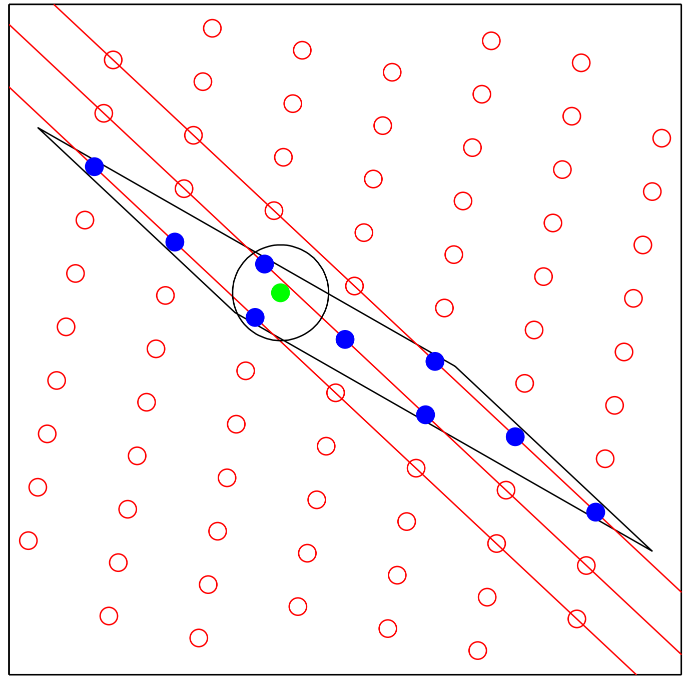
$\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

Can be view as a branch and bound algorithm on a tree

Sphere decoder example



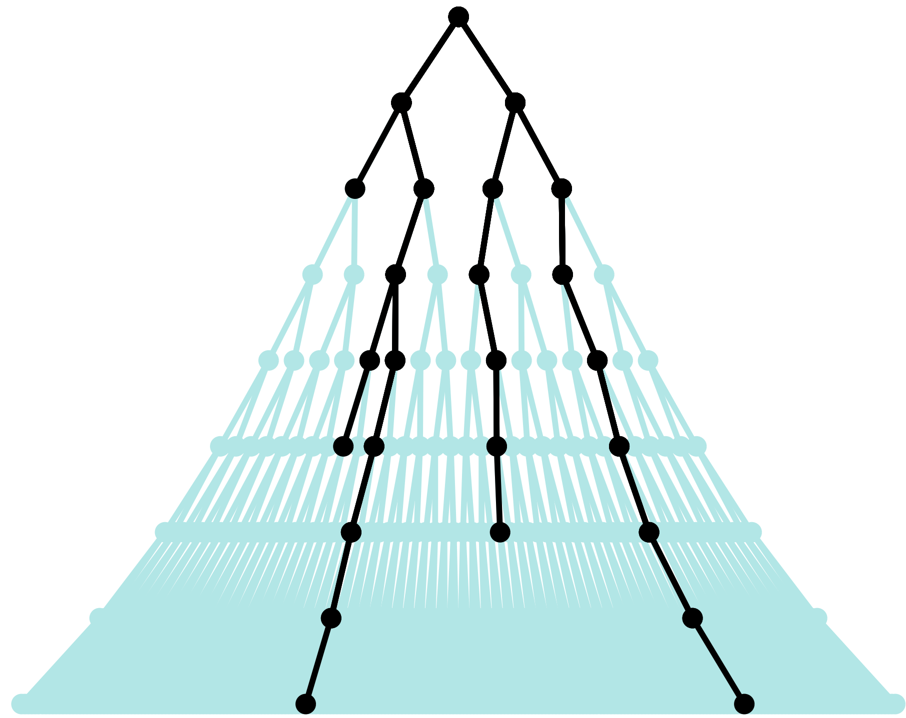
$\mathcal{T}$



$\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

Fading (and rate) influenced the branching behavior

Sphere decoding for large problems - complexity savings



Large gains to be had when solving large dimensional problems

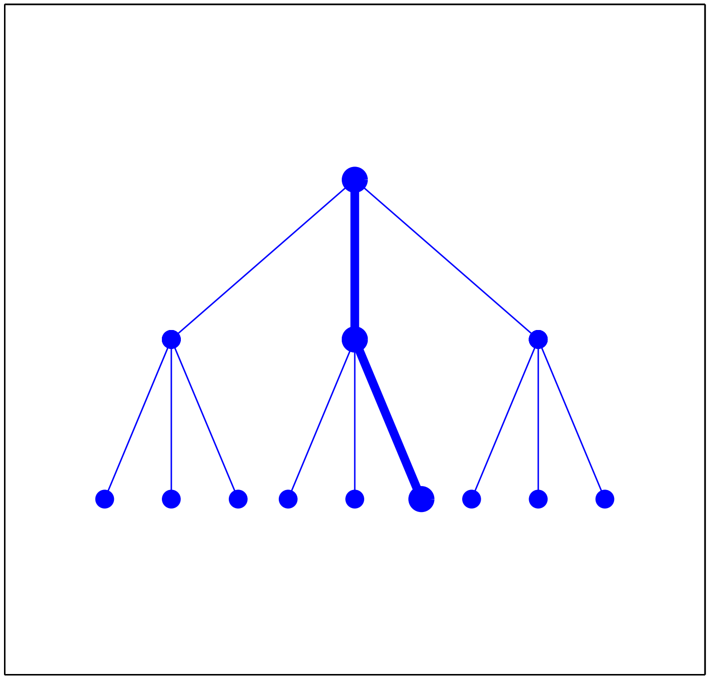
- How to select the search radius ξ
  - ⋆ nowadays a *non-issue* due to adaptive radius updates and the Schnorr-Euchner implementation (Algorithm II[30], started with $\xi = \infty$)
- Left pre-processing: Instead of applying the sphere decoder to $\boldsymbol{M}$ we can apply it to (the MMSE preprocessed matrix)

$$\tilde{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{H} \\ \sigma \boldsymbol{I} \end{bmatrix}$$

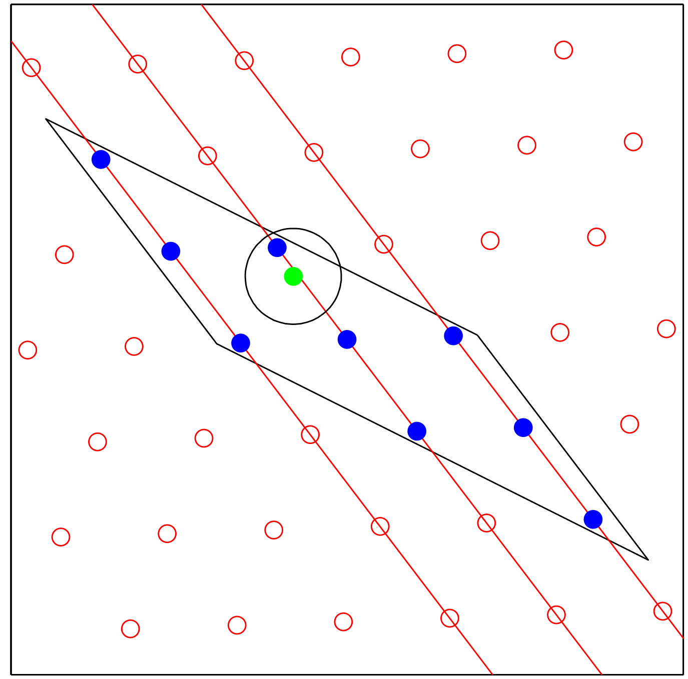
- Right pre-processing: Instead of applying the sphere decoder to $\boldsymbol{M}$ we can apply it to $\boldsymbol{M}\boldsymbol{U}$ or $\tilde{\boldsymbol{M}}\boldsymbol{U}$ where $\boldsymbol{U}$ is an LR (unimodular) matrix

[30] M. O. Damen, H. El Gamal, and G. Caire, "On maximum-likelihood detection and the search for the closest lattice point", *Trans. IT*, Oct. 2003

Sphere decoder example (continued)



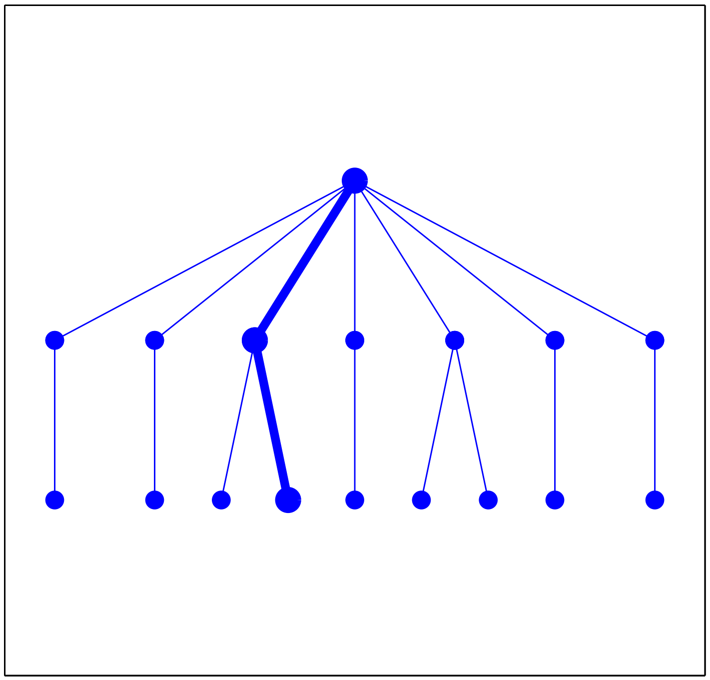
$\mathcal{T}$



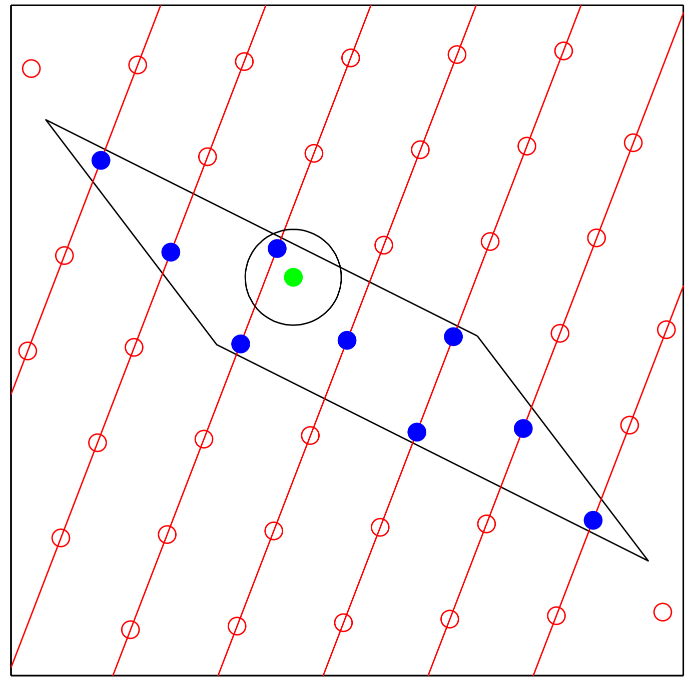
$\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

SD can be view as a branch and bound algorithm on a tree
Layers aligned along natural lattice basis

Sphere decoder example (continued)



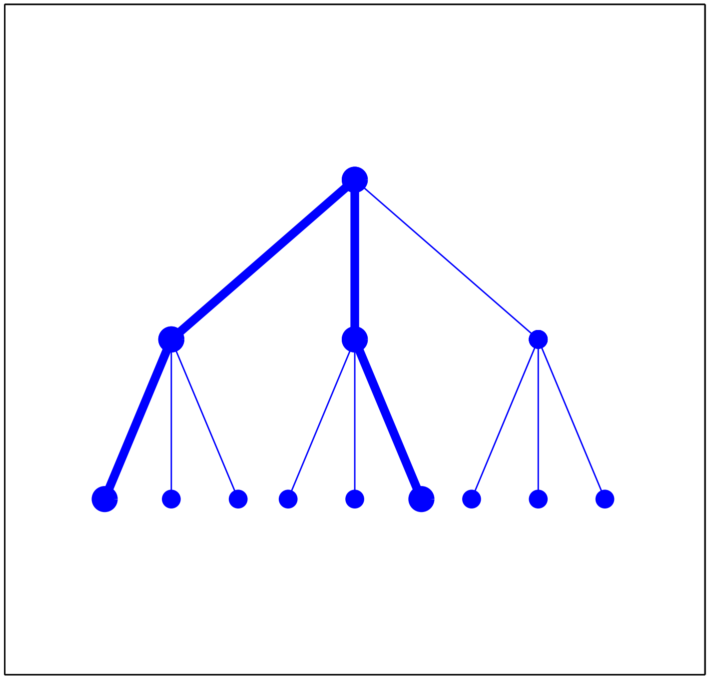
$\mathcal{T}$



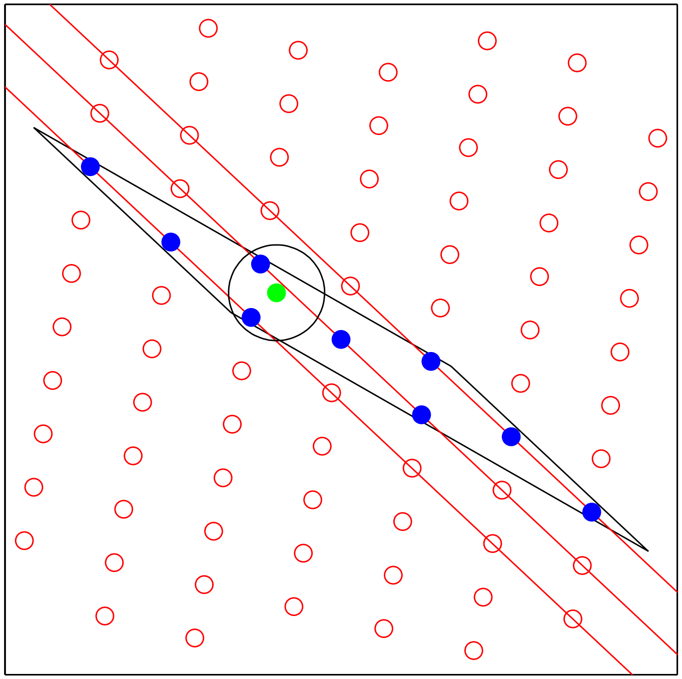
$\mathbf{y} = \boldsymbol{M}\boldsymbol{U}\boldsymbol{z} + \boldsymbol{w}$

There is a degree of freedom in how to choose the lattice layers
This is the concept of lattice reduction (LR)

Sphere decoder example (continued)



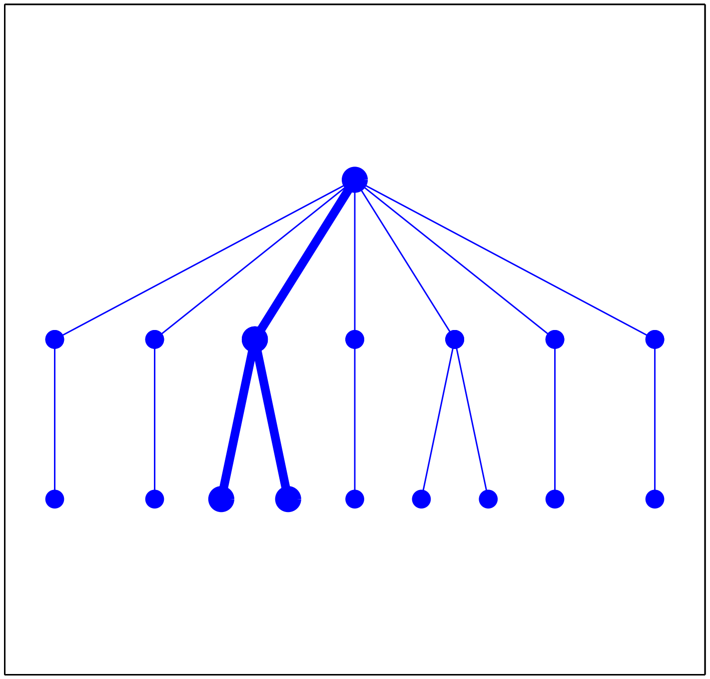
$\mathcal{T}$



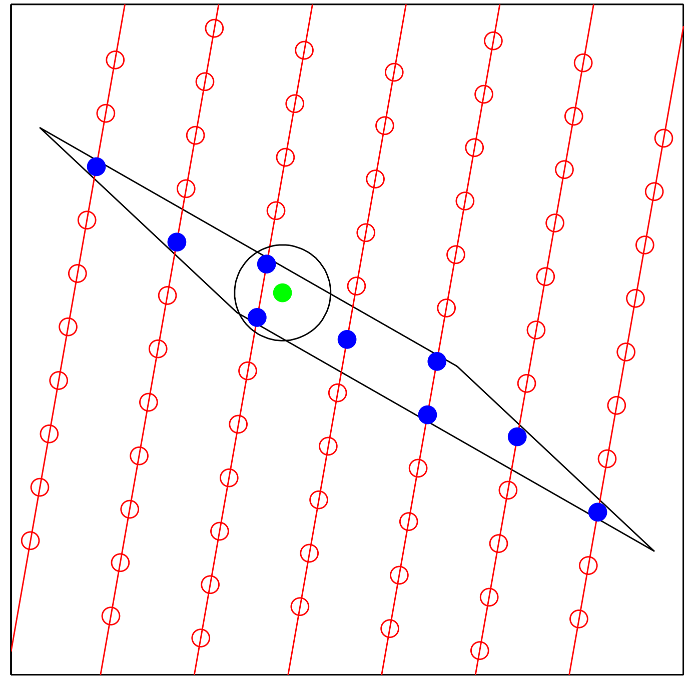
$\mathbf{y} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{w}$

LR can provide increased complexity robustness towards fading

Sphere decoder example (continued)



$\mathcal{T}$



$\mathbf{y} = \boldsymbol{M}\boldsymbol{U}\boldsymbol{z} + \boldsymbol{w}$

LR can provide increased complexity robustness towards fading
. . . but hard to keep track of the codebook boundary

Sphere decoder base variants

- Exact implementation of ML decoder

$$\hat{\boldsymbol{z}}_{\text{ML}} = \arg\min_{\hat{\boldsymbol{z}} \in \mathcal{Z}} \|\mathbf{y} - \boldsymbol{M}\hat{\boldsymbol{z}}\|^2$$

- Exact implementation of (naive) lattice decoder

$$\hat{\boldsymbol{z}}_{\text{NLD}} = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{Z}^n} \|\mathbf{y} - \boldsymbol{M}\hat{\boldsymbol{z}}\|^2$$

- Exact implementation of (MMSE preprocessed) lattice decoder

$$\hat{\boldsymbol{z}}_{\text{MMSE}-\text{LD}} = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{Z}^n} \|\mathbf{y} - \boldsymbol{M}\hat{\boldsymbol{z}}\|^2 + \sigma^2\|\hat{\boldsymbol{z}}\|^2 = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{Z}^n} \|\tilde{\boldsymbol{y}} - \boldsymbol{F}\hat{\boldsymbol{z}}\|^2$$

where $\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \boldsymbol{M}^{\mathrm{T}}\boldsymbol{M} + \sigma^2\boldsymbol{I}$ and $\tilde{\boldsymbol{y}} = \boldsymbol{F}^{-\mathrm{T}}\boldsymbol{M}^{\mathrm{T}}\mathbf{y}$

Interesting complexity performance behavior in all cases!

Classical sphere decoder complexity results

- Closest lattice point (vector) problem (CVP) is NP-hard
  - ⋆ we should not expect any miracles
- Most work on SD complexity assume an i.i.d. Rayleigh model for $\boldsymbol{H}$ and no code (i.e., $\boldsymbol{x}$ is a vector of uncoded symbols)
- Huge improvement in the average complexity for moderate values of n and high SNR (small search radius)[31]
- Average complexity still grows exponentially in n, even under optimal symbol ordering and radius selection[32] [33].
- Work on complexity probability tail exponent of (naive) lattice implementation[34]

---

[31] B. Hassibi and H. Vikalo, "On the sphere-decoding algorithm I. Expected complexity," *Trans. SP*, Aug., 2005.

[32] J. Jaldén and B. Ottersten, " On the complexity of sphere decoding in digital communications," *Trans. SP*, Apr., 2005.

[33] J. Jaldén and B. Ottersten, "On the limits of sphere decoding," *ISIT*, Sept., 2005.

[34] D. Seethaler, J. Jaldén, C. Studer, and H. Bölcskei, "Tail behavior of sphere-decoding complexity in random lattices," *ISIT*, June, 2009.

- Maximum likelihood decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2$$

  - ⋆ full search consider $|\mathcal{X}| \doteq \rho^{rT}$ codeword hypothesis
  - ⋆ ML SD worst case complexity exponent is generally

$$c_{\max}(r) = \lim \frac{\log\left(\sup_{\boldsymbol{H},\mathbf{y}} F(\mathbf{y},\boldsymbol{H})\right)}{\log \rho} = rT$$

- With a DMT optimal code we have $P_e \doteq \rho^{-d_{\mathrm{out}}(r)}$
- Consider a time-limited sphere decoder which stops after visiting $C = \rho^c$ nodes and calls an error
  - ⋆ let $\Psi(x)$ be given by

$$\Psi(c) \triangleq \lim_{\rho \to \infty} \frac{\log \mathrm{P}\left(F(\mathbf{y},\boldsymbol{H}) \geq \rho^c\right)}{\log \rho} \quad \Leftrightarrow \quad \mathrm{P}\left(F(\mathbf{y},\boldsymbol{H}) \geq \rho^c\right) \doteq \rho^{-\Psi(c)}$$

- If $\Psi(c) > d_{\mathrm{out}}(r)$ then $\mathrm{P}\left(F(\mathbf{y},\boldsymbol{H}) \geq \rho^c\right) \ll P_e$ at large ρ (negligible loss)
- If $\Psi(c) < d_{\mathrm{out}}(r)$ then $\mathrm{P}\left(F(\mathbf{y},\boldsymbol{H}) \geq \rho^c\right) \gg P_e$ at large ρ

The sphere decoder complexity exponent

$$c^\star(r) \triangleq \inf_c \{c \mid \Psi(c) > d_{\text{out}}(r)\}$$

Interpretations

- $\rho^{c^\star(r)}$ is (in the exponent) the tightest runtime constraint that can be placed on the sphere decoder without loosing diversity in the decoding process
- The optimal DMT diversity $d_{\text{out}}(r)$ is achievable with complexity exponent $c = c^\star(r)$ using a time-limited sphere decoder
- More generally, any triplet (r, d, c) in

$$\left\{(r, d, c) \mid d \leq \min\big(d_{\text{out}}(r), \Psi(c)\big), c \geq 0\right\}$$

is achievable using time-limited sphere decoders

All of the above quantities can an actually be obtained in closed form!

Benefits of high SNR asymptotics and mathematical tools

High data rates

- High rate assumption that follows with high SNR makes discrete problems amendable to continuous approximations
  - ⋆ codewords and layers → codeword and layer densities
  - ⋆ discrete counting problems → scaled volumes

Large deviations techniques

- The theory of large deviations (rare events) turn intractable probability integrals into tractable optimization problems
  - ⋆ For sequences of probability measures μ_ϵ we can make statements like

$$\lim_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\mathcal{B}) = - \inf_{x \in \mathcal{B}} I(x)$$

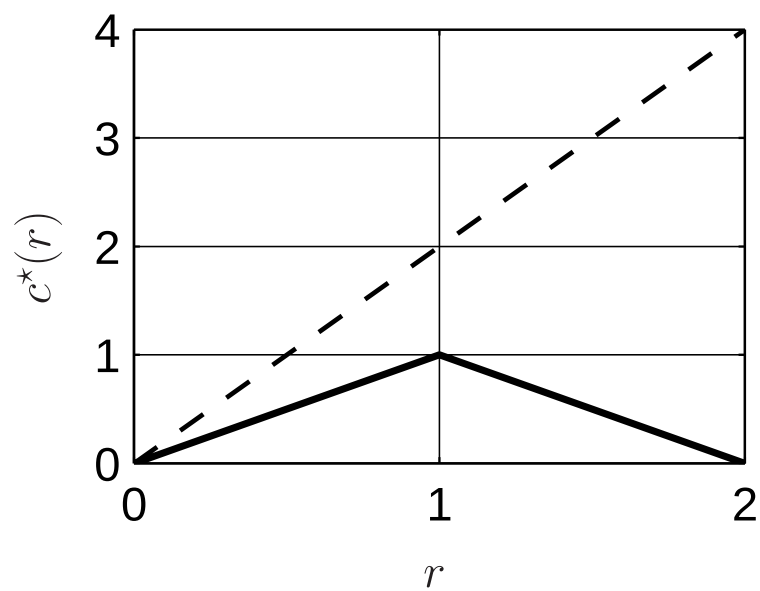
  - ⋆ In the end we get (reasonably simple) linear optimization problems

Complexity exponent for point-to-point channels

The complexity exponent of any DMT optimal full-rate $n_{\mathrm{T}} \times T = 2 \times 2$ linear code for the $n_{\mathrm{R}} \times n_{\mathrm{T}} = 2 \times 2$ MIMO channel is[35]

$$c^{\star}(r) = \min(r, 2-r)$$

for $r \in [0,2]$. The result does not depend on n_{R} if $n_{\mathrm{R}} \geq n_{\mathrm{T}}$



THE COMPLEXITY EXPONENT IS NOT MONOTONE IN r!

[35] J. Jaldén and P. Elia, "Sphere Decoding Complexity Exponent for Decoding Full-Rate Codes Over the Quasi-Static MIMO Channel", *Trans. IT*, Sept. 2012.

Connection to information theoretic channel outages

- The *singularity level* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n_\mathrm{T}})$ of the channel is

$$\alpha_i \triangleq -\frac{\log \lambda_i(\boldsymbol{H}^\mathrm{H}\boldsymbol{H})}{\log \rho} \quad \Leftrightarrow \quad \lambda_i(\boldsymbol{H}^\mathrm{H}\boldsymbol{H}) = \rho^{-\alpha_i}$$

  where $\lambda_1(\boldsymbol{H}^\mathrm{H}\boldsymbol{H}) \leq \ldots \leq \lambda_{n_\mathrm{T}}(\boldsymbol{H}^\mathrm{H}\boldsymbol{H})$ are the eigenvalues of $\boldsymbol{H}^\mathrm{H}\boldsymbol{H}$

- The high SNR outage probability is[36]

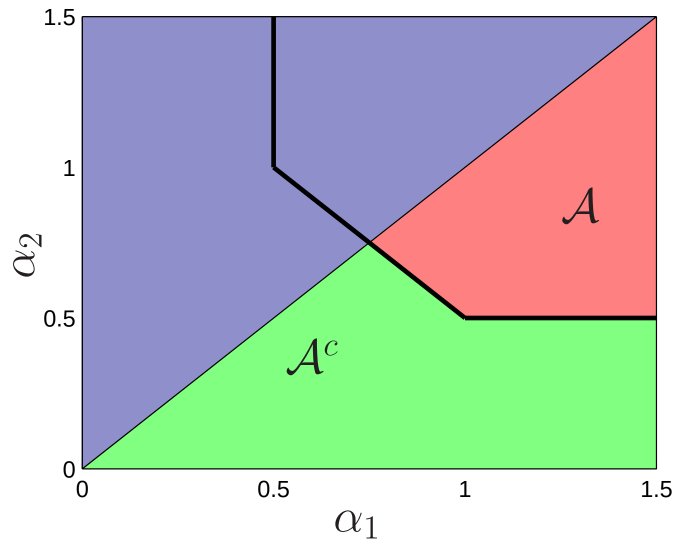
$$p_\mathrm{out} \doteq \mathrm{P}\left(\boldsymbol{\alpha} \in \mathcal{A}\right), \quad \mathcal{A} \triangleq \left\{ \boldsymbol{\alpha} \,\middle|\, \sum_{i=1}^{n_\mathrm{T}} (1-\alpha_i)^+ < r, \right\}$$

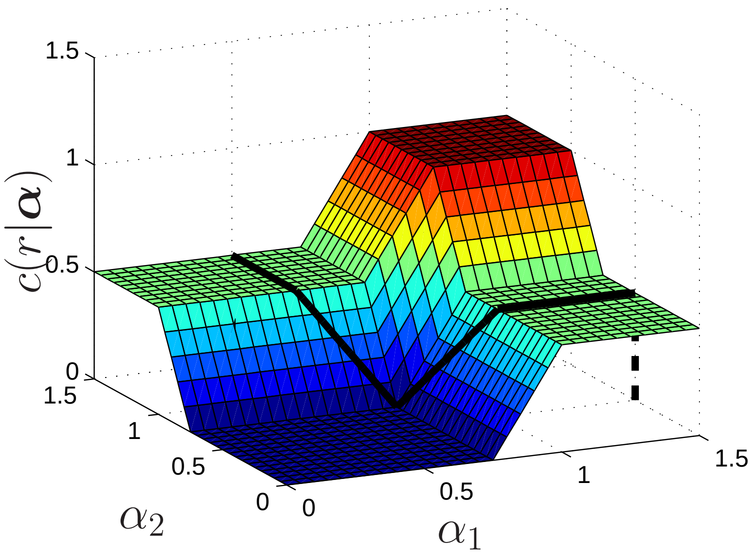
  For near ML performance, we (essentially) need to decode for all singularity levels that are not in outage, i.e., $\boldsymbol{\alpha} \in \mathcal{A}^c$

[36] L. Zheng and D. N. C. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels," *Trans. IT*, May 2003.

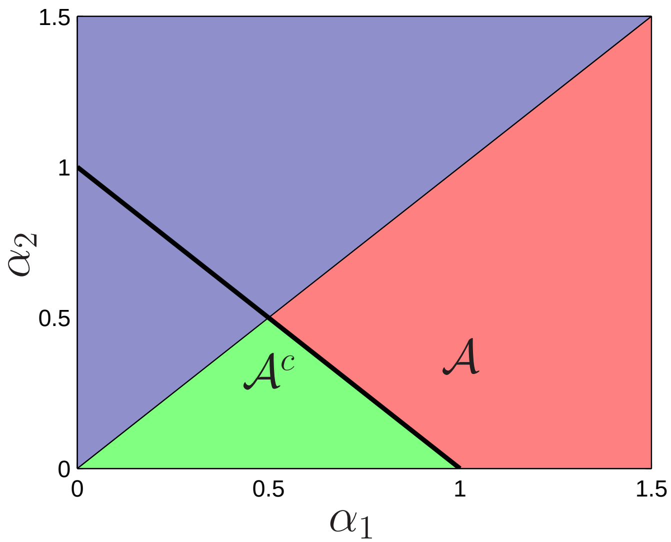
Comparison between outages and decoding complexity

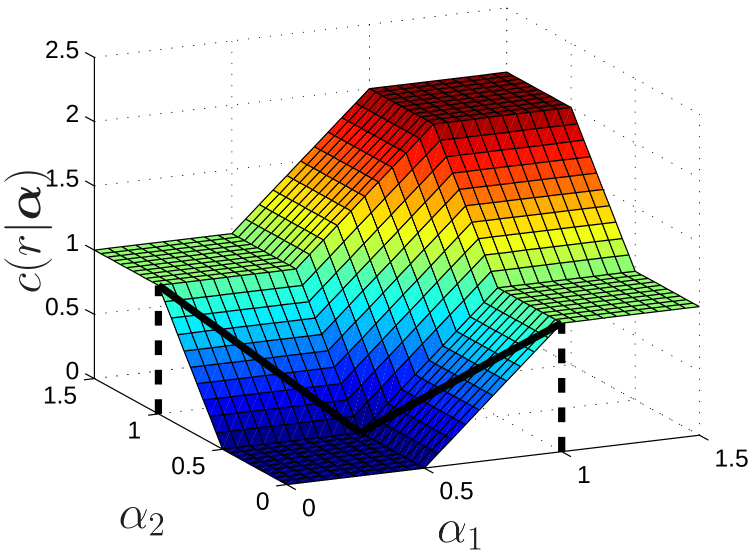
2×2 code at multiplexing gain $r = 1/2 \Rightarrow c^{\star}(r) = 1/2$





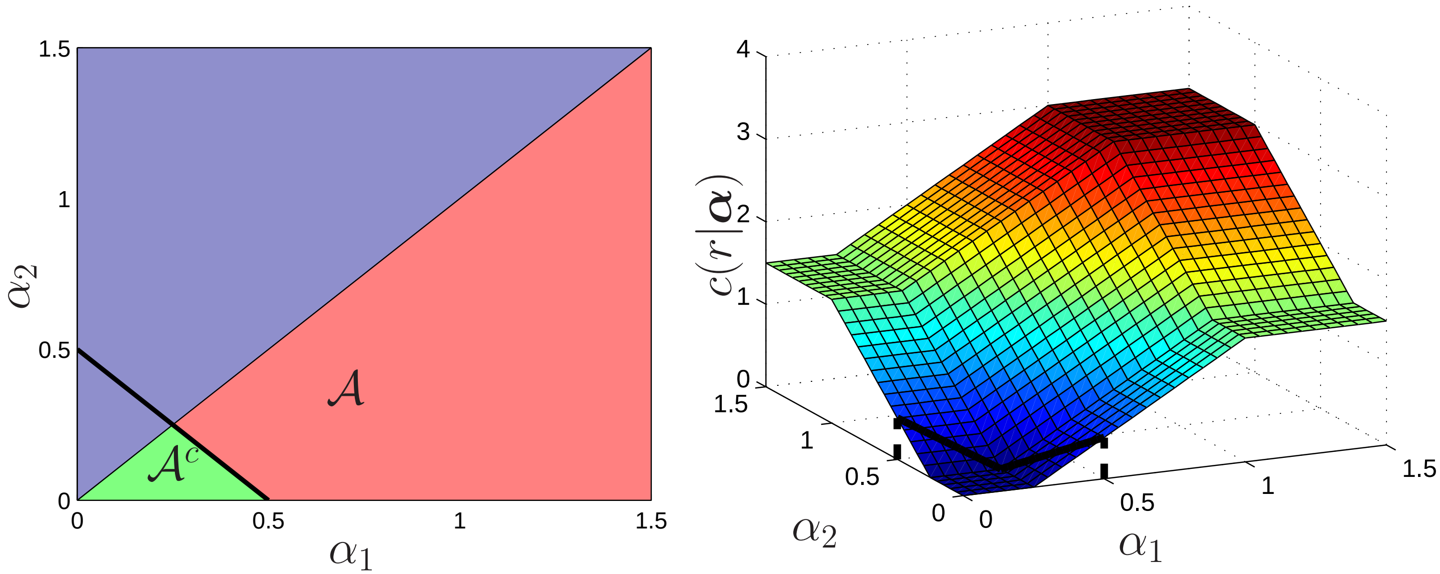
2×2 code at multiplexing gain $r = 1 \Rightarrow c^{\star}(r) = 1$





Comparison between outages and decoding complexity

2×2 code at multiplexing gain $r = 3/2 \Rightarrow c^{\star}(r) = 1/2$



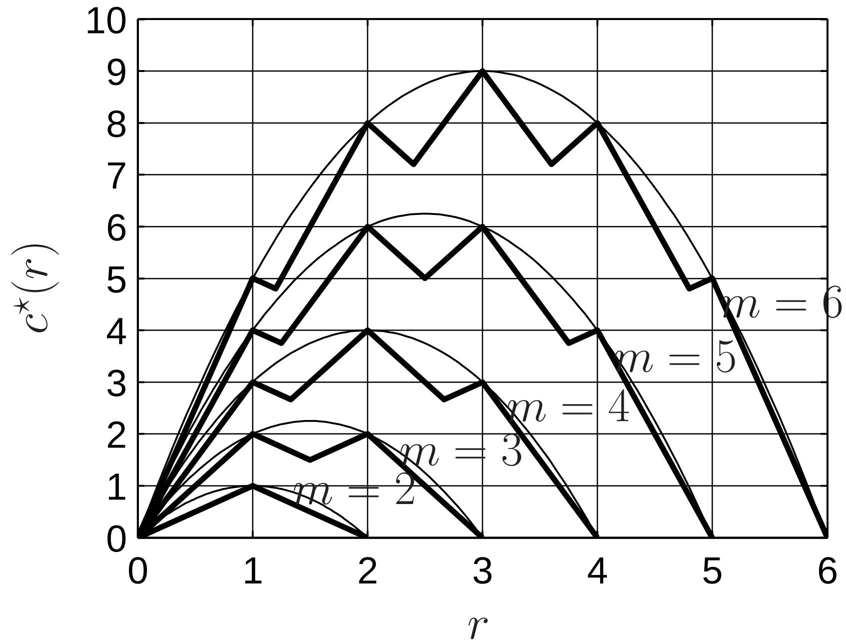
The complexity exponent of any threaded algebraic DMT optimal full-rate $n_{\mathrm{T}} \times T = n \times n$ code for the $n_{\mathrm{R}} \times n_{\mathrm{T}} = m \times m$ MIMO channel is is

$$c^{\star}(r) = r(m - \lfloor r \rfloor - 1) + \big(m\lfloor r \rfloor - r(m-1)\big)^{+}$$

for $r \in [0, m]$, where $\lfloor \cdot \rfloor$ rounds down and $(\cdot)^{+} = \max(0, \cdot)$, which simplifies to

$$c^{\star}(r) = r(m - r)$$

for $r \in \mathbb{N}$.



Summary of Part I

- Lattice codes and universal lattice decoders
- Complexity in the DMT setting
  - ⋆ definition of the complexity exponent
  - ⋆ "simple" closed for solutions for increased insight

OUR ACTUAL DECODER RECOMMENDATION

A time-limited, Schnorr-Euchner sphere decoder, applied to a regularized and LLL reduced basis matrix

Extensions...

This is just the beginning...

- General complexity regulating policies
  - ⋆ intelligently trade off complexity and reliability
- Refined measures of decoding performance
  - ⋆ the complexity of lattice decoding with zero asymptotic SNR gap
- The role of feedback
- Multiuser settings
- ... and much more in Part II

End of Part I

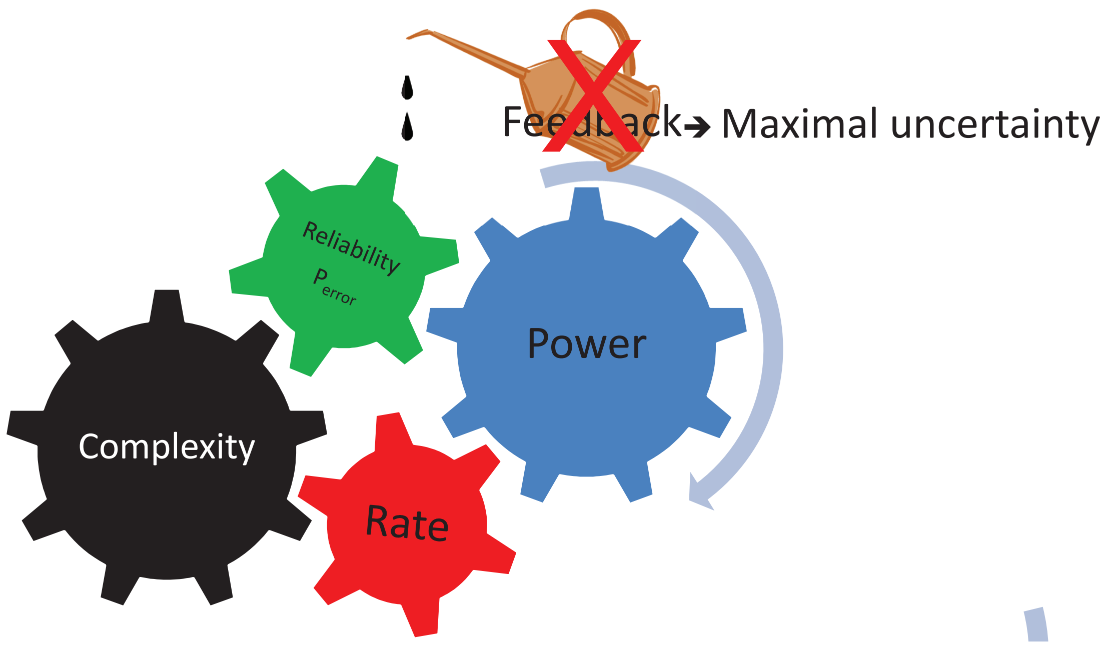
Questions on part I?

THANK YOU

PART II

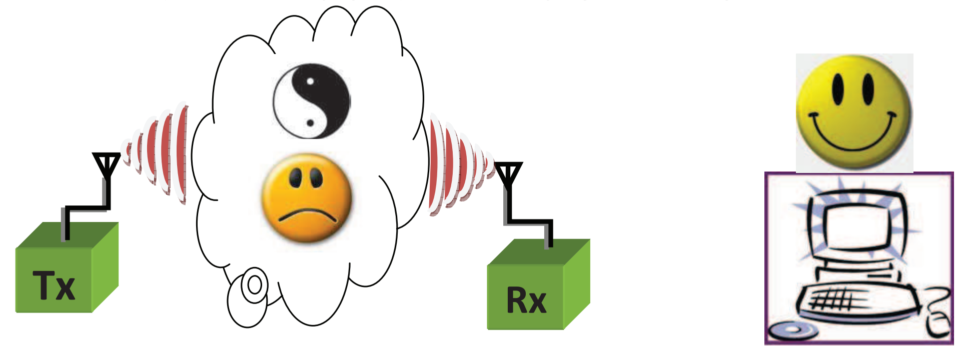
Recall: Complexity vs Performance in non-ergodic MIMO

First: Complexity vs Performance in non-ergodic MIMO

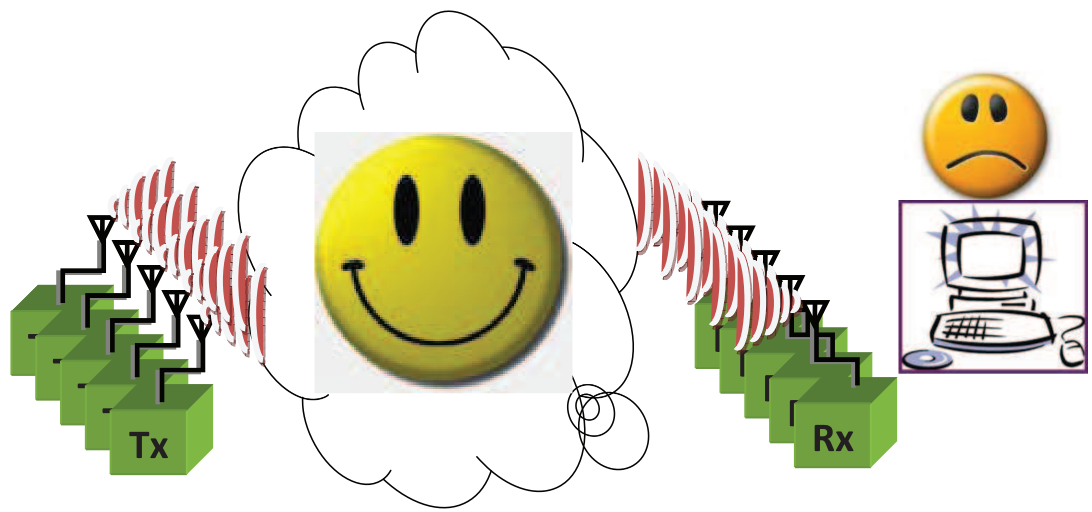


Recall: Outage-limited MIMO; large gains and costs

SINGLE ANTENNA CHANNEL IS UNCERTAIN: SLOW AND UNRELIABLE



MULTIPLE ANTENNA SYSTEMS: FASTER, MORE RELIABLE, BUT MORE COMPLEX



Recall: Performance-complexity question in MIMO

- Performance and complexity at the heart of gains-costs

$$\big(\text{SNR}, \quad \text{rate } R, \quad \text{reliability } P_{err}, \quad \text{complexity } C\big)$$

⋆ How to construct $\boldsymbol{x}$? How to process $\mathbf{y}$?
⋆ A long standing open problem

$$C_{\max}$$

MAXIMUM ALLOWABLE COMPUTATIONAL RESOURCES (PER T CHANNEL USES)

- chip size, number of flops (after that effort must terminate), etc.
- generally $P_{\text{err}} \uparrow$ as $C_{\max} \downarrow$
- Keep in mind: Complexity fluctuates with channel

Small example: $C_{\text{max}} = 132957$ flops

- Can you achieve $(P_{\text{err}}, R, \rho)$ with $C_{\text{max}} = 2000$ flops?
  - ⋆ *No! Too common early-terminations for search based decoders ($N(H)$ varies) - or too weak linear receivers*
- Can you do it with $C_{\text{max}} = 100000$ flops?
  - ⋆ *No, but we are getting there.*
- How about with 132957 flops?
  - ⋆ *Yes!*
- How about with 132956 flops?
  - ⋆ *Nope!*.
- OK, for $(P_{\text{err}}, R, \rho)$ you need $C_{\text{max}} = 132957$ flops.
  Else $(P_{\text{err}}, R, \rho)$ is not achievable.

Intuition: optimality in MIMO

$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w} = \boldsymbol{H}\boldsymbol{G}\boldsymbol{z} + \boldsymbol{w}$$

- Lattice

$$\Lambda \triangleq \{\boldsymbol{G}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$

- Variably dense lattice

$$\Lambda_r \triangleq \rho^{-\frac{rT}{n}}\Lambda$$

- $\mathcal{R} \subset \mathbb{R}^n$: shaping region picks out codewords

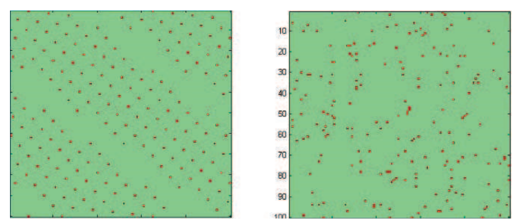
$$\boldsymbol{x} \in \mathcal{X}_r = (\rho^{-\frac{rT}{n}}\Lambda) \cap \mathcal{R}$$

Intuition: optimality in MIMO$_1$

- Performance delivered by the lattice design $\mathcal{X}$ and decoder $\mathcal{D}$:

$$d_{\mathcal{X},\mathcal{D}}(r) \triangleq - \lim_{\rho\to\infty} \frac{\log \mathrm{P}\left(\hat{\boldsymbol{x}}_{\mathcal{D}} \neq \boldsymbol{x}\right)}{\log \rho} \geq - \lim_{\rho\to\infty} \frac{\log \mathrm{P}\left(\boldsymbol{H} \in \mathcal{O}\right)}{\log \rho}$$

$$\mathcal{O} = \{\boldsymbol{H} : \frac{1}{T} \log\det\left(\mathbf{I} + \beta \boldsymbol{H}\boldsymbol{H}^{\dagger}\right) < R\}, \text{ some fixed } \beta,$$



- Optimality when code-channel lattice has statistically good distance properties
  - ⋆ SUCCESS WHENEVER IT IS INFORMATION-THEORETICALLY POSSIBLE!
  - ⋆ Recall: complexity generally prohibitive

DMT optimal decoding solution

LLL-BASED LR-AIDED REGULARIZED LINEAR DECODER

(Jaldén-Elia 2009, Elia-Jaldén ITW-2010)

(drawing from [Yao-Wornell],[Windpassinger-Fischer],[El Gamal,Caire,Damen]...)

- 1. Shed bounding region
- 2. Regularize - penalize far away elements
- 3. Policy: don't lattice-reduce if channel too "non-orthogonal"
- 4. Linear detection

Theorem: *[Jaldén-Elia 09] For a very general class of MIMO channels, the above achieves DMT optimal decoding,* for any code.

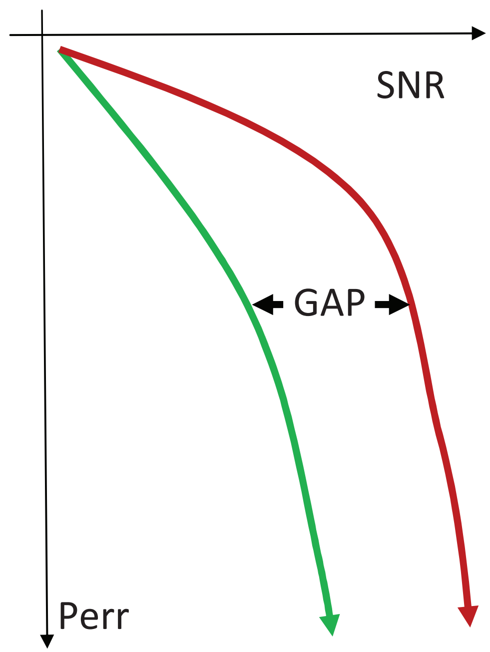
Theorem: *Codes based on cyclic division algebras, and the LLL-based LR-aided linear implementations of the regularized lattice decoders provide DMT optimality over ... a broad range of MIMO settings (relaying, OFDM, MIMO ARQ, quasi-static MIMO, etc).*

Theorem: *Worst-case complexity at most* linear *in the rate (sum-rate).High-SNR optimal performance with at most $O(n^2)$ flops per bit.*

BUT... there may be a large error gap

Bounding the error-gap to optimal/exact decoding solutions

- Error exponents $d(r)$ could allow large gap to optimal performance



Recall: Analysis of more powerful transceivers

Needed to explore more powerful decoding solutions, and thus needed to be able to capture their performance vs. complexity behavior

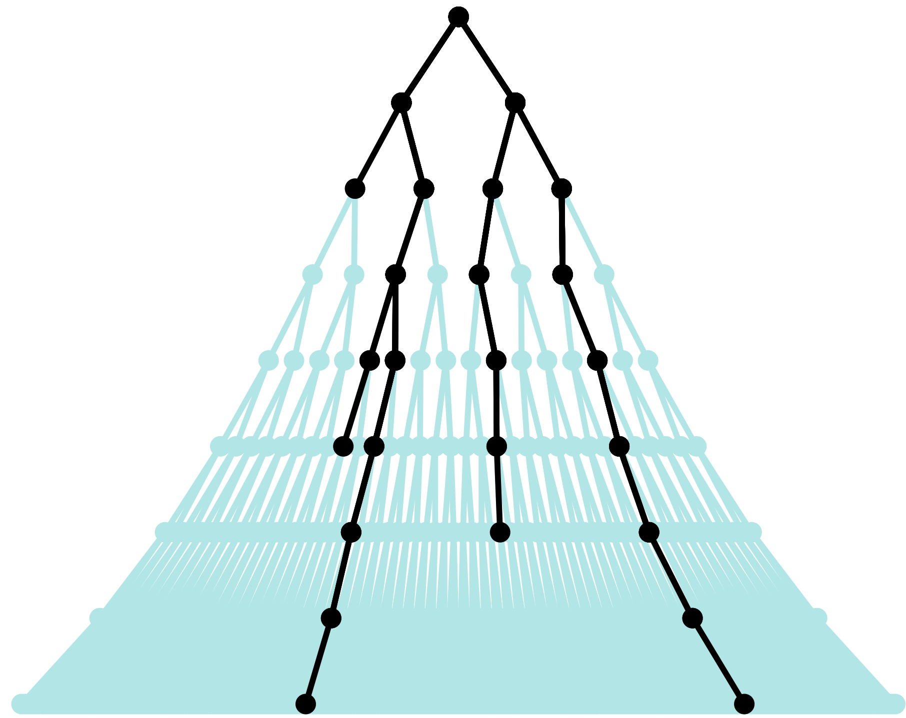
ML and high-performance lattice decoding

- Maximum likelihood (minimum error probability) sphere-type decoder
  - ⋆ More efficient than a full search
  - ⋆ Better actual performance than LR-aided linear solution
  - ⋆ More costly than LR-aided linear solution

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\min_{\hat{\boldsymbol{x}} \in \mathcal{X}_r} \left\| \mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}} \right\|^2 , \qquad \mathcal{X}_r = \Lambda_r \cap \mathcal{R}$$

- Need the mathematical machinery to capture the cost of such high performance

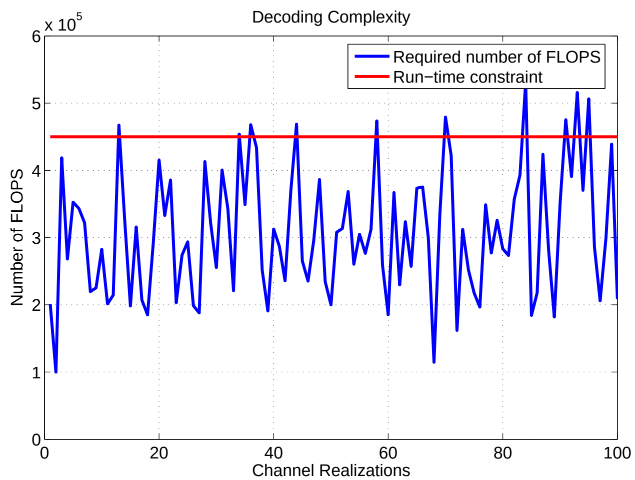
Sphere decoding for large problems - complexity savings



Large gains to be had when solving large dimensional problems

Looked Performance-Complexity Tradeoff

- Generally algorithmic complexity fluctuates with channel
  - ⋆ Channels affect received constellation density and hence complexity
- Generally $P_{\text{err}} \uparrow$ as $C_{\max} \downarrow$



Instantaneous algorithmic complexity fluctuations

Complexity exponent

$$c(r) := \lim_{\rho\to\infty} \frac{\log C_{\max}}{\log \rho},$$

$$C_{\max} \doteq \rho^{c(r)} = 2^{R\frac{c(r)}{r}} \dot{\leq} \rho^{rT} = |\mathcal{X}|$$

$$c(r) > 0 \implies C_{\max} \text{ exponential in } R \text{ (and often in } RT)$$

and also recall

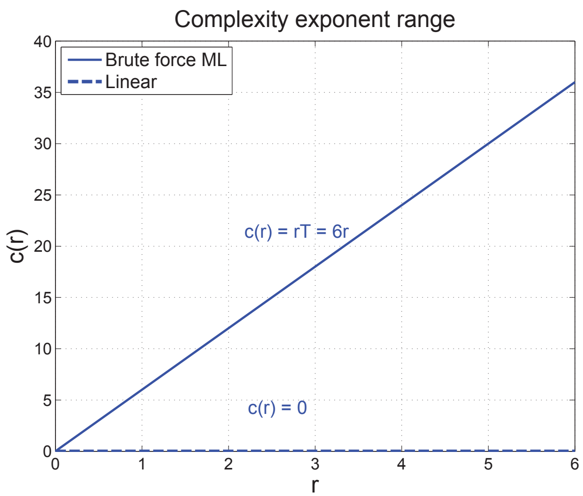
$$d(r) := -\lim_{\rho\to\infty} \frac{\log P_{\text{err}}}{\log \rho}$$

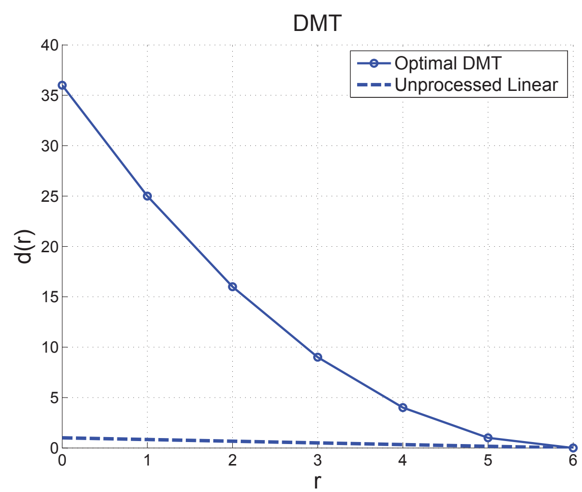
Practical ramifications of error and complexity exponents

$$c(r) := \lim_{\rho\to\infty} \frac{\log C_{\max}}{\log \rho}, \qquad d(r) := -\lim_{\rho\to\infty} \frac{\log P_{\text{err}}}{\log \rho}$$

- Reliability and complexity naturally polynomial in ρ

$$C_{\max} : \rho^0 \to K \cdot |\text{Code}| \approx 2^{RT} \approx \rho^{rT}, \qquad P_{\text{err}} : \rho^0 \to \rho^{-d_{\text{opt}}(r)}$$



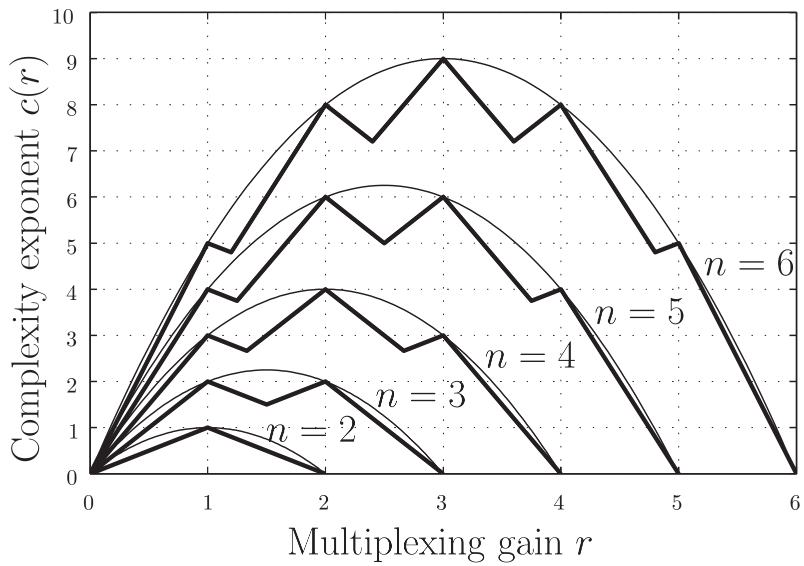


Universal Bounds - Quasi static

Theorem: *$c(r)$ is upper bounded as (piecewise linear)*

$$c(r) \leq \bar{c}(r) = \frac{T}{n_T} r(n_T - r), \quad r = 0, 1, \cdots, n_T$$

for all fading statistics, all full rate lattice designs, and all $\underbrace{\text{decoding order policies}}_{?}$.



Observation: still need extraordinary complexity

Still need extraordinary complexity to achieve good performance - to achieve a vanishing gap to ML.

Let us try different ... tricks

- Different, less constrained decoders
- Different codes and different decoding ordering policies
- Lattice reduction solutions
- Feedback

Regularized lattice decoding

Regularized lattice decoding

- Recall: ML decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg \min_{\hat{\boldsymbol{x}} \in \Lambda_r \cap \mathcal{R}} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2$$

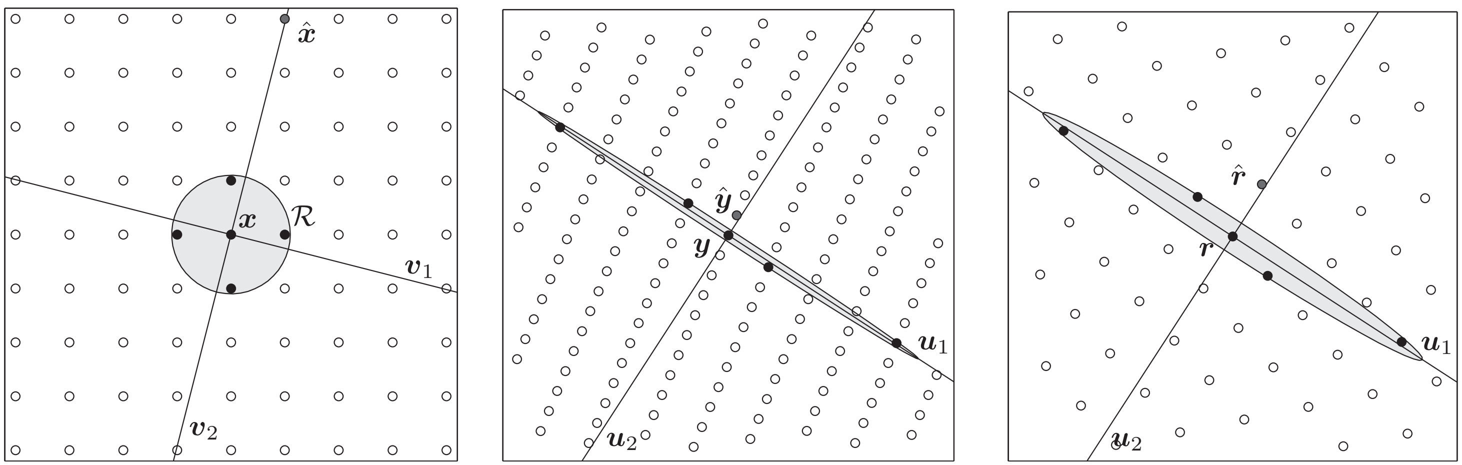
- Recall: equivalently ML decoder

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg \min_{\hat{\boldsymbol{x}} \in \Lambda_r} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2 + I_{\mathcal{R}}(\hat{\boldsymbol{x}}) \quad \text{where} \quad I_{\mathcal{R}}(\hat{\boldsymbol{x}}) = \begin{cases} 0 & \hat{\boldsymbol{x}} \in \mathcal{R} \\ \infty & \hat{\boldsymbol{x}} \notin \mathcal{R} \end{cases}$$

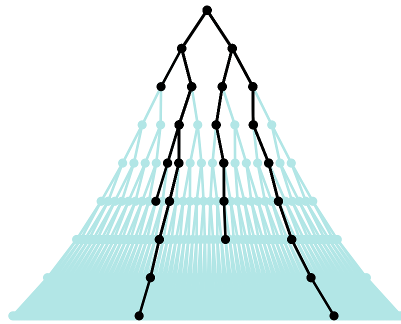
- Recall: regularized lattice decoder (RLD)

$$\hat{\boldsymbol{x}}_{\mathrm{RLD}} = \arg \min_{\hat{\boldsymbol{x}} \in \Lambda_r} \|\mathbf{y} - \boldsymbol{H}\hat{\boldsymbol{x}}\|^2 + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{x}$$

Regularized lattice decoder illustration



Cost of vanishing gap to EXACT lattice decoding



$$\hat{\mathbf{s}}_{r-ld} = \arg\min_{\hat{\mathbf{s}} \in \mathbb{Z}^\kappa} \|\mathbf{r} - \mathbf{R}\hat{\mathbf{s}}\|^2$$

$$c(r) = 0 \quad g_{\text{lattice}} \leq 2^{\kappa/2} \quad C_{\text{lattice}}(1) \triangleq \inf\{\lim_{\rho\to\infty} \frac{\log C_{\max}}{\log \rho} \;:\; g_{\text{lattice}} = 1\} = ?$$

Theorem: *The complexity exponent for MMSE preprocessed lattice sphere decoding any full-rate threaded code (quasi-static regular MIMO), is equal that of ML-based bounded SD with or without regularization.*

Corollary: *Irrespective of the fading statistics and of the full rate code applied, the complexity exponent of MMSE preprocessed lattice SD and ML-based SD is upper bounded by*

$\overline{c}(r) = \frac{T}{n_T}\left(r(n_T - \lfloor r \rfloor - 1) + (n_T \lfloor r \rfloor - r(n_T - 1))^+\right) \longrightarrow \frac{T}{n_T} r(n_T - r).$

Equivalence of ML and lattice decoding

Theorem: *(Equivalence of complexity of ML and lattice decoding)*
ML based sphere decoding and regularized lattice sphere decoding share the same complexity exponent for a very broad setting (share bounds and 'tightness')

Enhanced Theorem: *ML- and regularized lattice-based SD share the same* $c(r), d(r)$ *... for a very broad setting*

$\Rightarrow$ ALL FOLLOWING RESULTS WILL HOLD FOR ML AS WELL AS FOR (REGULARIZED) LATTICE SPHERE DECODING

Theorem *(ML and Lattice SD) (Singh-Elia-Jaldén Trans-IT 2012):* $c(r)$ *of achieving a diversity gain* $d(r)$ *is upper bounded as*

$$c(r) \leq \bar{c}(r) \triangleq \max_{\boldsymbol{\mu}} \sum_{i=1}^{m} \left(\frac{rT}{m} - \frac{1}{2}(1-\mu_i)^+ \right)^+$$
$$s.t. \quad I(\boldsymbol{\mu}) \leq d(r),$$
$$\mu_1 \geq \cdots \geq \mu_m \geq 0,$$

for all fading statistics, all full rate lattice designs, and all decoding ordering policies [37] *.*

- $m \times n$ $(n \geq m)$, $\mu_j \triangleq -\frac{\log \sigma_j(\mathbf{H}^H\mathbf{H})}{\log \rho}$, $j = 1, \cdots, m$, rate function $I(\boldsymbol{\mu})$

[37] Decoding order policy specifies the order in which transmitted symbols are decoded.

Cost of vanishing gap to exact ML and exact lattice

For all existing codes - quasi static

Theorem: *The ML and Lattice SD complexity exponent $c(r)$ is upper bounded at integer $r = k$ as*

$$c(k) \leq \bar{c}(k) = \frac{Tk(n_{\mathrm{T}} - k)}{n_{\mathrm{T}}}.$$

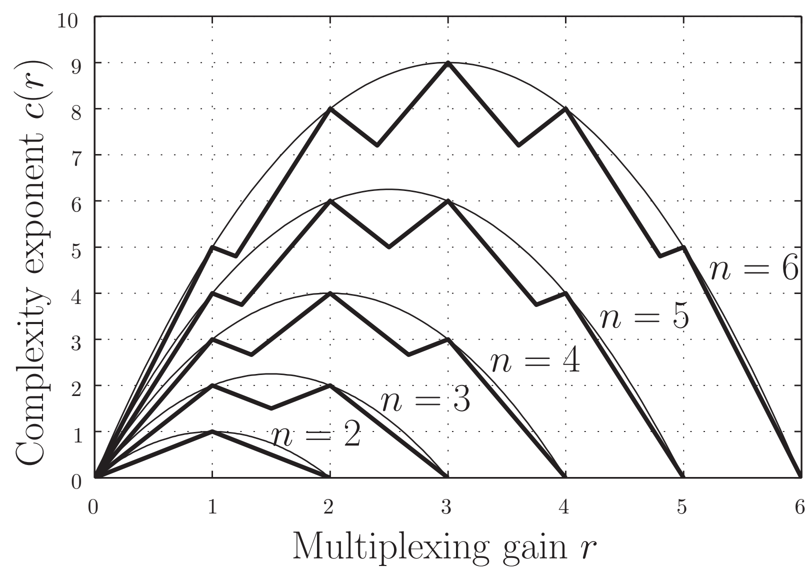
For general r, the above is

$$\bar{c}(r) = \frac{T}{n_{\mathrm{T}}}\Big(r(n_{\mathrm{T}} - \lfloor r \rfloor - 1) + \big(n_{\mathrm{T}}\lfloor r \rfloor - r(n_{\mathrm{T}} - 1)\big)^{+}\Big).$$

The above is a universal upper bound, irrespective of full rate code, of decoding ordering, and irrespective of fading statistics.

BUT, ARE THE BOUNDS CHARACTERISTIC OF THE ACTUAL COMPLEXITY OF LATTICE SEARCH?

IS THIS SUFFICIENT COMPLEXITY, ALSO NECESSARY?



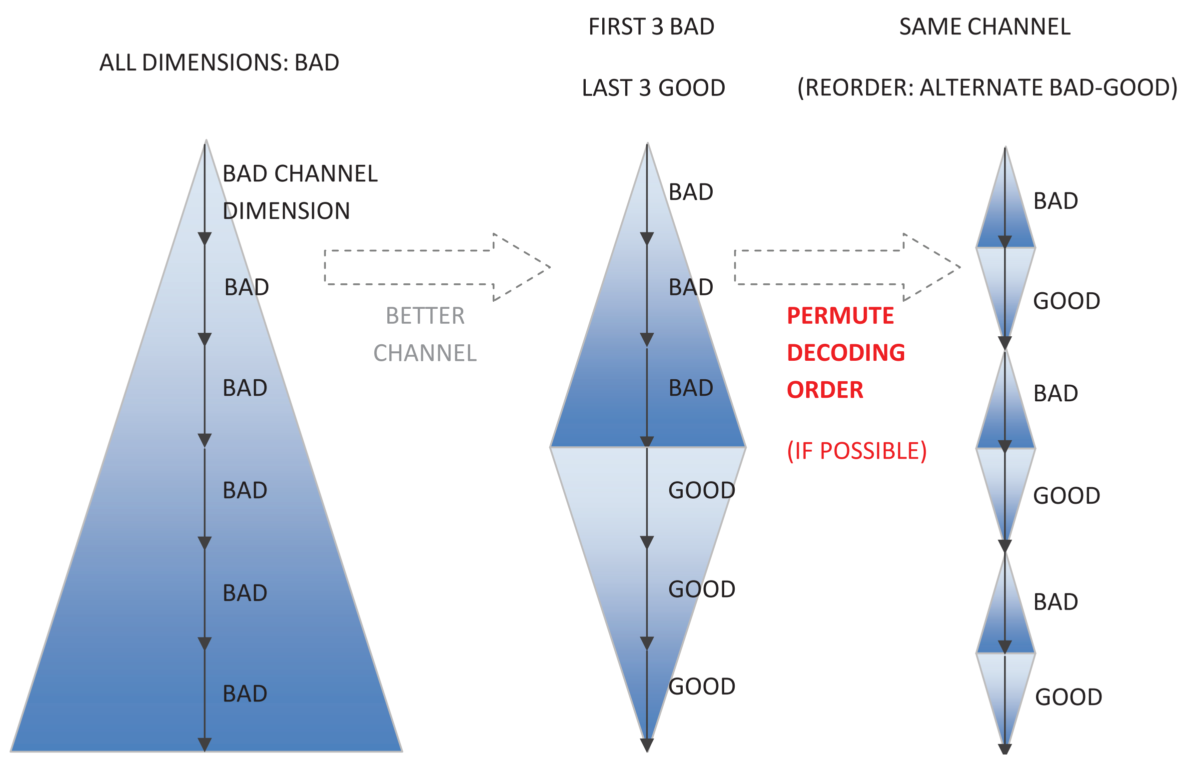
Complexity Exponent for general setting - ordering

Lemma: *Irrespective of channel fading statistics and of the full-rate or below-full-rate code applied, for every realization of channel* $\boldsymbol{M}$ *there exists a channel dependent column permutation matrix* $\boldsymbol{\Pi}$ *such that the ML-based sphere decoder with decoding order* $\boldsymbol{\Pi}$ *has the complexity exponent*

$$\begin{aligned} \tilde{c}(r) \triangleq \max_{\boldsymbol{\mu}} \; & \sum_{i=1}^{\kappa} \min\left(\frac{rT}{\kappa} - \frac{1}{2}(1-\mu_i), \frac{rT}{\kappa}\right)^{+} \\ s.t. \;\; & I(\boldsymbol{\mu}) \leq d(r), \\ & \mu_1 \geq \cdots \geq \mu_\kappa \geq 0. \end{aligned}$$

where $\boldsymbol{\mu} \triangleq (\mu_1, \cdots, \mu_\kappa)$ *satisfies the large deviation principle with rate function* $I(\boldsymbol{\mu})$*.*

Issue: lattice codes and decoding-ordering policies



Are there codes and ordering policies that allow ‘zipping’?

Tightness of universal bound

Proposition: *(General MIMO) Irrespective of channel fading statistics and of the lattice design applied, there exists a fixed decoding order for which the bounds are tight.*

Theorem: *(Quasi-static, Rayleigh, $n_R \geq n_T$) With probability 1 in the choice of the DMT optimal lattice design, the bounds are tight for all (static or dynamic) ordering policies.*

Theorem: *(Quasi-static, Rayleigh, $n_R \geq n_T$) Under a 'richness of codes' assumption[38], with probability 1 in the choice of the lattice design, the bounds are tight for all ordering policies.*

[38] ∃ sufficiently many lattice generator matrices of a certain (suboptimal) DMT performance, so that the entries of the generator matrix accept a continuous distribution across the real numbers.

Theorem: *Given any threaded code, decoded with the natural column ordering or under any other threat-wise grouping, then* $c(r) = \bar{c}(r)$.

At least some good news:

Theorem: *For MISO time-selective channels, and any full-rate code, then* $c(r) = 0$ *for any* T.

WE HAVE RECEIVED BAD NEWS!!
MASSIVE COMPLEXITY FOR VANISHING GAP TO EXACT ML AND LATTICE DECODING

- For integer r then
$$c(r) = \frac{T}{n_{\mathrm{T}}} r(n_T - r)$$
- complexity in the order of
$$2^{\frac{1}{4} n_{\mathrm{T}} T \log \rho} = \rho^{n_{\mathrm{T}} T/4} = \sqrt{|\mathcal{X}|}$$
- exponential in the number of codeword bits
$$C_{\max} \doteq 2^{RT(\frac{n_{\mathrm{T}} - r}{n_{\mathrm{T}}})}$$
- Natural solution: Lattice Reduction

LR-aided Regularized Lattice SD Complexity

- Lattice reduction techniques previously used to improve error-performance of suboptimal MIMO decoders

From

$$\hat{\mathbf{s}}_{r-ld} = \arg\min_{\hat{\mathbf{s}} \in \mathbb{Z}^{\kappa}} \left\| \mathbf{r} - \mathbf{R}\hat{\mathbf{s}} \right\|^2$$

to the new

$$\hat{\mathbf{s}}_{r-lr-ld} = \arg\min_{\hat{\mathbf{s}} \in \mathbb{Z}^{\kappa}} \left\| \mathbf{r} - \mathbf{R}\mathbf{T}\mathbf{T}^{-1}\hat{\mathbf{s}} \right\|^2, \quad (3)$$

- $\mathbf{T} \in \mathbb{Z}^{\kappa \times \kappa}$ is unimodular
- generally better conditioned channel matrix $\mathbf{RT}$.
- new model: $\tilde{\mathbf{r}} = \tilde{\mathbf{R}}\tilde{\mathbf{s}} + \mathbf{w}''$

$$\tilde{\mathbf{s}}_{r-lr-ld} = \arg\min_{\tilde{\mathbf{s}} \in \mathbb{Z}^{\kappa}} \left\| \tilde{\mathbf{r}} - \tilde{\mathbf{R}}\tilde{\mathbf{s}} \right\|^2, \quad (4)$$

Strength of lattice reduction algorithm

$$\boldsymbol{R} = \begin{bmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ 0 & R_{2,2} & R_{2,3} & R_{2,4} \\ 0 & 0 & R_{3,3} & R_{3,4} \\ 0 & 0 & 0 & R_{4,4} \end{bmatrix} \xrightarrow[mmse + LR]{} \quad \tilde{\mathbf{R}}$$

Lemma: *(Singh-Elia-Jaldén) The smallest singular value $\sigma_{min}(\tilde{\mathbf{R}}_{\mathbf{k}})$ of $\tilde{\mathbf{R}}_{\mathbf{k}}$, after MMSE preprocessing and LLL lattice reduction, satisfies*

$$\mathrm{P}\left(\sigma_{min}(\tilde{\mathbf{R}}_k) \dot{<} \rho^{\frac{-\epsilon T}{\kappa}}\right) \dot{\leq} \rho^{-d_L(r-\epsilon)}, \ \textit{for all } r \geq \epsilon > 0, \quad k \geq 1.$$

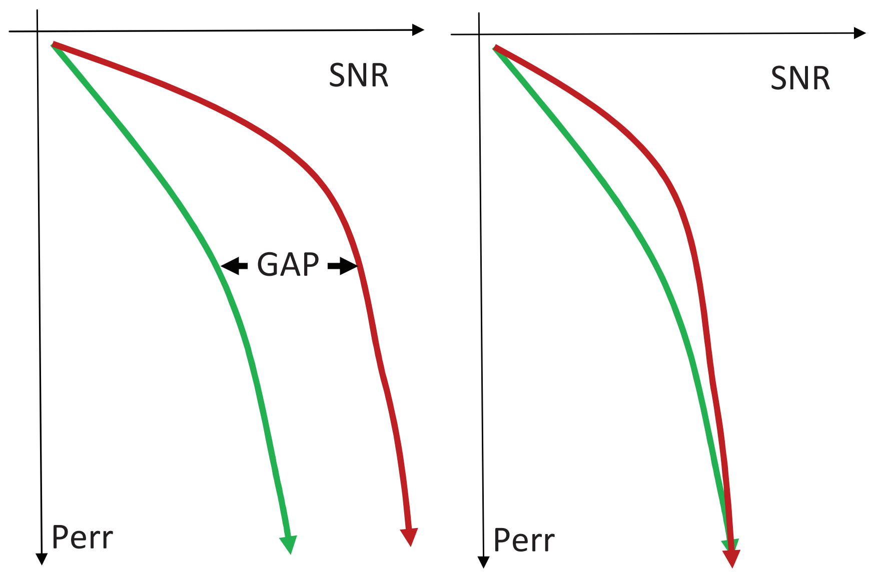
Achieving a vanishing gap at subexponential complexity

Theorem: *(Singh-Elia-Jaldén) LR-aided MMSE preprocessed lattice sphere decoding introduces a zero complexity exponent, and achieves a vanishing gap to the exact implementation of lattice decoding.*

- First ever lattice decoding solution that provably achieves both a vanishing gap to the error-performance of the exact solution of regularized lattice decoding, as well as a computational complexity that is subexponential in the rate and in the problem dimensionality
  - ⋆ for the most general outage-limited MIMO setting
  - ⋆ all mimo scenarios, all reasonable fading statistics, all codes

Achieving a vanishing gap at subexponential complexity[1]

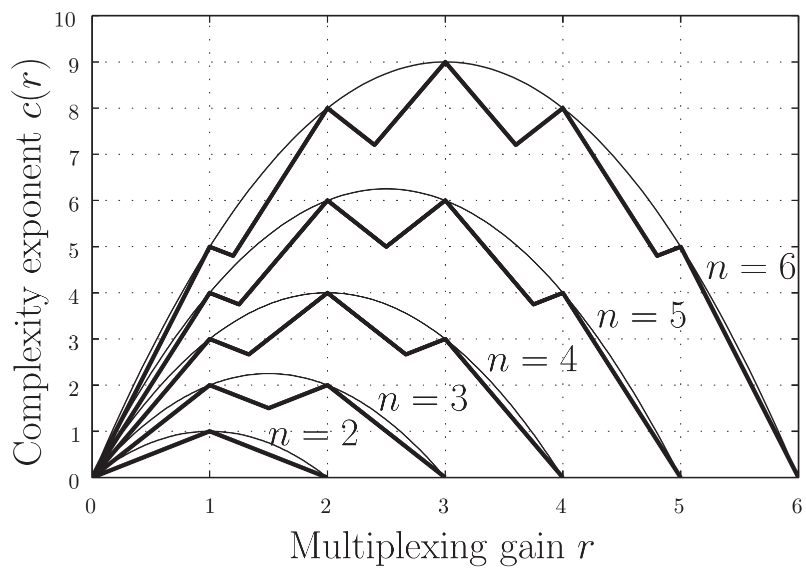
- Vanishing gap to error-performance of exact lattice decoding
- Subexponential computational complexity



LR Problem: sometimes not applicable

BUT, LR can sometimes not be applicable
Especially in the presence of an inner code and soft decoding, which is often an absolute must for the industry.

Hence, non-LR solutions still of importance.



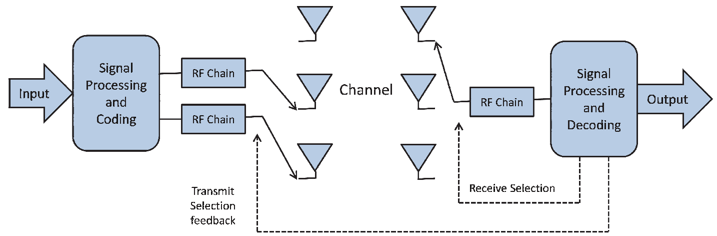
Performance-complexity ramifications of feedback in outage limited MIMO communications

Use feedback to reduce complexity

Use antenna selection to reduce size of system

$$(n_T \times n_R) \rightarrow (l_T \times l_R)$$

Complexity reductions using antenna selection



ANTENNA SELECTION

- Use $\log_2 \binom{n_\mathrm{T}}{l_T}$ bits of feedback to reduce system size

$$(n_\mathrm{T} \times n_\mathrm{R}) \longrightarrow (l_T \times l_R)$$

  - ⋆ While maintaining $d^*_{n_\mathrm{T} \times n_\mathrm{R}}(r)$

- Smaller system means less complexity

- We only focus on a very specific case: the performance, after antenna selection, remains DMT optimal ($d(r) = d^*_{n_T \times n_R}(r)$)
- We consider only the greedy selection algorithms of Varanasi et al.

Complexity-Reduction using Feedback for Antenna Selection

- Let $N \triangleq \min\left(l_R, l_T\right) = l_T$
- Let $P = \arg\min_p \frac{(n_\mathrm{R}-p)(n_\mathrm{T}-p)}{N-p}$ such that $0 \le P \le N-1, p \in \mathbb{Z}$
- Let i.i.d. Rayleigh
- Let $n_\mathrm{R} \ge n_\mathrm{T}$

Theorem: *(Varanasi et al.) Pruning an $n_\mathrm{T} \times n_\mathrm{R}$ MIMO system to an $l_T \times l_R$ system, can maintain the optimal $d^*_{n_\mathrm{T}\times n_\mathrm{R}}(r)$ for all $r \le P$.*

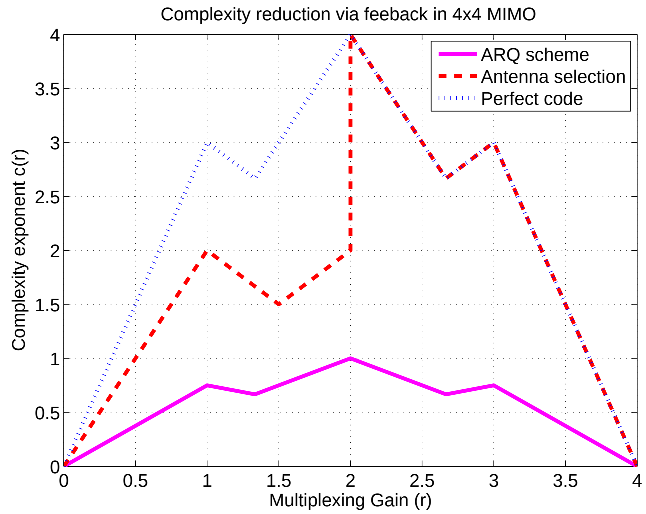
Proposition: *(Singh-Elia-Jaldén) The minimum* $c(r)$ *(over all antenna selection algorithms, all lattice designs and all halting and decoding order policies) required to achieve the optimal DMT* $d^*_{n_\mathrm{T}\times n_\mathrm{R}}(r)$*, is upper bounded as (piecewise linear - integer* r*)*

$$c(r) \leq \overline{c}_{as}(r) = r(N_r - r), \text{ for } r = 0, 1, \cdots, n_\mathrm{T}.$$

where

$$N_r = \arg \min_{N' \in \{1, \cdots, n_T\}} \left[\left(\arg \min_{p \in \{0, \cdots, N'-1\}} \frac{(n_\mathrm{T} - p)(n_\mathrm{R} - p)}{N' - p} \right) = \lceil r \rceil \right] \qquad (5)$$

Complexity-Reduction using Feedback for Antenna Selection$_2$



Complexity savings: antenna selection (Note: N_r varies with r)

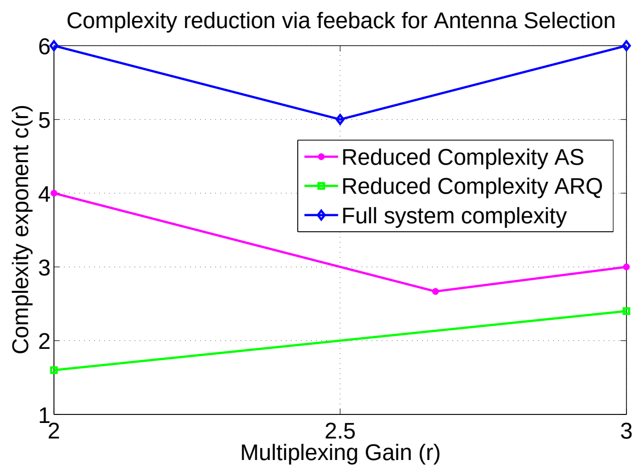
Complexity reduction with antenna selection

Example: Start with 5×6 MIMO system. Antenna select down to $l_T = 4, l_R = 4$ $(P = 3)$. Then for $2 \le r < 3$, the pruned system gives

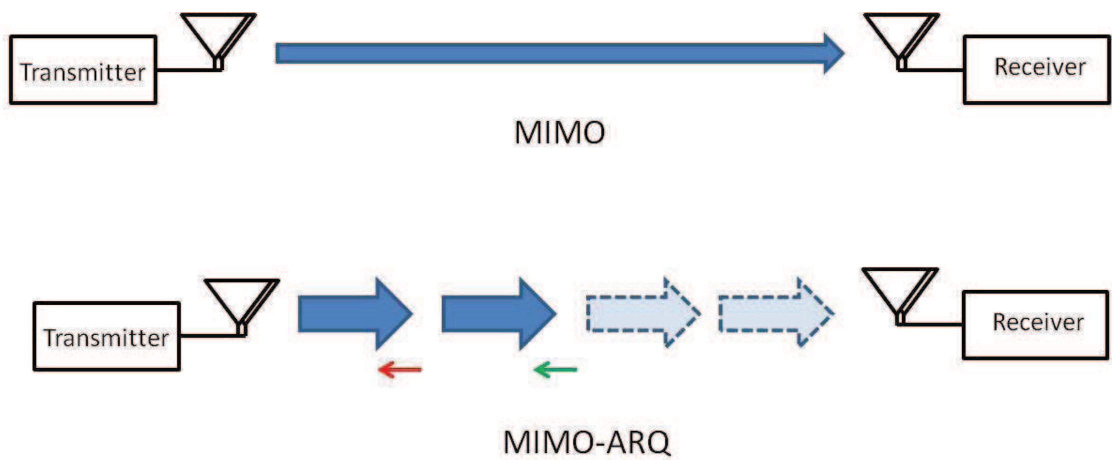
$$c_{as}(r) = \begin{cases} 2(4-r) & \text{for } 2 \le r \le \frac{8}{3} \\ r & \text{for } \frac{8}{3} < r < 3, \end{cases}$$

which is less than that of the unpruned system

$$c_{ML-SD}(r) = \begin{cases} 2(5-r) & \text{for } 2 \le r \le \frac{5}{2} \\ 2r & \text{for } \frac{5}{2} < r < 3. \end{cases}$$

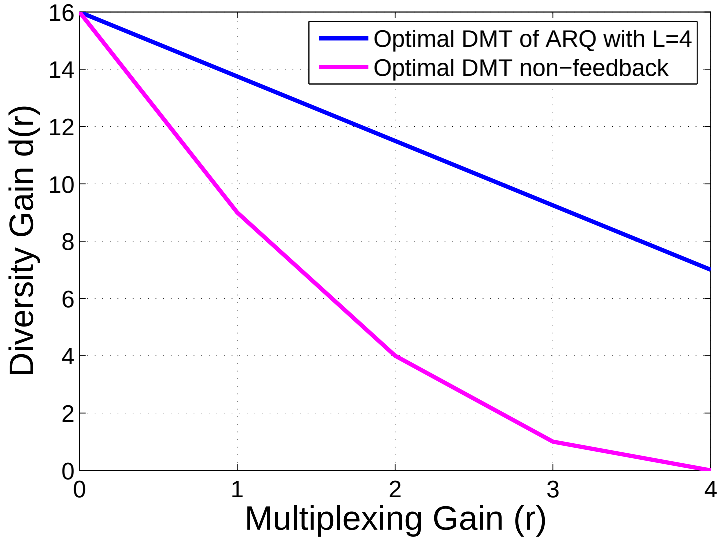


MIMO ARQ Feedback for high performance and low complexity


Transmitter
Receiver
MIMO
Transmitter
Receiver
MIMO-ARQ

Previous work[39] has shown that with L rounds of ARQ

- $d^*(r)$ (original optimal DMT) $\longrightarrow d^*(r/L)$
- $d^*(r/L)$ (feedback-aided DMT) often $d^*(r/L) >> d^*(r)$

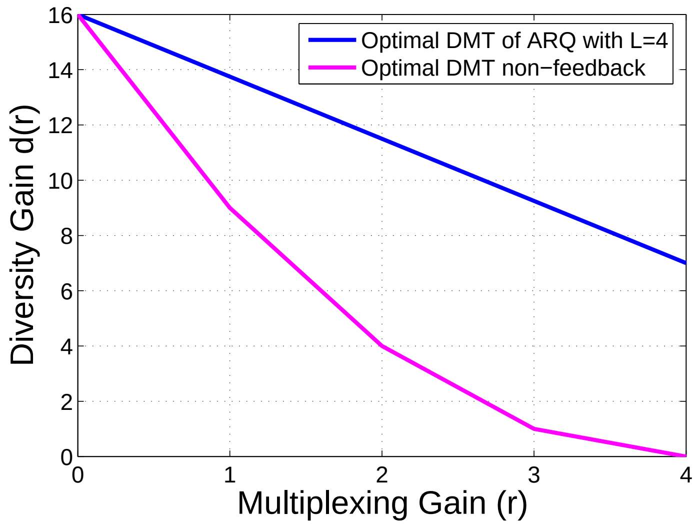


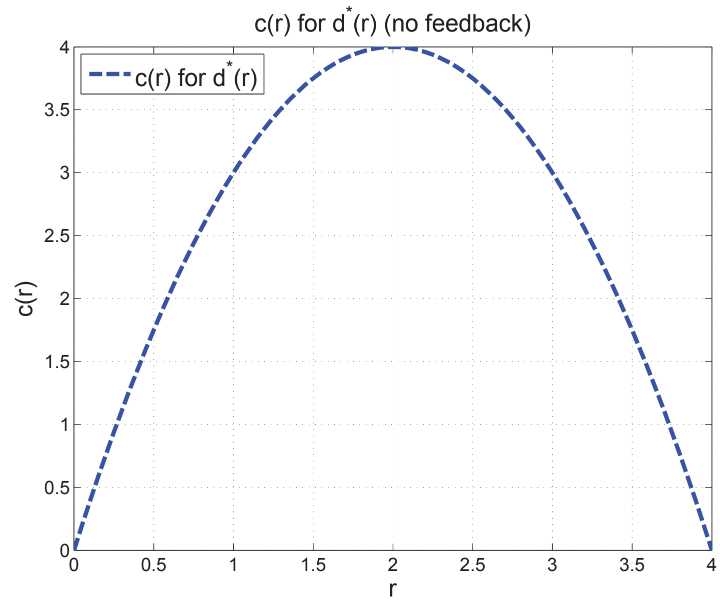
[39] El Gamal, Caire, Damen

Two interesting questions:

- What is the feedback-aided complexity to achieve DMT $d^*(r)$?
- What is the complexity to achieve the feedback-aided DMT $d^*(r/L)$?

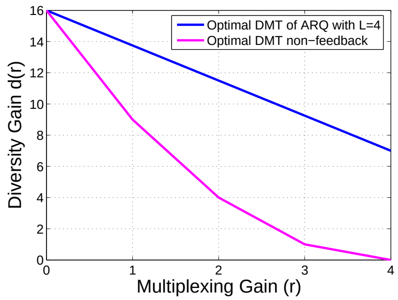
Example:

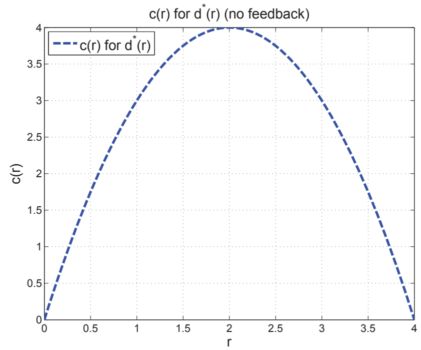




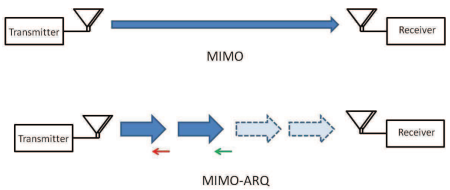
Feedback-aided complexity for optimal DMT $d^*(r)$

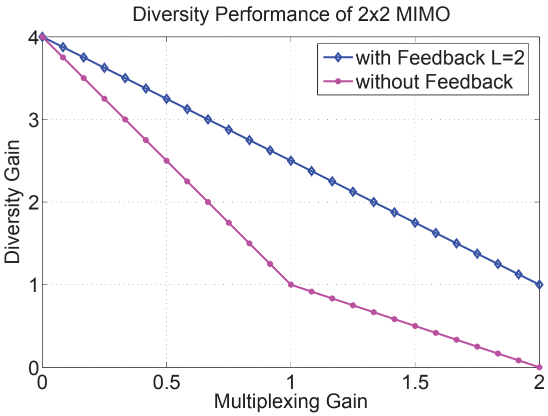
(i.e., Use feedback to reduce complexity, without sacrificing performance)

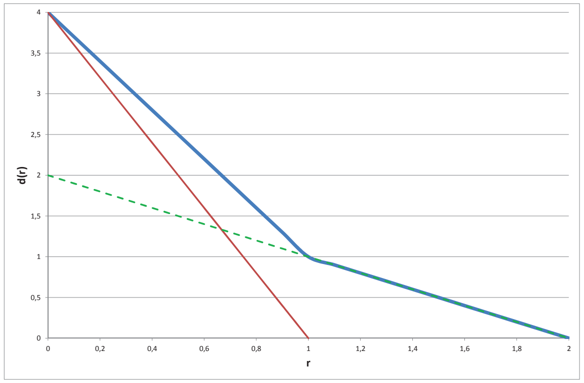




First attempt: use simpler codes





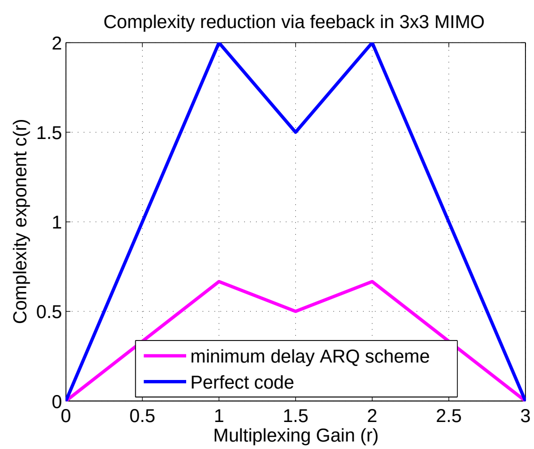


Feedback-aided complexity for optimal DMT $d^*(r)_1$

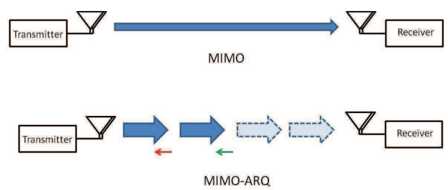
Corollary: *(First attempt) (quasi-static iid Regular $n_\mathrm{R} \geq n_\mathrm{T}$, $LT = n_\mathrm{T}$) Minimum $c(r)$ for $d^*(r)$, (minimized over all lattice designs, all L-round ARQ schemes, all halting and decoding order policies), bounded as (piece-wise linear $r = 0, 1, \cdots, n_T$)*

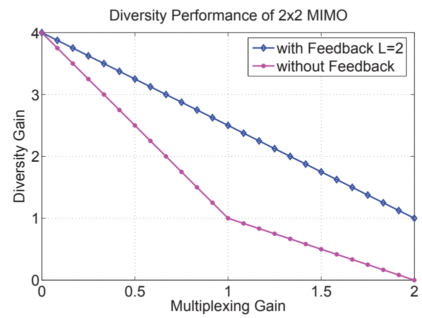
$$c(r) \leq \bar{c}_{red}(r) = \frac{1}{n_T} r(n_T - r).$$

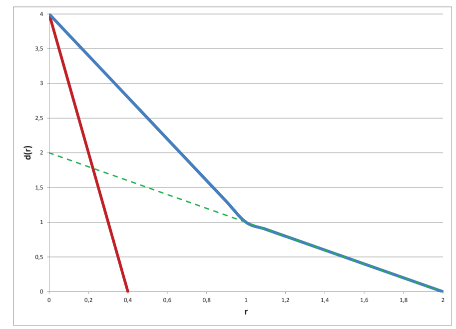
- Compare to $c(r) = r(n_T - r)$
- Important role of "aggressive intermediate halting policies"



Second attempt



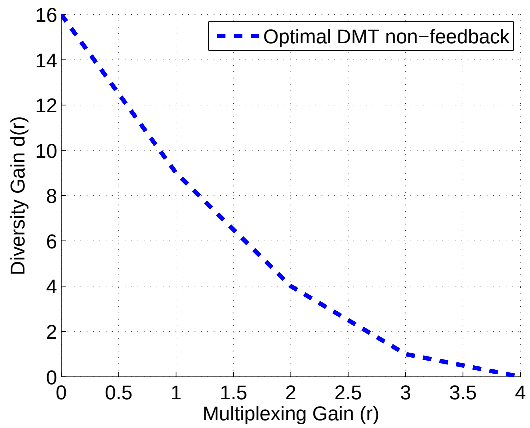




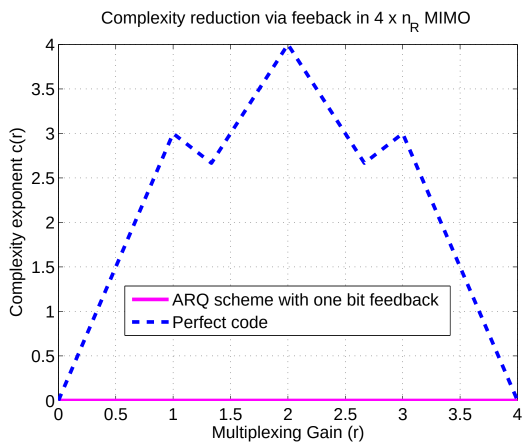
- ARQ scheme with two rounds, $T_1 = 1$ and $T = n_{\mathrm{T}}^2 + 1$
  - ⋆ First round: high-rate uncoded (high rate, no diversity)
  - ⋆ second round: orthogonal design with rate-$\frac{1}{n_{\mathrm{T}}}$ (ultra low rate, full diversity)
- Decoding policy
  - ⋆ First round decode iff really good channel, halt decoding if $|\sigma_{min}(\mathbf{H})| \leq \rho^{-\epsilon}$ for some $\epsilon > 0$.
  - ⋆ Second round full decoding

Theorem: *(Second Attempt - longer delay) (quasi-static i.i.d. Regular $n_{\rm R} \geq n_{\rm T}$)*
Sphere decoding with one-bit of ARQ feedback and a computational constraint activated at ρ^x flops achieves optimal DMT $d^*(r)$ for any $x > 0$.

EXAMPLE:



Diversity Gain



Complexity Exponent

Complexity cost for feedback-aided DMT $d^*(r/L)$

Recall: Generally $P_{\text{ERR}} \uparrow$ as $C_{\max} \downarrow$

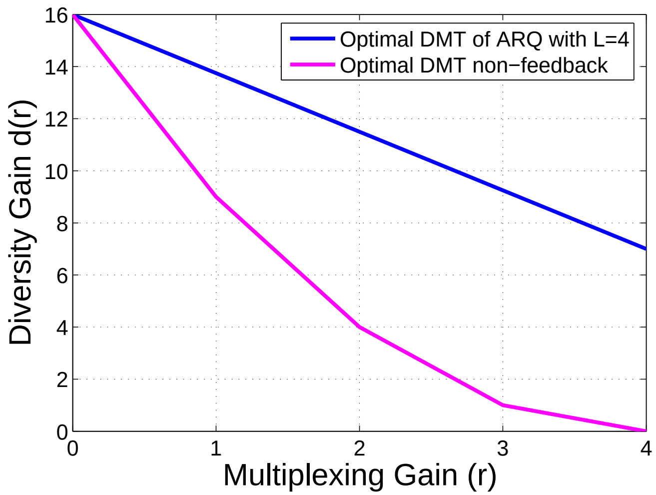
Killing two birds with one bit of feedback

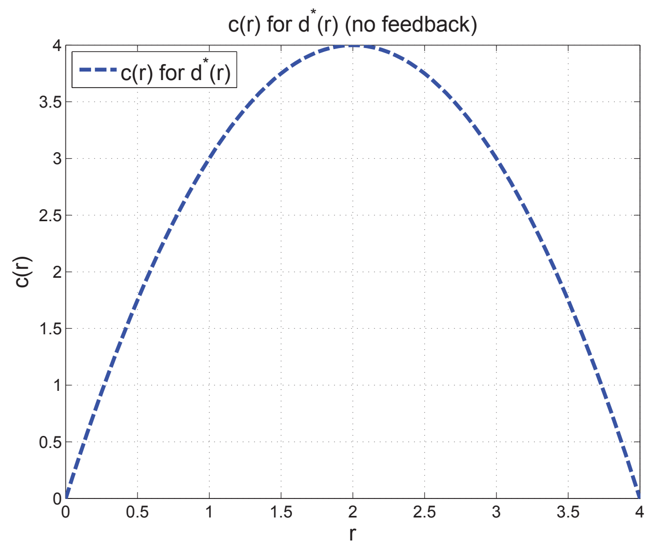
One-bit feedback $\rightarrow C_{\max} \downarrow$ also $P_{\text{ERR}} \downarrow$

Complexity cost for feedback-aided DMT $d^*(r/L)$

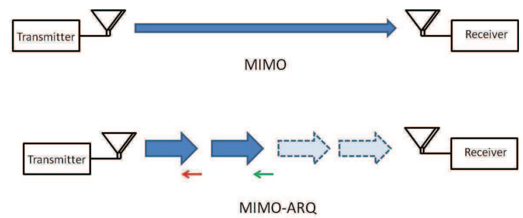
SEEKING $c(r)$ NEEDED TO ACHIEVE $d^*(r/L)$

Recall:





Proposed Communication Scheme



- ARQ scheme with two rounds, $T_1|T$ and $T = n_{\mathrm{T}}$
- Incremental redundancy lattice designs (powerful in each round)
- Decoding policy
  - ⋆ First round decode iff really good channel
    halt decoding if $|\sigma_{min}(\mathbf{H})| \leq \rho^{-\epsilon}$ for some $\epsilon > 0$.

$$\mathrm{P}\left(r_\ell\right)_{e,\ell} \dot{\leq} \mathrm{P}\left(r_L\right)_{e,L}, \ell = 1, \ldots L-1$$

  - ⋆ Much reduced complexity due to channel singularity level
- Lim-optimal decoding in the last L-th round

$$\mathrm{P}\left(r_L\right)_{e,L} \doteq \rho^{-d_{ARQ,L}(r_L)}$$

  - ⋆ Second round halt after $\rho^{\overline{c}_{dmd}(r)}$ flops

Complexity in general MIMO settings

- Complexity for ℓ-th round

$$c_\ell(r) \triangleq \max_{\boldsymbol{\mu}} \left(1 - \frac{\ell}{L}\right) rT + \ell T \sum_{j=1}^{n_T} \left(\frac{r}{L n_T} - (1-\mu_j)^+\right)^+,$$

$$\text{s.t.} \quad I(\boldsymbol{\mu}) \le d(r_\ell),$$
$$\mu_1 \ge \cdots \ge \mu_{n_T} \ge 0,$$

- The overall complexity exponent is given by

$$c_{ARQ}(r) = \max\left(c_1(r), \cdots, c_L(r)\right)$$

- High computational complexity cost due to
  - ⋆ beyond-full-rate decoding
  - ⋆ high diversity gain achieved

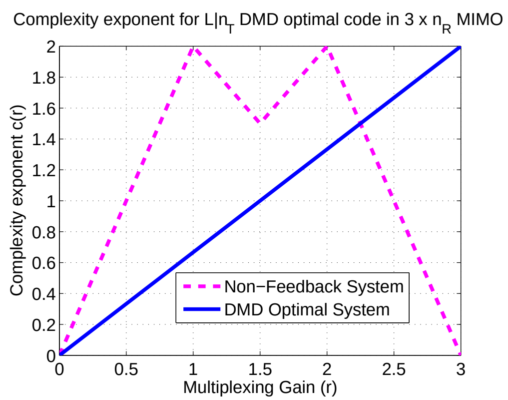
Complexity reduces with feedback despite increased $d^*(r/L)$

Theorem: *($L|n_\mathrm{T}$, quasi-static, $n_\mathrm{R} \geq n_\mathrm{T}$)*
Minimum $c(r)$ to achieve optimal $d^*(r/L)$ is bounded as ((mult. of L))

$$c(r) \leq \bar{c}_{dmd}(r) = \frac{r n_\mathrm{T}}{L^2}\left(L - \frac{r}{n_\mathrm{T}}\right).$$

Corollary: *The above with $L = n_\mathrm{T}$ gives*

$$c(r) \leq \bar{c}_{DMD}(r) = \left(1 - \frac{1}{n_T}\right) r.$$

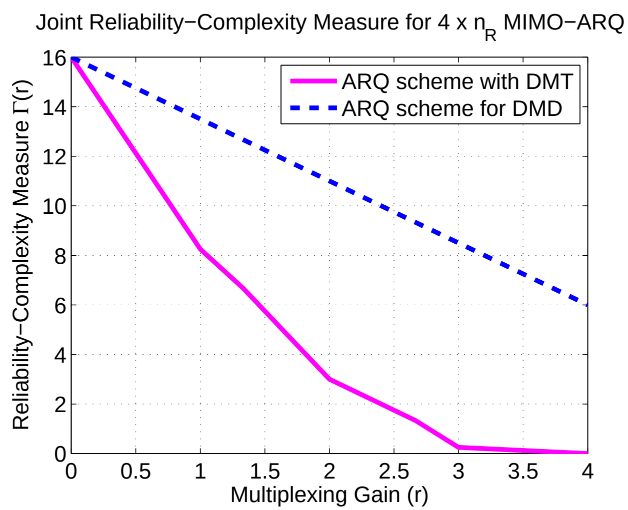


$3 \times n_\mathrm{R}$

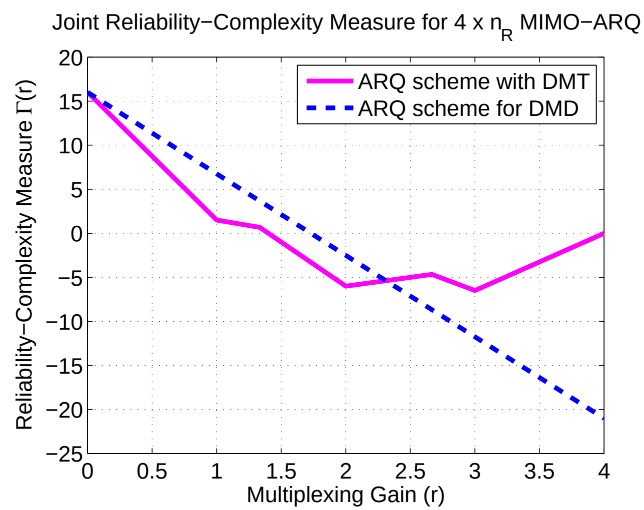
Have feedback: Go for basic DMT or feedback-aided DMT?

Joint performance-complexity measure

$$\Gamma(r) = d(r) - \gamma c(r)$$



$\gamma = 1$



$\gamma = 10$

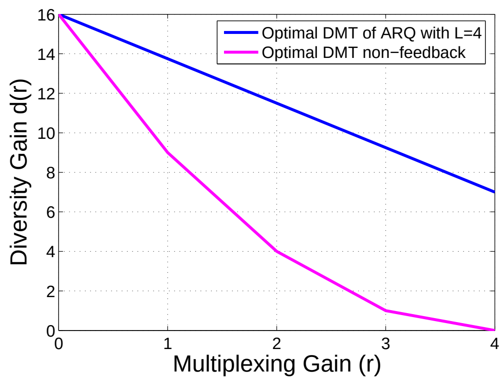
Joint reliability-complexity measure for DMT and DMD optimal ARQ schemes

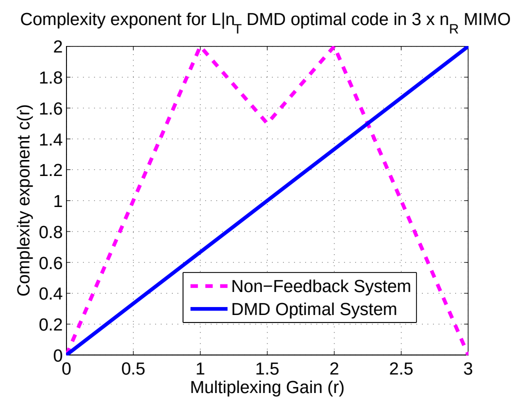
Complexity for $d^*(r/L)$ is still very high

Recall-Corollary: *Can achieve* $d^*(r/n_{\mathrm{T}})$ *with*

$$c(r) \leq \bar{c}_{dmd}(r) = \left(1 - \frac{1}{n_T}\right) r.$$

EXAMPLE:





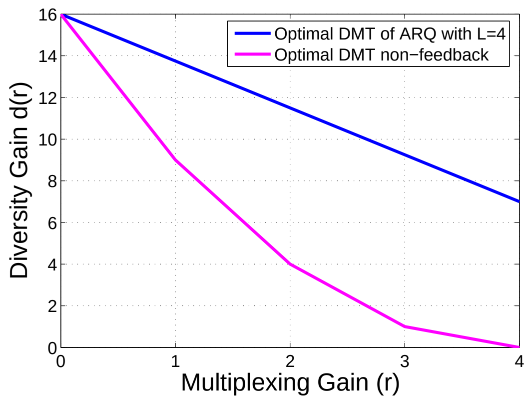
- Feedback reduces complexity up to $r = \frac{n_{\mathrm{T}}^2}{n_{\mathrm{T}}+1}$
- High complexity for $r \approx n_{\mathrm{T}}$

$$C_{\max} \to \rho^{n_{\mathrm{T}}-1}$$

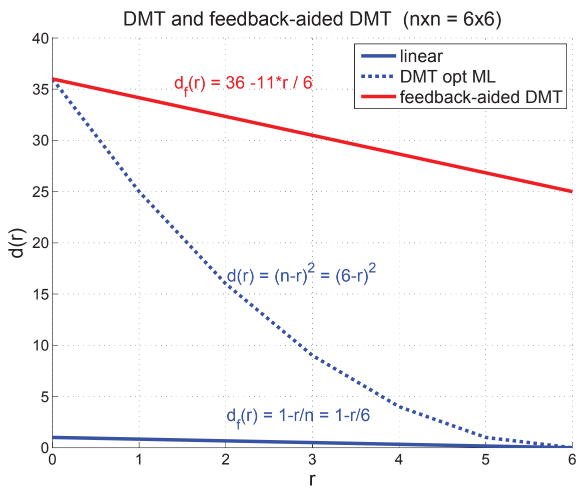
- SEEK HELP OF LR FOR ERGODIC-LIKE BEHAVIOR

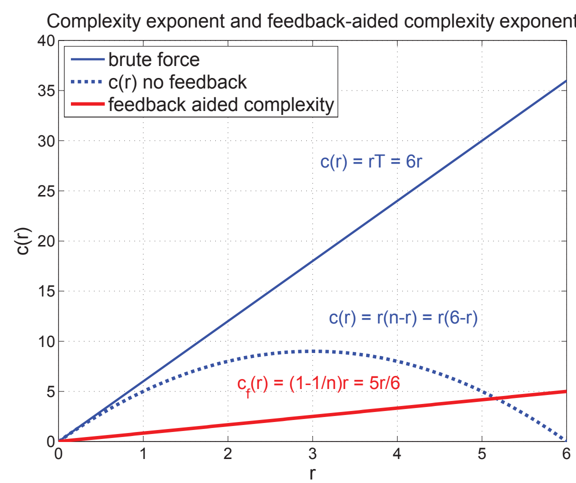
DESIRABLE TO ACHIEVE ERGODIC LIKE BEHAVIOR
WITH MINIMAL FEEDBACK AND MINIMAL COMPLEXITY

- Want to achieve high $d(r)$ for very high r
- Want to achieve it with reduced $c(r)$



Theorem *(Trans IT June 2012 and ISIT 2013):*
LR-aided regularized lattice sphere decoding with an aggressive first-round halting policy, with LR- and outage-based last-round halting policies, and with a single bit of feedback, introduces a zero complexity exponent, and achieves the optimal $d(r/n)$ *(ergodic-like).*

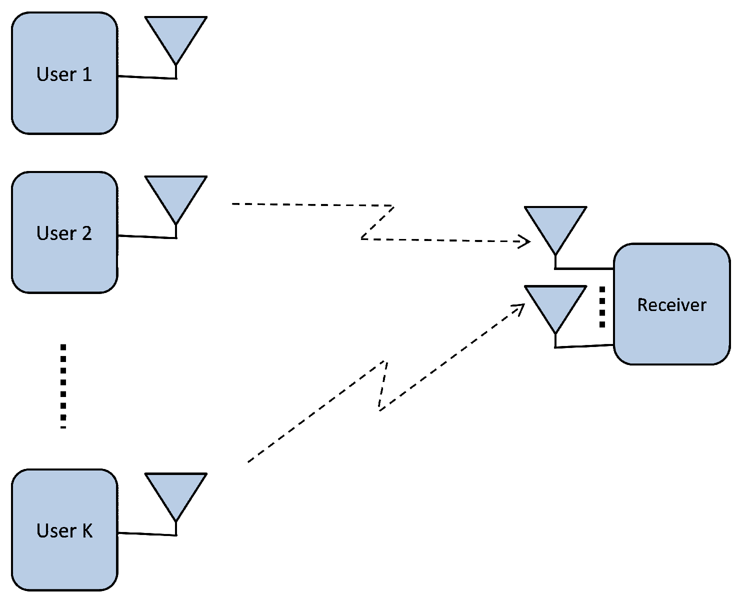




- First algorithm to achieve a vanishing gap to the exact solution of (regularized) lattice decoding, with subexponential computational complexity

Different directions

Complexity in Multiple Access Channels



Complexity results for Multiple Access Channel - Symmetric Case

- Interested in very specific problem
  - ⋆ K users with n_T antennas each
  - ⋆ destination with n_R antennas

$$d_{mac}(r) = \begin{cases} d^*_{n_T,n_R}(\frac{r}{K}), & r \leq \min\left(Kn_T, \frac{n_R K}{K+1}\right) \\ d^*_{Kn_T,n_R}(r), & \min\left(Kn_T, \frac{n_R K}{K+1}\right) < r \leq \min\left(Kn_T, n_R\right) \end{cases}$$

- Interested in complexity for optimal DMT with joint ML/lattice decoding
- Draw from only known MIMO-MAC optimal codes (Lu, Hollandi,...)

Proposition: *(Singh et al.) The optimal complexity exponent of ML-based SD joint decoder is upper bounded as*

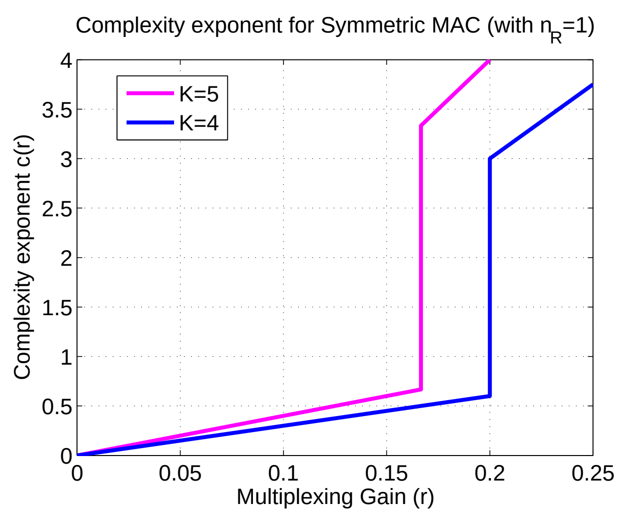
$$c_{mac}(r) = \max_{\boldsymbol{\mu}} Kn_T \sum_{j=1}^{Kn_T} \left(\frac{r}{Kn_T} - (1 - \mu_j) \right)^+$$
$$s.t. \quad I(\boldsymbol{\mu}) \leq d_{mac}(r), 1 \geq \mu_1 \geq \cdots \geq \mu_{Kn_T} \geq 0.$$

MIMO MAC

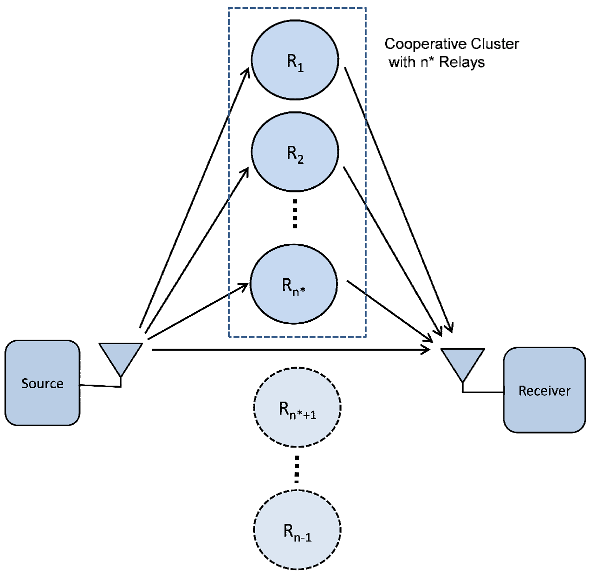
(K user MAC, $n_{\mathrm{T}} = 1$, $n_{\mathrm{R}} = 1$, r per user, Rayleigh, K odd)

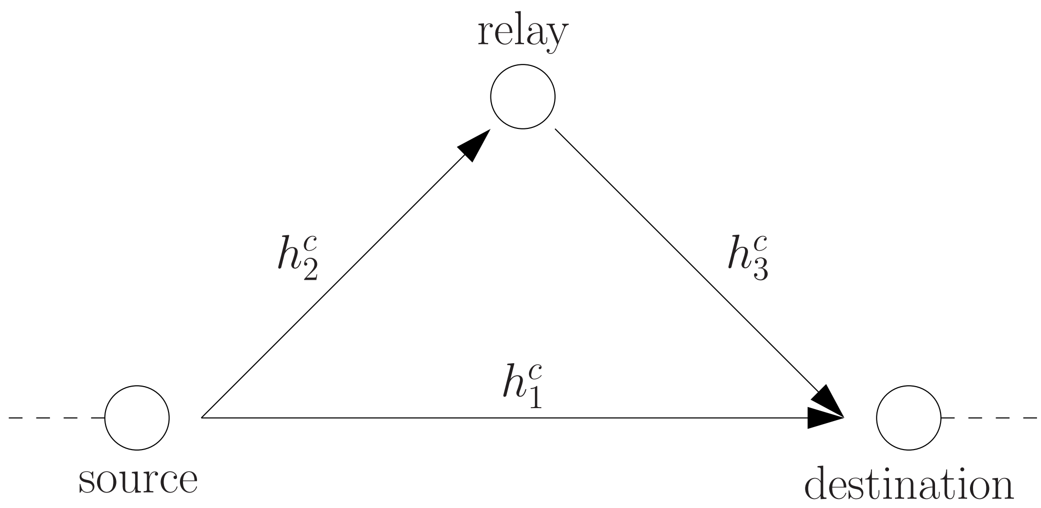
Corollary: *(Best known upper bound)*
The minimum $c(r)$ (over all lattice designs and halting and decoding order policies) to achieve the optimal MAC-DMT, is upper bounded as

$$c(r) \leq \overline{c}_{mac}(r) = \begin{cases} (K-1)r & \text{for } r \leq \frac{1}{K+1}, \\ (K-1)Kr & \text{for } \frac{1}{K+1} < r \leq \frac{1}{K}. \end{cases}$$



Complexity in Cooperative communications





$$\mathbf{y}_t^c = \begin{bmatrix} \sqrt{\rho} h_1^c & 0 \\ \rho b h_2^c h_3^c & \sqrt{\rho} h_1^c \end{bmatrix} \boldsymbol{x}_t^c + \begin{bmatrix} 0 \\ \sqrt{\rho} b h_3^c \end{bmatrix} w_t^c + \boldsymbol{v}_t^c, \qquad |b|^2 = \frac{1}{\rho |h_2|^2 + 1}$$

$$\downarrow$$

$$\mathbf{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w}$$

SPECIFIC PROBLEM ADDRESSED

- One source, $n_T - 1$ relays, one destination
  - ⋆ All nodes having one antenna
- Protocol: Orthogonal Amplify Forward (OAF)
- Interested in complexity for achieving the optimal OAF DMT

$$d_{oaf}(r) = \begin{cases} n_T\left(1 - \frac{(2n_T - 1)r}{n_T}\right), & \text{for} \quad 0 \leq r \leq \frac{1}{2}, \\ 1 - r, & \text{for} \quad \frac{1}{2} < r \leq 1. \end{cases}$$

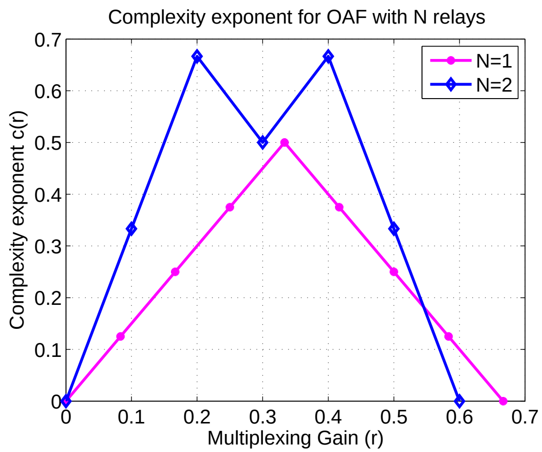
- The complexity exponent is given by

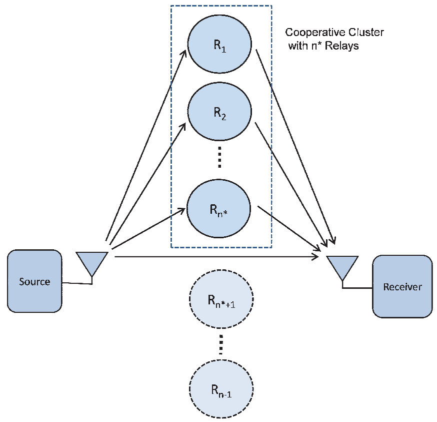
$$c_{oaf}(r) = \max_{\boldsymbol{\mu}} \sum_{j=1}^{n_T} \left(\frac{2n_T - 1}{n_T} r - (1 - \mu_j) \right)^+ ,$$
$$\text{s.t.} \quad \sum_{j=1}^{n_T} \mu_j \leq n_T (1 - \frac{2n_T - 1}{n_T} r),$$
$$1 \geq \mu_1 \geq \cdots \geq \mu_{n_T} \geq 0.$$

$$c_{oaf}(r) = c_{miso}(\frac{2n_T - 1}{n_T} r)$$

$$c_{oaf}(r) = (2n_T - 1) r (1 - \frac{2n_T - 1}{n_T} r) \quad \text{for} \quad r = 0, \frac{1}{2n_T - 1}, \cdots, \frac{n_T}{2n_T - 1}$$

Complexity analysis for OAF Relaying[1]





- Complexity exponential in number of relays
- Complexity constraints force relay selection

Seek best cooperative protocols and best relay-selection protocols to improve performance-complexity tradeoff

Decoding in Massive MIMO: a wide open problem

- Interesting work on low-complexity detection in large-MIMO (A. Chockalingam, B. Sundar Rajan, et al., and others)
  - ⋆ LR-based solutions
  - ⋆ random sampling based solutions

- Interesting performance analysis for decoders (Moustakas, R Kumar, Caire, Mertikopoulos...)

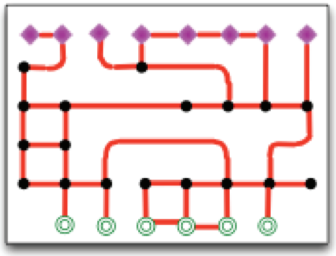
- Interesting and wide-open challenge

Apply statistical approach of performance-vs-complexity in different large systems

Theoretical underpinnings and practical designs of green radios

Work by Grover, Sahai, Goldsmith, Ganesan...

- Effort to analyze complexity of coding
- Emphasis on short-distance communication systems
  - ⋆ require processing power that dominates transmit power

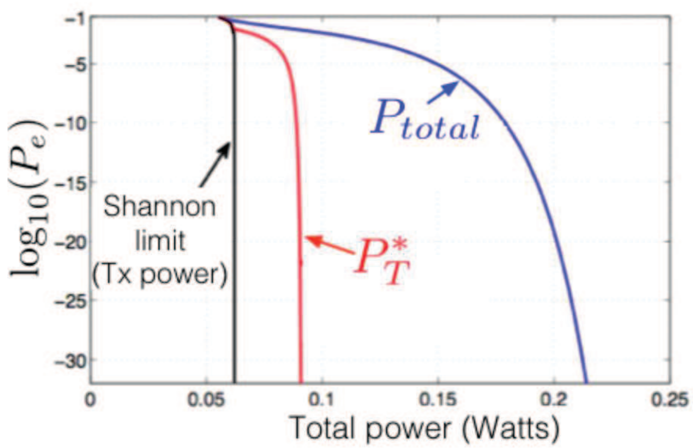


The VLSI model of implementation
[Thompson '80]

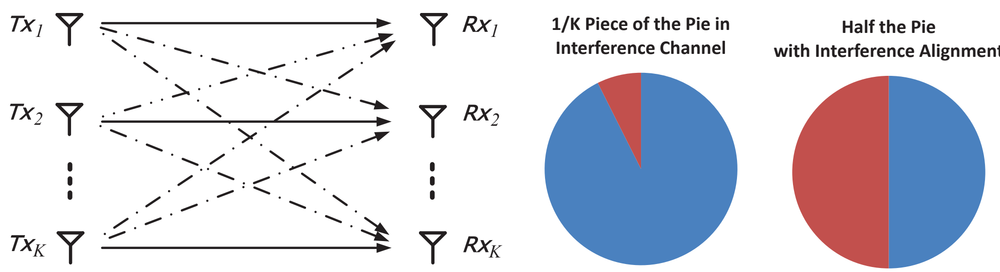
- Emphasis on limiting encoding and decoding power
- Measure is communication complexity, not Turing complexity

Communication complexity in green radios[1]

- Derived bounds on encoding/decoding power
- Modify traditional Shannon capacity view point
- Reveal tradeoff between transmit and encoding/decoding power
- Insight: When computational nodes dominate processing power, to minimize total power, one must fundamentally stay away from capacity.
  - ⋆ Capacity-approaching LDPC codes optimize over transmit power, but require large decoding power.
  - ⋆ TODO: Find such codes that require reduced decoding power.



Complexity in interference alignment (Cadambe and Jafar)



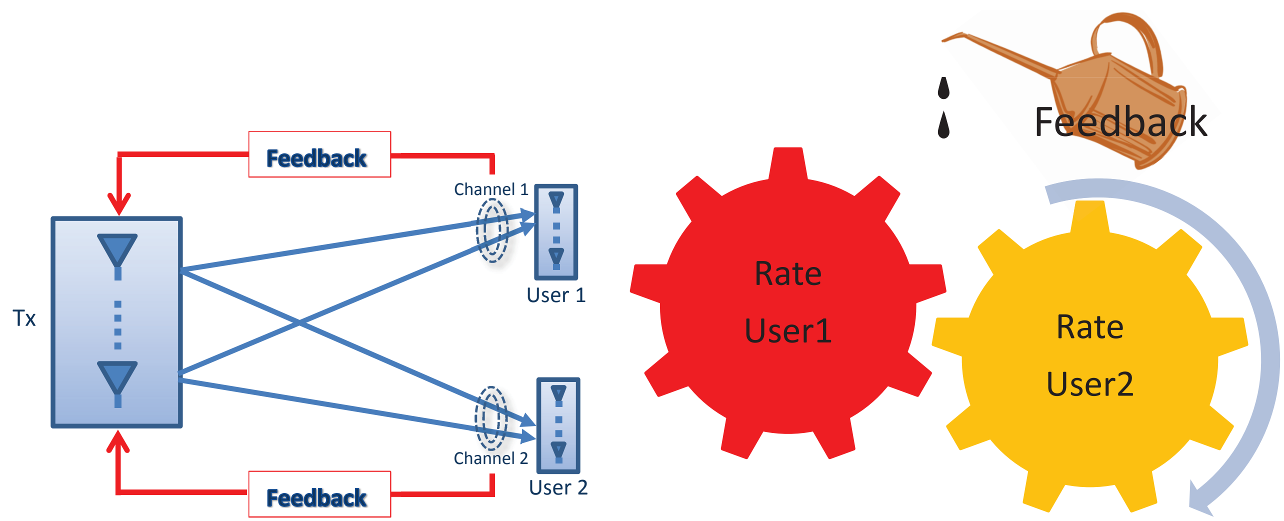
- Maximizing sum DoF for general MIMO (without symbol extension) is NP-hard (in number of user pairs)
  - ⋆ DoF max is NP-hard if ≥ 3 rx-tx antennas
  - ⋆ Polynomial-time if ≤ 2 rx-tx antennas
- Conjecture (Razaviyayn et al): for symmetric network, polynomial-time algor. may exist
- Recent (Ma et al.) Conjecture holds only in a very limited sense
  - ⋆ polynomial-time if ≤ 2 rx-tx antennas (generally NP-hard otherwise)

Apply statistical optimization and complexity methods

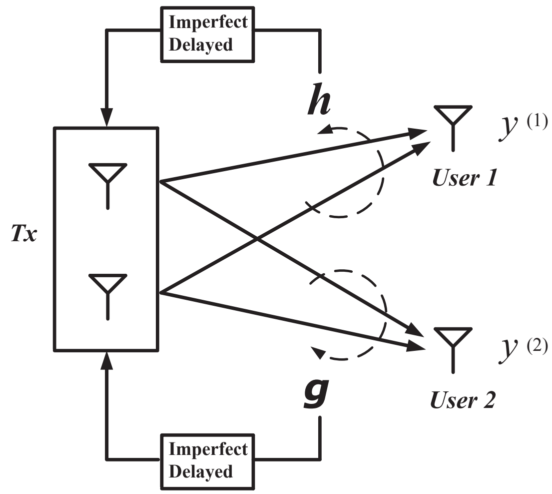
Complexity and feedback in multiuser communications

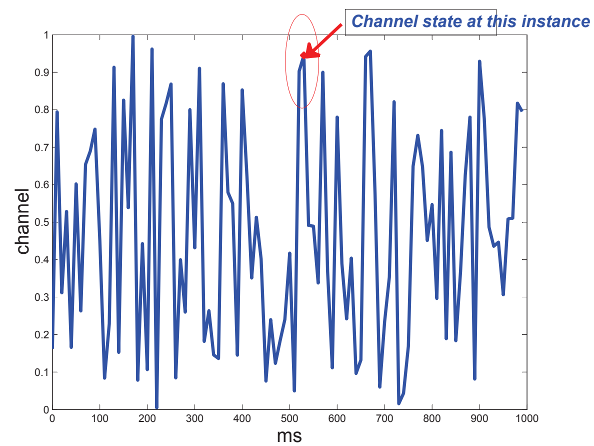
FEEDBACK (CSIT) IS CRUCIAL: INTERFERENCE $\downarrow$ RATES $\uparrow$

- RECENT ADVANCES IN UNDERSTANDING AND MEETING THE LONG ELUSIVE FUNDAMENTAL TRADEOFF BETWEEN PERFORMANCE AND FEEDBACK IN CLASSICAL MULTIUSER CHANNELS
- COMPLEXITY SHOWS ITS UGLY FACE AGAIN



Challenge in performance-vs-feedback problem



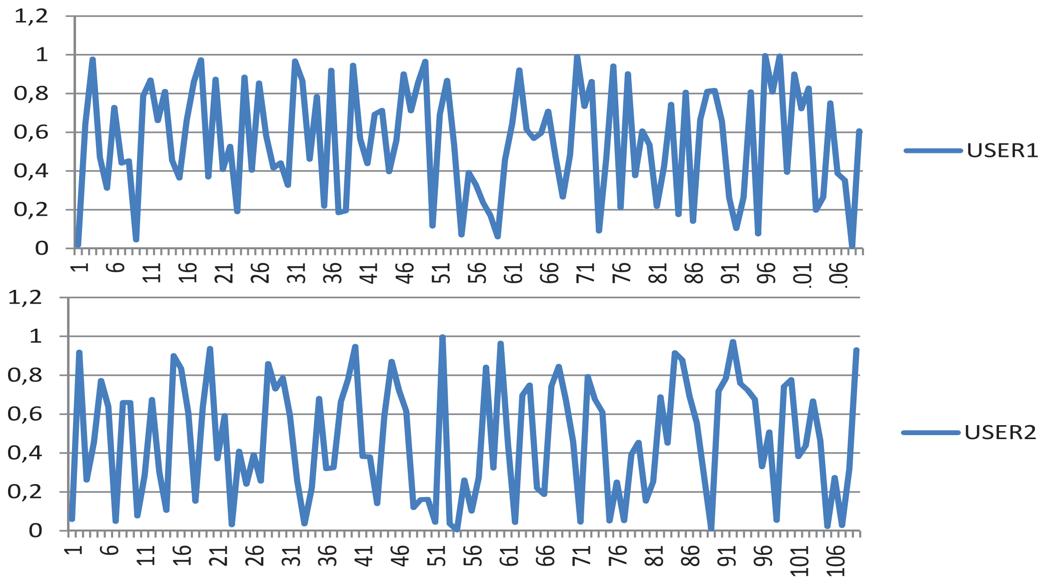


- Transmit: $(\overbrace{\text{Inverse-channel}}^{\text{Feedback}} \times \text{Message}) \Rightarrow$ separates users' messages
  - ⋆ Channel × Inverse-channel × Message → Message OK
- BUT, channel changes: Feedback can be imperfect, limited and delayed
  - ⋆ Channel × Approximately-inverse-channel × Message → ®‡♠∅⅂ ≗

STEP 1: COMMUNICATION OF DURATION n (n IS LARGE)

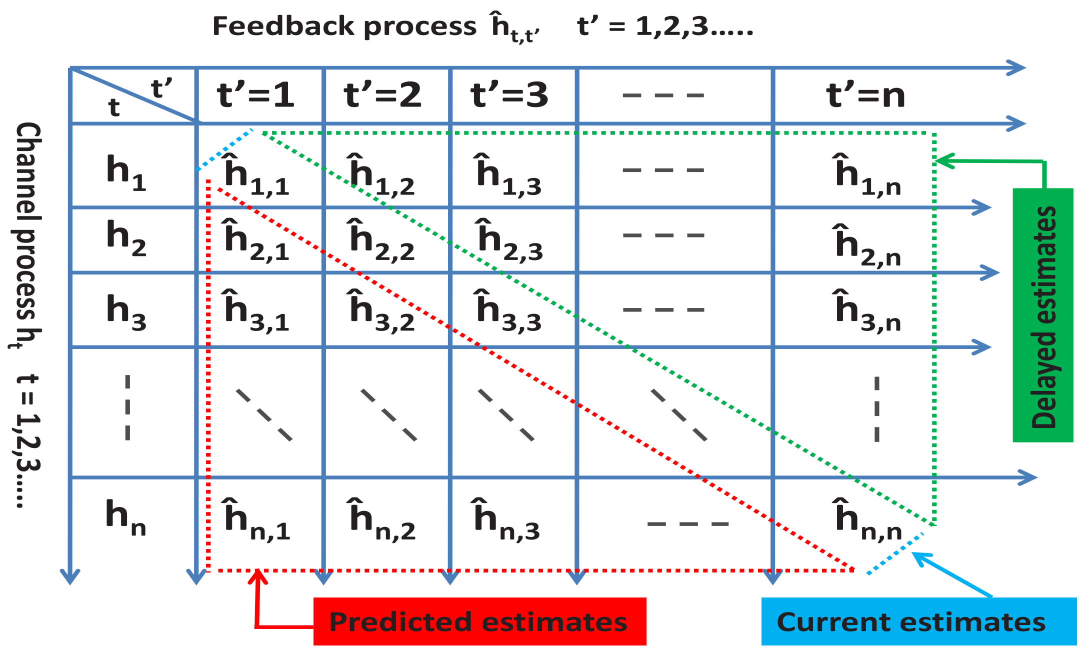
STEP 2: COMMUNICATION ENCOUNTERS AN ARBITRARY CHANNEL PROCESS

$$\begin{array}{llllll} \text{user } 1: & \boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3 & \cdots & \boldsymbol{h}_n \\ \text{user } 2: & \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 & \cdots & \boldsymbol{g}_n \end{array}$$

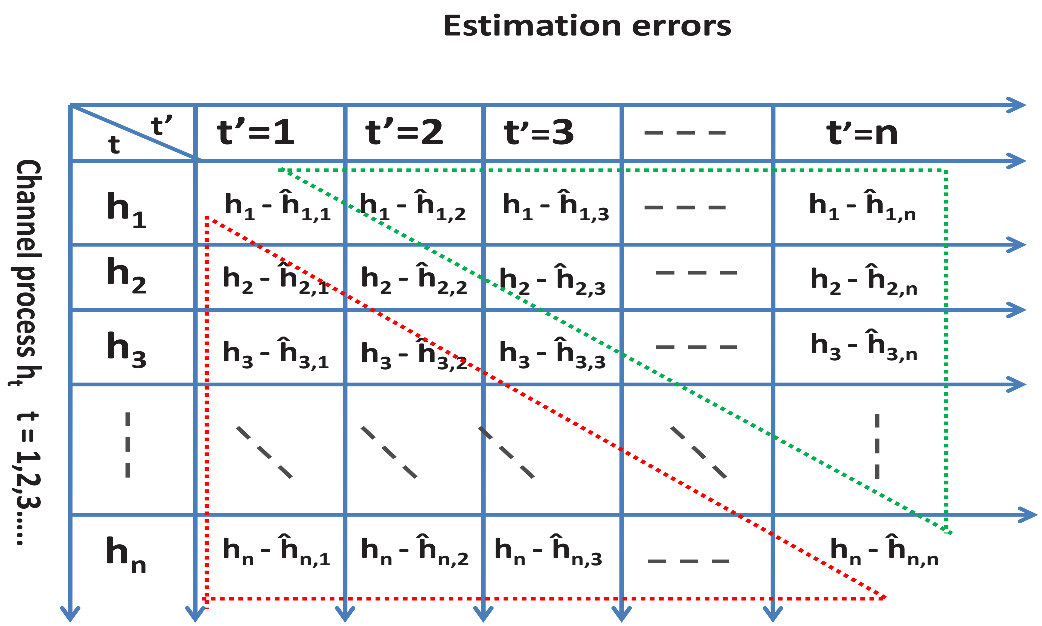


Step 3: An arbitrary feedback process

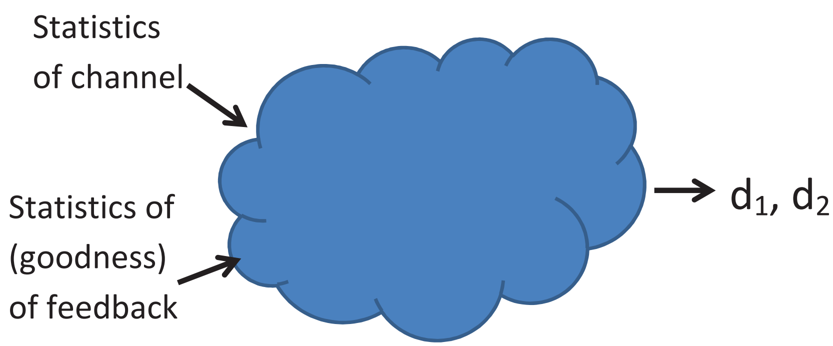
What do we know - at any time t'– about any channel $\mathbf{h_t}$?

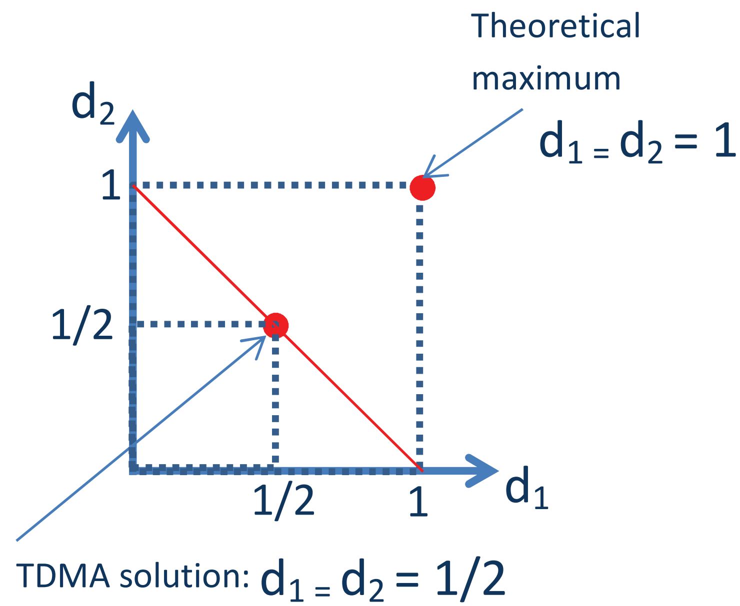


Step 4: A 'primitive' measure of feedback 'goodness'



Recall: performance in degrees-of-freedom (DoF)





$$d_i = \lim_{P \to \infty} \frac{R_i}{\log P}, \;\; i = 1, 2$$

- (R_1, R_2): achievable rate pair $\quad R_i \approx d_i \log P$

Performance/Feedback limits (Chen-Elia Trans-IT Dec. 2013)

Theorem: *(Chen-Elia 2013) The DoF region*

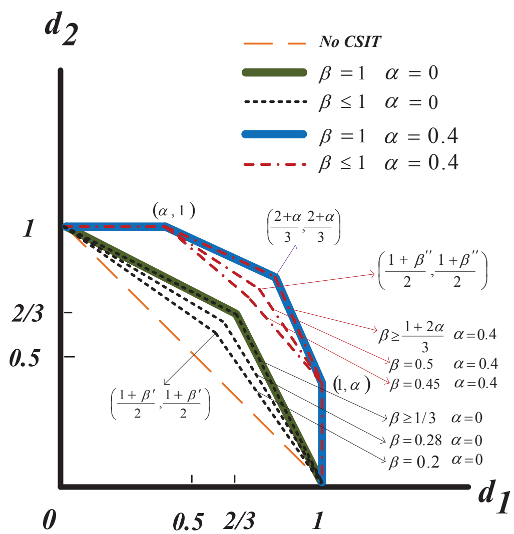
$$d_1 \leq 1, \quad d_2 \leq 1$$

$$2d_1 + d_2 \leq 2 + \bar{\alpha}^{(1)}$$

$$2d_2 + d_1 \leq 2 + \bar{\alpha}^{(2)}$$

$$d_1 + d_2 \leq \frac{1}{2}(2 + \bar{\beta}^{(1)} + \bar{\beta}^{(2)})$$

is achievable and is optimal for ... sufficiently good CSIT (To explain).



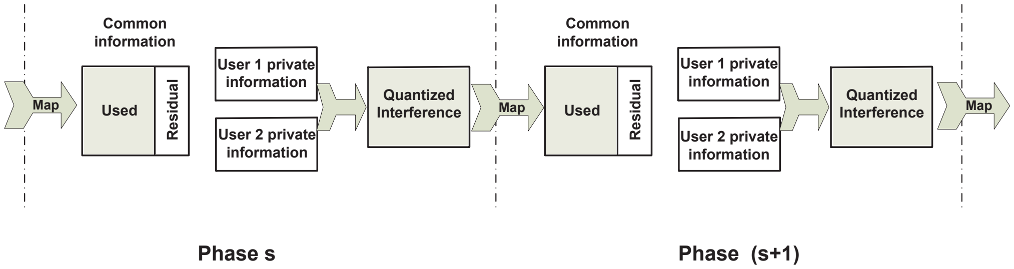
$$\bar{\alpha} \triangleq -\frac{1}{n}\sum_{t=1}^{n} \lim_{P\to\infty} \frac{\log \mathbb{E}[||\boldsymbol{h}_t - \hat{\boldsymbol{h}}_{t,t}||^2]}{\log P}$$

$$\bar{\beta} \triangleq -\frac{1}{n}\sum_{t=1}^{n} \lim_{P\to\infty} \frac{\log \mathbb{E}[||\boldsymbol{h}_t - \hat{\boldsymbol{h}}_{t,t+\eta}||^2]}{\log P};$$

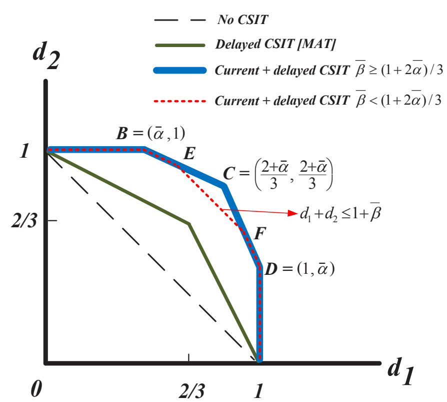
High complexity Block Markov schemes

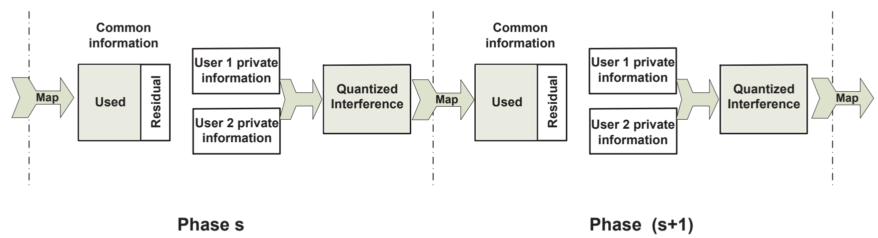
Universal encoding-decoding scheme

Schemes exploit imprecise, delayed or premature feedback



- High complexity and delay for achieving optimal DoF
  - ⋆ Complexity a function of $\bar{\alpha}, \bar{\beta}$
  - ⋆ Hint!! Complexity can dramatically reduce for specific cases of fading statistics, feedback statistics, feedback periodicity..





TODO: Reduce prohibitive complexity of encoding with imperfect and delayed feedback

Concluding remarks

- In general MIMO settings, dimensionality should be respected but not cause paralyzing fear
  - ⋆ Algorithms are much faster now
- Proper analysis can result in substantial insight
  - ⋆ Can help proper planning of network resources
- Complexity is a sizable parameter that is often left unattended
- In multiuser communications, many theoretical promises remain unfulfilled due to prohibitive algorithmic complexity
- New tools allow for insightful analysis of fundamental performance-complexity questions in many different areas
- In an area like telecommunications, such tools need be stochastic
- Discrete mathematics help. Feedback helps.
- Complexity is here to stay

We thank you

Bibliography

Bibliography[1]

SOME UNDERLYING MATHEMATICS

• A. A. Albert, *Structure of Algebras*, Coll. Publ., Vol. 24, Amer. Math. Soc., Providence, R. I., 1961.

• W. Scharlau, *Quadratic & Hermitian Forms (Grundlehren Der Mathematischen Wissenchaften Series, Vol 270)* , Springer-Verlag, 1984.

• R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge, UK: Cambridge University Press, 1985.

• Paul J. McCarthy, *Algebraic extensions of fields*, New York: Dover Publications, 1991.

• D. A. *Marcus, Number Fields* (Universitext), Springer Verlag, New York, 1977.

• Robert L. Long, *Algebraic Number Theory*, Marcel Dekker, New York, 1977.

• S. Lang, *Algebraic Number Theory*, New York: Springer-Verlag: Graduate texts in mathematics, 1994.

• Paulo Ribenboim, *Classical Theory of Algebraic Numbers*, New York: Springer-Verlag: Universitext, 2001.

SOME FUNDAMENTALS

• T. Richardson and R. Urbanke, Modern Coding Theory. Cambridge University Press, 2008.

• David Tse and Pramod Viswanath, Fundamentals of wireless communication, Cambridge University Press, New York, NY, 2005.

• L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *Trans-IT*, vol. 49, no. 5, pp. 1073-1096, May 2003.

• I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, Nov.-Dec. 1999.

• E. Biglieri, J. Proakis, and S. Shamai (Shitz), "Fading channels: Information-theoretic and communications aspects," *Trans-IT*, vol. 44, no. 6, pp. 2619-2692, Oct. 1998.

• A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, 2d ed. New York: Springer-Verlag, 1998.
• L.H. Ozarow, S. Shamai, and A.D. Wyner, "Information Theoretic Considerations for Cellular Mobile Radio," *IEEE Trans. Veh. Technol.*, 1994, vol. 43, no. 2, pp. 359-378.

ENCODING COMPLEXITY

• P. Elia and J. Jaldén, "Fundamental Rate-Reliability-Complexity Limits in Outage Limited MIMO Communications, ISIT-2010.
• P. Koteshwar and B.S. Rajan, "Reduced ML-Decoding Complexity, Full-Rate STBCs for 4 Transmit Antenna Systems," ISIT-2010.
• W. Zhang, L. Shi, and X.-G. Xia, "A Systematic Design of Space-Time Block Codes with Reduced-Complexity Partial Interference Cancelation Group Decoding," ISIT-2010.
• P. Koteshwar and B.Sundar Rajan, "Reduced ML-Decoding Complexity, Full-Rate STBCs for 4 Transmit Antenna Systems," ISIT-2010.
• B. Smith, M. Ardakani, W. Yu, and F. R. Kschischang, "Design of Low-Density Parity-Check Codes with Optimized Complexity-Rate Tradeoff," accepted for publication *in IEEE Trans. on Commun.,* June 2009.
• H. Pfister and I. Sason, "Accumulate-repeat-accumulate codes: Capacity-achieving ensembles of systematic codes for the erasure channel with bounded complexity," *Trans-IT*, vol. 53, no. 6, pp. 2088 - 2115, June 2007.
• I. Sason and G. Wiechman, "On achievable rates and complexity of LDPC codes over parallel channels: bounds and applications," *Trans-IT*, vol. 53, no. 2, pp. 580 - 598, February 2007.
• M. Ardakani, W. Yu, B. Smith, and F. R. Kschischang, "Complexity-Optimized Low-Density Parity-Check Codes," *in Proc. of the 43rd Annual Allerton Conf. on Commun., Control and Computing*, Sep. 2005.

• H. Pfister, I. Sason and R. Urbanke, "Capacity-achieving ensembles for the binary erasure channel with bounded complexity," *Trans-IT*, vol. 51, no. 7, pp. 2352 - 2379, July 2005.
• I. Sason and R. Urbanke, "Complexity versus performance of capacity-achieving irregular repeat-accumulate codes on the erasure channel," *Trans-IT*, vol. 50, no. 6, pp. 1247 - 1256, June 2004.
• H. Pfister, I. Sason and R. Urbanke, "Capacity-achieving ensembles for the binary erasure channel with bounded complexity," ISIT-2004.
• A. Khandekar and R. J. McEliece, "On the complexity of reliable communication on the erasure channel," ISIT-2001.
• S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE Journal on Selected Areas in Communications,* vol. 16, pp. 1451-1458, Oct. 1998.

HIGH PERFORMANCE COOPERATIVE PROTOCOLS

• S. O. Gharan, A. Bayesteh, and A. K. Khandani, "Asymptotic analysis of amplify and forward relaying in a parallel MIMO relay network," http://arxiv.org/abs/cs/0703151
• P. Elia, K. Vinodh , M. Anand, and P. V. Kumar, "D-MG Tradeoff and Optimal Codes for a Class of AF and DF Cooperative Communication Protocols," *Trans-IT*, vol. 55, no. 7, July 2009.
• D. Gunduz, A. Goldsmith, and H. V. Poor, "Diversity-multiplexing tradeoffs in MIMO relay channels," *in Proc. IEEE Global Communications Conf. (Globecom)*, New Orleans, LA, November 2008.
• S. Pawar, A. S. Avestimehr and D. N. C. Tse, "Diversity-Multiplexing Tradeoff of the Half-Duplex Relay Channel", in *Proc. Allerton Conf. Communication, Control and Computing*, 2008.

• A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "Approximate Capacity of Gaussian Relay Networks" *Proc. IEEE Intl. Symp. Inform. Theory*, Toronto, July 6–11, 2008.
• M. Yuksel and E. Erkip, "Multiple-Antenna Cooperative Wireless Systems: A Diversity-Multiplexing Tradeoff Perspective", *Trans-IT*, vol 53, no.10, pp 3371-3393, Oct. 2007.
• S. Yang and J.-C. Belfiore, "On slotted amplify-and-forward cooperative diversity schemes," ISIT-2006.
• S. Yang and J.-C. Belfiore, "Towards the Optimal Amplify-and-Forward Cooperative Diversity Scheme," *Trans-IT*, vol. 53, no. 9, pp. 3114-3126, Sept. 2007.
• N. Prasad and M. K. Varanasi, "High performance static and dynamic cooperative communication protocols for the half duplex fading relay channel," in Proc. IEEE Globecom, San Francisco, CA, 2006.
• A. S. Avestimehr and D. N. C. Tse, "Outage-Optimal Cooperative Relaying", MSRI workshop, UC Berkeley, Apr. 2006.
• R. Knopp, "Two-Way Radio Networks with a Star Topology," *IEEE International Zurich Seminar on Communications*, Zurich, Feb. 2006, pp 154-157.
• K. Azarian, H. El Gamal, and P. Schniter, "On the Achievable Diversity-Multiplexing Tradeoff in Half-Duplex Cooperative Channels," *Trans-IT*, vol. 51, no. 12, pp. 4152–4172, Dec. 2005.
• M. Katz and S. Shamai, "Transmitting to colocated users in wireless ad hoc and sensor networks," *Trans-IT*, vol. 51, no. 10, pp. 3540-3563, Oct. 2005.
• G. Kramer, M. Gastpar, and P. Gupta, "Cooperative Strategies and Capacity Theorems for Relay Networks," *Trans-IT*, vol. 51, no. 9, pp. 3037-3063, September 2005.
• P. Mitran, H. Ochiai, and V. Tarokh, "Space-time diversity enhancements using collaborative communications," *Trans-IT*, vol. 51, no. 6, pp. 2041-2057, Jun. 2005.

• H.-F. Lu, C. Hollanti, R. Vehkalahti, and J. Lahtonen, "DMT optimal codes constructions for multiple-access MIMO channel," *Trans-IT*, vol. 57, no. 6, pp. 35943617, June 2011.
• H.-F. Lu, "Diversity-Multiplexing Tradeoff in MIMO Gaussian Interference Channels," ISIT 2010.
• S. Karmakar and M. Varanasi, "The diversity-multiplexing tradeoff of the symmetric MIMO half-duplex relay channel," ISIT 2010.
• S. Karmakar and M. Varanasi, "The diversity-multiplexing tradeoff of the MIMO Z interference channel," ISIT 2010.
• P. Elia and J. Jaldén, "General DMT optimality of LR-aided linear MIMO-MAC transceivers with worst-case complexity at most linear in sum-rate, ITW-2010.
• S. Kannan, B. P. Sasidharan, and P Vijay Kumar, "DMT of Multi-hop Cooperative Networks, ITW-2010.
• A. Raja and P. Viswanath, "Diversity-Multiplexing Tradeoff of the Two-User Interference Channel," ISIT 2009.
• C. Akcaba and H. Bölcskei, "On the Achievable Diversity-Multiplexing Tradeoff in Interference Channels," ISIT 2009.
• H. Ebrahimzad and A. K. Khandani, "On Diversity-Multiplexing Tradeoff of the Interference Channel," ISIT 2009.
• H.-F. Lu and C. Hollanti, "Diversity-multiplexing tradeoff-optimal code constructions for symmetric MIMO multiple-access channels," ISIT-2009.
• T. T. Kim and H. Vincent Poor, "Diversity Multiplexing Tradeoff in Adaptive Two-way Relaying," 2009. [Online]. Available: http://www.princeton.edu/thanhkim/relay twoway fb.pdf
• S. A. Pawar, K.R. Kumar, P. Elia, B.A. Sethuraman and P. V. Kumar, "Space-Time Codes Achieving the DMD Tradeoff of the MIMO-ARQ Channel," *Trans-IT*, vol. 55, no. 7, July

2009.
• P. Elia and P. Vijay Kumar, "Space-Time Codes that are Approximately Universal for the Parallel, Multi-Block and Cooperative DDF Channels," ISIT-2009.
• R. Vaze and R. W. Heath Jr, "On the Capacity and Diversity- Multiplexing Tradeoff of the Two-Way Relay Channel," 2008. [Online]. Available: arXiv:0810.3900
• R. N. Krishnakumar, N. Naveen, K. Sreeram and P. Vijay Kumar, "Diversity Multiplexing Tradeoff of Asynchronous Cooperative Relay Networks," in *Proc. Allerton Conf. Communication, Control and Computing*, 2008.
• D. Gunduz, A. Goldsmith, and H. Poor, "MIMO Two-way Relay Channel: Diversity-Multiplexing Tradeoff Analysis," in Proc. IEEE Asilomar Conf. on Signals, Systems and Computers, Oct. 2008, pp. 1474-1478.
• H.-F. Lu, "Constructions of multi block space-time coding schemes that achieve the diversity multiplexing tradeoff," *Trans-IT*, vol. 54, no. 8, pp. 3790-3796, Aug. 2008.
• P. Mitran, "The Diversity-Multiplexing Tradeoff for Independent Parallel MIMO Channels," ISIT-2008.
• Y.-H. Nam and H. E. Gamal, "On the optimality of lattice coding and decoding in multiple access channels," ISIT-2007.
• S. Wei, "Diversity-multiplexing tradeoff of asynchronous cooperative diversity in wireless networks," *Trans-IT*, vol. 53, no. 11, pp. 4150-4172, Nov. 2007.
• P. Coronel and H. Bölcskei, "Diversity-multiplexing tradeoff in selective fading MIMO channels," ISIT-2007.
• A. Medles and D. T. M. Slock, "Optimal diversity vs. multiplexing tradeoff for frequency selective MIMO channels," ISIT-2005.
• D. N. C. Tse, P. Viswanath, and L. Zheng, "Diversity-multiplexing tradeoff in multiple-access channels," *Trans-IT*, vol. 50, no. 9, pp. 1859-1874, Sep. 2004.

• H. El Gamal, G. Caire, and M. O. Damen, "The MIMO ARQ channel: Diversity-multiplexing-delay tradeoff," *Trans-IT*, vol. 50, no. 6, pp. 968–985, Jun. 2004.

• L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *Trans-IT*, vol. 49, no. 5, pp. 1073–1096, May 2003.

• Y. Jiang and M. K. Varanasi, "The RF-chain limited MIMO system part I: optimum diversity-multiplexing tradeoff," *IEEE Trans. Wireless Commun., vol. 8, no. 10, pp. 52385247, Oct. 2009.*

LOW COMPLEXITY COOPERATIVE PROTOCOLS

• J. P. K. Chu, R. S. Adve, and A. W. Eckford, "Relay selection for low complexity coded cooperation," Globecom 2007.

• A. Bletsas, A. Khisti, and M. Z. Win, "Low complexity virtual antenna arrays using cooperative relay selection," *in Proceedings of the 2006 international conference on Wireless communications and mobile computing,* 2006.

• A. Bletsas, A. Khisti, D. Reed, and A. Lippman, "A simple cooperative diversity method based on network path selection," *IEEE Journal on Selected Areas of Communication,* vol. 24, pp. 659-672, March 2006.

• J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *Trans-IT*, vol. 50, pp. 3062-3080, Dec. 2004.

• E. Zimmermann, P. Herhold, and G. Fettweis, "On the Performance of Cooperative Diversity Protocols in Practical Wireless Systems," *in Proc. 58th IEEE Vehic. Technol. Conf.*, Orlando, FL, Oct. 2003.

HIGH PERFORMANCE DMT-OPTIMAL DESIGNS FOR COOPERATIVE COMMUNICATIONS

- P. Elia, K. Vinodh , M. Anand, and P. V. Kumar, "D-MG Tradeoff and Optimal Codes for a Class of AF and DF Cooperative Communication Protocols," *Trans-IT*, vol. 55, no. 7, July 2009.
- K. Raj Kumar and G. Caire, "Coding and Decoding for the Dynamic Decode and Forward Relay Protocol," *Trans-IT*, vol. 55, no. 7, July 2009.
- P. Elia and P. Vijay Kumar, "Space-Time Codes that are Approximately Universal for the Parallel, Multi-Block and Cooperative DDF Channels," ISIT-2009.
- P. Elia, S. Kittipiyakul and T. Javidi, "Cooperative Diversity Schemes for Asynchronous Wireless Networks," *Springer-Verlag Journal of Personal Communications, Special Issue on Cooperative Wireless Systems*, vol. 43, no. 1, October 2007.
- S. Yang and J.-C. Belfiore, "Optimal space-time codes for the MIMO Amplify-and-Forward cooperative channel", *Trans-IT*, vol. 53, no. 2, pp. 647-663, Feb. 2007.

LOW COMPLEXITY DESIGNS FOR COOPERATIVE COMMUNICATIONS

- G. S. Rajan and B. S. Rajan, "Multi-group ML Decodable Collocated and Distributed Space Time Block Codes," *Trans-IT*. July 2010
- Harshan J. and B. Sundar Rajan, "High-Rate, Single-Symbol ML Decodable Precoded DSTBCs for Cooperative Networks," *Trans-IT* May 2009.
- Y. Jing and H. Jafarkhani, "Distributed differential space-time coding in wireless relay networks," *IEEE Trans. on Communications*, vol. 56, pp. 1092-1100, July 2008.

• Z. Li and X.-G. Xia, "A Simple Alamouti SpaceTime Transmission Scheme for Asynchronous Cooperative Systems", *IEEE Signal Processing Letters*, vol. 14, pp. 804, Nov 2007.

• G. Susinder Rajan and B.Sundar Rajan, "Algebraic distributed space-time codes with low ML decoding complexity," ISIT 2007.

• P. Elia, F. Oggier, and P. Vijay Kumar, "Asymptotically optimal cooperative wireless networks with reduced signaling complexity," *IEEE J. Select. Areas Commun.,* vol. 25, pp. 258-267, Feb. 2007.

• T. Kiran and B. S. Rajan, "Distributed space-time codes with reduced decoding complexity," ISIT 2006.

• A. W. Eckford, J. P. K. Chu, and R. Adve, "Low complexity cooperative coding for sensor networks using rateless and LDGM codes," ICC 2006.

• C. Ö. Oyman, J. N. Laneman and S. Sandhu, "Multihop Relaying for Broadband Wireless Mesh Networks: From Theory to Practice," *IEEE Commun. Mag.*, vol. 45, no. 11, pp. 116–122, Nov. 2007.

• R. U. Nabar, H. Bölcskei, and F. W. Kneubühler, "Fading Relay Channels: Performance Limits and Space-Time Signal Design," *IEEE J. Select. Areas Commun.*, vol. 22, no. 6, pp. 1099–1109, Aug. 2004.

• J. N. Laneman and G. W. Wornell, "Distributed Space–Time-Coded Protocols for Exploiting Cooperative Diversity in Wireless Networks," *Trans-IT*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.

HIGH PERFORMANCE, UNIFIED DMT-OPTIMAL MIMO DESIGNS

• K. Raj Kumar and G. Caire, "Space-time codes from structured lattices," *Trans-IT*, vol. 55, no. 2, pp. 547-556, Feb. 2009.

• C. Hollanti, J. Lahtonen, K. Ranto, and R. Vehkalahti, "On the densest MIMO lattices from cyclic division algebras," 2006, submitted to *Trans-IT*, available on arXiv:cs/0703052v1 [cs.IT].

• P. Elia, B. A. Sethuraman, and P. Vijay Kumar, "Perfect space-time codes for any number of transmit antennas," *Trans-IT*, vol. 53, no. 11, pp. 3853-3868, Nov. 2007.

• F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space-time block codes," *Trans-IT*, vol. 52, no. 9, pp. 3885-3902, Sept. 2006.

• S. Tavildar and P. Viswanath, "Approximately universal codes over slow fading channels," *Trans-IT*, vol. 52, no. 7, pp. 3233-3258, July 2006.

• H. El Gamal, G. Caire, and M. O. Damen, "Lattice coding and decoding achieve the optimal diversity-multiplexing tradeoff of MIMO channels," *Trans-IT*, vol. 50, no. 6, pp. 968-985, June 2004.

Lattices and Lattice Reduction

●H. Yao and G. W. Wornell, "Lattice-Reduction-Aided Detectors for MIMO Communication Systems," in *Proc. GLOBECOM*, Nov. 2002.

•C. Windpassinger and R. F. H. Fischer, "Low-Complexity Near-Maximum-Likelihood Detection and Precoding for MIMO Systems using Lattice Reduction," in *Proc. ITW*, Mar. 2003.

•J. Jaldén, D. Seethaler, and G. Matz, "Worst- and average-case complexity of LLL lattice reduction in MIMO wireless systems", ICASSP 2008

• D. Wubben, D. Seethaler, J. Jaldén, and G. Matz, "Lattice Reduction: A survey with applications,??", *in IEEE Signal Processing Magazine*, vol. 28, no. 3, pp. 70 - 91, May 2011.

Sphere Decoding

•U. Fincke and M. Pohst, "Improved Methods for Calculating Vectors of Short Length in a Lattice, Including a Complexity Analysis", *Mathematics of Computation*, Apr. 1985
•W. H. Mow., "Maximum Likelihood Sequence Estimation from the Lattice Viewpoint", *Trans. IT*, Sep. 1994
•E. Viterbo, E. Biglieri., "A universal decoding algorithm for lattice codes". Proc. GRETSI, Juanles- Pins, France, Sep. 1993
•E. Viterbo, J. Boutros. "A universal lattice code decoder for fading channels", *Trans. IT*, July 1999
•E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices", *Trans. IT*, Aug. 2002
•M. O. Damen, H. El Gamal, and G. Caire, "On maximum-likelihood detection and the search for the closest lattice point", *Trans. IT*, Oct. 2003
•A. D. Murugan, H. El Gamal, M. O. Damen, and G. Caire, "A unified framework for tree search decoding: rediscovering the sequential decoder", *Trans. IT*, Jan. 2006

Sphere Decoding - Faster Near Optimal Variants

•L.G. Barbero and J.S. Thompson, "A Fixed-Complexity MIMO Detector Based on the Complex Sphere Decoder," SPAWC 2006
•J. Jaldén and L.G. Barbero, B. Ottersten, and J.S. Thompson, "The Error Probability of the Fixed-Complexity Sphere Decoder," *Trans. SP*, July 2009
•S.D. Howard, S. Sirianunpiboon, and A.R. Calderbank, "Low complexity essentially maximum likelihood decoding of perfect space-time block codes," ICASSP 2009
•L.P. Natarajan and B.S. Rajan, "An Adaptive Conditional Zero-Forcing Decoder With Full-Diversity, Least Complexity and Essentially-ML Performance for STBCs," *Trans. SP*, Jan. 2013

Interference Alignment

●L. Ma, T. Xu, G. Sternberg, “Computational Complexity of Interference Alignment for Symmetric MIMO Networks”, IEEE Communications Letters, Vol. 17, No. 12, Dec 2013.
●M. Razaviyayn, M. Sanjabi, and Z. Luo, Linear transceiver design for interference alignment: complexity and computation, IEEE Trans. Inf. Theory, vol. 58, pp. 28962910, May 2012

Complexity and feedback in multiuser communications

●M. A. Maddah-Ali and D. N. C. Tse, “Completely stale transmitter channel state information is still very useful,” IEEE Trans. Inf. Theory, vol. 58, no. 7, pp. 4418 4431, Jul. 2012.
●R. Tandon, S. A. Jafar, S. Shamai, and H. V. Poor, “On the synergistic benefits of alternating CSIT for the MISO broadcast channel,” IEEE Trans. Inf. Theory, vol. 59, no. 7, pp. 4106 4128, Jul. 2013.
●X. Yi, S. Yang, D. Gesbert, and M. Kobayashi, “The degrees of freedom region of temporally-correlated MIMO networks with delayed CSIT,” IEEE Trans. Inf. Theory, vol. 60, no. 1, pp. 494 514, Jan. 2014.
●J. Chen and P. Elia, “Toward the performance vs. feedback tradeoff for the two-user MISO broadcast channel,” IEEE Trans. Inf. Theory, vol. 59, no. 12, pp. 8336 8356, Dec. 2013.

Recent publications on complexity, coding, and massive MIMO

●G. Abhinav and B. Sundar Rajan, ”Interference Alignment with Diversity for the 2x2 X-Network With Four Antennas,” To appear in IEEE Transactions on Information Theory.
●G.R. Jithamithra and B. Sundar Rajan, ”Construction of Block Orthogonal STBCs and Reducing their Sphere Decoding Complexity,” To appear in IEEE Transactions on Wireless Communications.

•K. Pavan Srinath and B. Sundar Rajan, ”Fast-Decodable MIDO Codes with large Coding Gain,” IEEE Transactions on Information Theory, Vol. 60, No.2, pp. 992-1017, Feb. 2014.
•G.R. Jithamithra and B. Sundar Rajan, ”Minimizing the Complexity of Fast Sphere Decoding of STBCs,” IEEE Transactions on Wireless Communications, Vol.12, No.12, pp. 6142-6153, 2013.
•L. P. Natarajan and B. Sundar Rajan, ”Asymptotically-Good, Multigroup Decodable Space-Time Block Codes,” IEEE Transactions on Wireless Communications, Vol.12, No.10, pp. 5035-5047, 2013.
•L. P. Natarajan, K. Pavan Srinath and B. Sundar Rajan, ”On the Sphere Decoding Complexity of High Rate Multigroup Decodable STBCs in Asymmetric MIMO Systems,” IEEE Transactions on Information Theory, Vol.59, No. 9, September 2013, pp.5959-5965.
•T. Datta, N. Ashok Kumar, A. Chockalingam, and B. Sundar Rajan, “A Novel Monte Carlo Sampling Based Receiver for Large-Scale Uplink Multiuser MIMO Systems,” IEEE Transactions on Vehicular Technology, Vol.62, No.7, pp.3019-3038, 2013.
•P. Som, T. Datta, N. Srinidhi, A. Chockalingam, and B. Sundar Rajan, Low-Complexity Detection in Large-Dimension MIMO-ISI Channels Using Graphical Models, IEEE Jl. Sel. Topics in Signal Processing (JSTSP): Special issue on Soft Detection for Wireless Transmission, vol. 5, no. 8, pp. 1497-1511, December 2011.
•P. Raviteja, T Lakshmi Narasimhan and A. Chockalingam, Detection in Large-Scale Multiuser SM-MIMO Systems: Algorithms and Performance, accepted in IEEE VTC-Spring, 2014.
•T. Lakshmi Narasimhan and A. Chockalingam, Detection and Decoding in Large-Scale MIMO Systems: A Non-Binary Belief Propagation Approach, VTC-Spring, 2014.
•S. Nagaraja, O. Dabeer, and A. Chockalingam, Large-MIMO Receiver based on Linear Regression of MMSE Residual, VTC-Fall, 2013.

•O. Dabeer, S. Nagaraja, and A. Chockalingam, Boosting MMSE receivers using AdaBoost, Proc. VTC-Fall, 2013.
•T. Datta, A. Chockalingam, and E. Viterbo, Gaussian Sampling Based Lattice Decoding, ISIT 2013.
•K. Singhal, T. Datta, and A. Chockalingam, Lattice Reduction Aided Detection in Large-MIMO Systems Proc. SPAWC 2013.
•T. L. Narasimhan, A. Chockalingam, and B. Sundar Rajan, Factor Graph Based Joint Detection/Decoding for LDPC Coded Large-MIMO Systems, Proc. IEEE VTC-Spring, 2012.
•T. Datta, N. Ashok Kumar, A. Chockalingam, and B. Sundar Rajan, A Novel MCMC Algorithm for Near-Optimal Detection in Large-Scale Uplink Mulituser MIMO Systems, ITA 2012.
•T. Datta, N. Srinidhi, A. Chockalingam, and B. Sundar Rajan, Low-Complexity Near-Optimal Signal Detection in Underdetermined Large-MIMO Systems, Proc. NCC 2012.
•A. Kumar, S. Chandrasekaran, A. Chockalingam, and B. Sundar Rajan, Near-Optimal Large-MIMO Detection Using Randomized MCMC and Randomized Search Algorithms, ICC 2011.
•N. Srinidhi, Tanumay Datta, A. Chockalingam, and B. Sundar Rajan, Layered Tabu Search Algorithm for Large-MIMO Detection and a Lower Bound on ML Performance, GLOBECOM 2010

Communication complexity in green radios

•P. Grover, "Bounds on the Tradeoff between decoding complexity and rate for codes on graphs," Information Theory Workshop 2007
•P. Grover and A. Sahai, "Time-division multiplexing for Green Broadcasting," (extended version) ISIT 2009.

•P. Grover, K. A. Woyach and A. Sahai, "Towards a communication-theoretic understanding of system-level power consumption," IEEE Journal of Selected Areas in Communication (JSAC) Special Issue on Energy-Efficient Wireless Communications.
•P. Grover and A. Sahai, "Fundamental bounds on the interconnect complexity of decoder implementations," CISS 2011.
•P. Grover, A. Goldsmith and A. Sahai, "Fundamental limits on complexity and power consumption in coded communication," ISIT 2012.
•K. Ganesan and P. Grover, "Decoding power can increase significantly with increased wire-lengths and code performance: empirical results," Design Automation Conference (DAC) 2011, Work In Progress Session.
•K. Ganesan, P. Grover, and J. Rabaey, "The power cost of over-designing codes," 2011 IEEE Workshop on Signal Processing Systems, Beirut, Lebanon.
•K. Ganesan, Y. Wen, P. Grover, A. Goldsmith and J. Rabaey, "Mixing theory and experiments to choose the greenest code-decoder pair," Globecom 2012.