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Confidence intervals for the Shapley–Shubik power index in Markovian games

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Abstract We consider simple Markovian games, in which several states succeed each other over time, following an exogenous discrete-time Markov chain. In each state, a different simple static game is played by the same set of players. We investigate the approximation of the Shapley–Shubik power index in simple Markovian games (SSM). We prove that an exponential number of queries on coalition values is necessary for any deterministic algorithm even to approximate SSM with polynomial accuracy. Motivated by this, we propose and study three randomized approaches to compute a confidence interval for SSM. They rest upon two different assumptions, static and dynamic, about the process through which the estimator agent learns the coalition values. Such approaches can also be utilized to compute confidence intervals for the Shapley value in any Markovian game. The proposed methods require a number of queries which is polynomial in the number of players in order to achieve a polynomial accuracy.

Keywords Shapley–Shubik power index · Shapley value · confidence interval · Markovian game

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1 Introduction

Cooperative game theory is a powerful tool to analyse, predict, and influence the interactions among several players capable to stipulate deals and form subcoalitions in order to pursue a common interest. Under the assumption that the grand coalition, comprising all the players, is formed, it is a delicate issue to share the payoff earned by the grand coalition among its participants.

Introduced by Lloyd S. Shapley in his seminal paper [30], the Shapley value is one of the best known payoff allocation rules in a cooperative game with transferable utility (TU). It is the only allocation procedure fulfilling three reasonable conditions of symmetry, additivity and dummy player compensation (see [30] for details), under a superadditive assumption on the coalition values. The significance of the Shapley value is witnessed by the breadth of its applications, spanning from pure economics [3] to Internet economics [20, 32, 7], politics [4], and telecommunications [17].

The concept of Shapley value is particularly meaningful also when applied to simple games [31], in which the coalition values are binary. They model winning/losing scenarios. In this case, the Shapley value is commonly referred to as the Shapley–Shubik power index. A specific instance of simple games are weighted voting games, in which each player possesses a different amount of resources and a coalition is effective, i.e. its value is 1, whenever the sum of the resources shared by its participants is higher than a certain quota; otherwise, its value is 0. The Shapley–Shubik index in weighted voting games proves to be particularly suitable to assess *a priori* the power of the members of a legislation committee, and has many applications to politics (see [34] for an overview). Our results on simple games, except the ones in Sect. 4, also apply to weighted voting games.

The computation of the Shapley value for each player $j = 1, \dots, P$ involves the assessment of the increment of the value of a coalition brought on by the presence of player j, over all 2^{P-1} possible coalitions. Hence, it is clear that the complexity of the Shapley value in the number of players P is a crucial issue. Mann and Shapley himself [21] were the first to suggest to adopt a Monte–Carlo procedure to approximate the Shapley–Shubik index. They first proposed a very simple algorithm, randomly generating a succession of players' permutations and evaluate the incremental value of player j with respect to the coalition formed by its preceding players in each permutation. The Shapley–Shubik index is approximated as the average of such increments. Then they empirically showed that the "cycling scheme" described below is characterized by a smaller variance. First, a target player is singled out, and the remaining players are placed in a random order. Then, this order is put through all of its cyclic permutations, and the target player is inserted in each position in each permutation. Thus, $P(P-1)$ permutations are generated, and for each of them the incremental value of player j with respect to the coalition formed by its preceding players is assessed. For this cycling approach, deriving a confidence interval for the Shapley–Shubik index seems to be a hard task. Hence, Bachrach *et al.* adopted in [8] the first Monte–Carlo procedure described above to compute a confidence interval. This approximation method, presented for simple games, can be easily generalized to any game.

The bulk of the literature on cooperative games focuses on static games. However, politics or economics is more like a process of continuing negotiation and bargaining. This motivates the introduction of dynamic cooperative game theory (see, e.g., [15], [19]). In this work, we consider that the game is not played one-shot, but rather over an infinite horizon: There exists a finite set of static cooperative games that come one after the other, following a discrete-time homogeneous Markov process. We call this interaction model repeated over time as the Markovian game. Our Markovian game model arises naturally in all situations in which several individuals keep interacting and cooperating over time, and an exogenous Markov process influences the value of each coalition, and consequently also the power of each player within coalitions. A very similar model, but with non-transferable utilities, was considered in [29]. Our model can also be viewed as a particular case of the cooperative Markov decision process described in [5], or in [27], in which the transition probabilities among the states do not depend on the players' actions.

We take into account the average and the discount criterion to compute the payoff earned by each player in the long-run Markovian game. In this article we extend the approach by Bachrach *et al.* in [8] to compute a confidence interval for the Shapley value in Markovian games. In [8], the authors considered a simple static game and proved that any deterministic algorithm which approximates one component of the Banzhaf index with accuracy better than $c/\sqrt{P}$ needs $\Omega(2^P/\sqrt{P})$ queries, where $c > 0$ and P is the number of players. Hence, when P grows large, it is crucial to find a suitable way to approximate the power index with a manageable number of queries. Hence, in [8] a confidence interval for the Banzhaf index and Shapley–Shubik power index in simple games has been developed, based on Hoeffding's inequality. In this article, we assume that the estimator agent knows the transition probabilities among the states. We first show that it is still beneficial to utilize a randomized approach to approximate the Shapley–Shubik index in simple Markovian games (SSM) for a number of players P sufficiently high. Then we propose three methods to compute a confidence interval for the SSM that also apply to the Shapley value of *any* Markovian game. Next we will essentially demonstrate that, asymptotically in the number of steps of the Markov chain and by exploiting the Hoeffding's inequality, the estimator agent does not need to have access to the coalition values in all the states at the same time. Indeed, it suffices for the estimator agent to learn the coalition values in each state along the course of the game to "well" approximate SSM.

Let us make an overview the content of this article. We provide some useful definitions, background results, and motivations of our dynamic model in Sect. 2. In Sect. 3, we motivate the significance of our Markovian model. In Sect. 4, we study the trade-off between complexity and accuracy of deterministic algorithms approximating SSM. An exponential number of queries is necessary for any deterministic algorithm even to approximate SSM with polynomial accuracy. Motivated by this, we propose three different randomized approaches to compute a confidence interval for SSM. Their complexity does *not* even depend on the number of players. Such approaches also hold for the classic Shapley value of any cooperative Markovian game (ShM). In Sect. 5, we provide the expression of our first confidence interval, SCI, which relies on the static assumption that the estimator agent has access to the coalition values in all the states at the same time, even before the Markov process initiates. Although SCI relies on an impractical assumption, it is still a valid benchmark for the performance of the approaches yielding the confidence intervals described in Sects. 6.1 and 6.2, dubbed DCI1 and DCI2 respectively. DCI1 and DCI2 also hold under the more realistic dynamic assumption that the estimator agent learns the value of coalitions along the course of the game. In Sect. 6.1, we propose a straightforward way to optimize the tightness of DCI1. In Sect. 7, we compare the three proposed approaches in terms of tightness of the confidence interval. Finally, in Sect. 8, we provide a trade-off between complexity and accuracy of our randomized algorithm, holding for any cooperative Markovian game.

We remark that the extension of our approaches to the Banzhaf index [10] is straightforward.

Some notation remarks. If $\mathbf{a}$ is a vector, then $\mathbf{a}_i$ is its ith component. If A is a random variable (r.v.), then A_t is its tth realization. Given a set S, $|S|$ is its cardinality. The expression $b^{(s)}$ indicates that the quantity b, standing possibly for the Shapley value, Shapley–Shubik index, coalition value, feasibility region, etc., is related to the static game played in state s. The expression $\Pr(B)$ stands for the probability of event B. The indicator function is written as $\mathbb{I}(.)$. With some abuse of terminology, we will refer to a confidence interval or to the approach utilized to compute it without distinction.

2 Markovian Model and Background results

In this article, we consider cooperative Markovian games with transferable utility (TU). Let P be the number of players and let $\mathscr{P} = \{1,\dots,P\}$ be the grand coalition of all players. We have a finite set of states $S = \{s_1,\dots,s_{|S|}\}$. In state s, each coalition $\Lambda \subseteq \mathscr{P}$ can ensure for itself the value $v^{(s)}(\Lambda)$ that can be shared in any manner among the players under the TU assumption. Hence, in each state $s \in S$, the game $\Psi^{(s)} \equiv (\mathscr{P}, v^{(s)})$ is played. Let $\mathscr{V}^{(s)}(\Lambda)$ be the half-space of all feasible allocations for coalition Λ in the TU game $\Psi^{(s)}$, i.e., the set of real $|\Lambda|$-tuple $\mathbf{a} \in \mathbb{R}^{|\Lambda|}$ such that $\sum_{i=1}^{|\Lambda|} \mathbf{a}_i \le v^{(s)}(\Lambda)$. We suppose that the coalition values are superadditive, i.e.,

$$v^{(s)}(\Lambda_1 \cup \Lambda_2) \ge v^{(s)}(\Lambda_1) + v^{(s)}(\Lambda_2), \quad \forall \Lambda_1, \Lambda_2 \subseteq \mathscr{P},\ \Lambda_1 \cap \Lambda_2 = \emptyset.$$

The succession of the states follows a discrete-time homogeneous Markov chain, whose transition probability matrix is $\mathbf{P}$. Let $\mathbf{x}^{(s)} \in \mathbb{R}^P$ be a payoff allocation among the players in the single stage game $\Psi^{(s)}$. Under the β-discounted criterion, where $\beta \in [0;1)$, the discounted allocation in the Markovian dynamic game Γ_{s_k}, starting from state s_k, can be expressed as

$$\sum_{t=0}^{\infty} \beta^t \mathbb{E}\left(\mathbf{x}^{(S_t)}\right) = \sum_{i=1}^{|S|} \nu_i^{(\beta)}(s_k)\, \mathbf{x}^{(s_i)}$$

where S_t is the state of the Markov chain at time t and $\nu^{(\beta)}(s_k)$ is the kth row of the nonnegative matrix $(\mathbf{I} - \beta \mathbf{P})^{-1}$. We stress that β can be interpreted as the probability that the game terminates, at any step. Under the average criterion, if the transition probability matrix $\mathbf{P}$ is *irreducible*, then the allocation in the long-run game Γ_{s_k} can be written as

$$\limsup_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}\left(\mathbf{x}^{(S_t)}\right) = \sum_{i=1}^{|S|} \pi_i \, \mathbf{x}^{(s_i)}$$

where π is the stationary distribution of the matrix $\mathbf{P}$.

We define $\mathscr{V}(\Lambda, \Gamma_s)$ as the set of feasible allocations in the long-run game Γ_s for coalition Λ, coinciding with the Minkowski sum:

$$\mathscr{V}(\Lambda, \Gamma_s) \equiv \sum_{i=1}^{|S|} \sigma_i(s) \, \mathscr{V}^{(s_i)}(\Lambda).$$

where $\sigma_i(s) \equiv v_i^{(\beta)}(s)$ if the β-discounted criterion is adopted, and $\sigma_i(s) \equiv \pi_i$ under the average criterion.

Proposition 1 ([5]) *$\mathscr{V}(\Lambda, \Gamma_s)$ is equivalent to the set $\mathscr{A}$ of real $\mathbb{R}^{|\Lambda|}$-tuples* $\mathbf{a}$ *such that $\sum_{i=1}^{|\Lambda|} \mathbf{a}_i \leq v(\Lambda, \Gamma_s)$, where $v(\Lambda, \Gamma_s) = \sum_{i=1}^{|S|} \sigma_i(s) \, v^{(s_i)}(\Lambda)$, for all $s \in S$, $\Lambda \subseteq \mathscr{P}$.*

Thanks to Proposition 1, it is legitimate to define $v(\Lambda, \Gamma_s)$ as the value of coalition $\Lambda \subseteq \mathscr{P}$ in the long-run game Γ_s. Let us define the Shapley value in static games [30].

Definition 1 The *Shapley value* $\mathrm{Sh}^{(s)}$ in the static game played in state $s \in S$ is a real P-tuple whose jth component is the payoff allocation to player j:

$$\mathrm{Sh}_j^{(s)} = \sum_{\Lambda \subseteq \mathscr{P}/\{j\}} \frac{|\Lambda|!(P-|\Lambda|-1)!}{P!} \left[v^{(s)}(\Lambda \cup \{j\}) - v^{(s)}(\Lambda) \right].$$

Now, we are ready to define the Shapley value in the Markovian game Γ_s, $\mathrm{ShM}(\Gamma_s)$, that can be expressed, thanks to Proposition 1 and to the standard linearity property of the Shapley value, as

$$\mathrm{ShM}_j(\Gamma_s) = \sum_{i=1}^{|S|} \sigma_i(s) \, \mathrm{Sh}_j^{(s_i)}, \quad \forall s \in S, \ 1 \leq j \leq P. \tag{1}$$

In the next sections, we will exploit Hoeffding's inequality [16] to derive basic confidence intervals for the Shapley value of Markovian games.

Theorem 1 (Hoeffding's inequality) *Let $A_1, \ldots, A_n$ be n independent random variables, where $A_i \in [a_i, b_i]$ almost surely. Then, for all $\varepsilon > 0$,*

$$\Pr\left(\sum_{i=1}^{n} A_i - \mathbb{E}\left[\sum_{i=1}^{n} A_i \right] \geq n\,\varepsilon \right) \leq 2 \exp\left(- \frac{2n^2 \varepsilon^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right).$$

In this work, several results are shown in the case of *simple Markovian games*. They are Markovian games with transferable utility in which the value of each coalition in each state can only take on binary values, i.e., 0 and 1. Simple games model winning/losing situations, in which winning coalitions have unitary value. We say that player i is *critical* for coalition $\Lambda \subseteq \mathscr{P} \backslash \{i\}$ in state s if $v^{(s)}(\Lambda \cup \{i\}) - v^{(s)}(\Lambda) = 1$. The Shapley value applied to simple static games is commonly referred to as Shapley–Shubik power index (SS). We define SSM as the Shapley value in simple Markovian games. Of course, the relation between SS and SSM is analogous to expression (1).

All the results in this paper for simple Markovian games, except the ones in Sect. 4, are also valid for weighted voting Markovian games, which we define in the following.

Definition 2 A *weighted voting Markovian game* is a Markovian game in which each single stage game $\Psi^{(s)}$ is associated to the triple $(P, T^{(s)}, \mathbf{w}^{(s)})$, where $1, \dots, P$ are the players, $\mathbf{w}^{(s)} \in \mathbf{R}^P$ is the set of weights, and $T^{(s)}$ is a threshold. The binary coalition values $v^{(s)}$ in state s are such that $v^{(s)}(\Lambda) = 1$ whenever $\sum_{i \in \Lambda} \mathbf{w}_i^{(s)} \geq T^{(s)}$ and $v^{(s)}(\Lambda) = 0$ whenever $\sum_{i \in \Lambda} \mathbf{w}_i^{(s)} < T^{(s)}$.

3 Motivations of the Markovian model

Many interaction situations among different individuals are not one-shot, but continue over time. Moreover, the environment in which interactions take place is dynamic, and this may influence the negotiation power of each individual. Under these assumptions, the value of each coalition varies over time. In economics, clear examples of this situation are the continuing bargaining among countries, firms, or management unions. This pragmatic reasoning spurred the research on dynamic cooperative games in the last decade (see, e.g., [15], [19]). Our Markovian model is a specific instance of a dynamic cooperative game, in which the evolution of the coalition values over time follows an exogenous Markov chain on a finite state space. A concrete example of our model, in which the coalition values are not bound to be binary though, can be found in [6], where a wireless multiple access channel is considered, and several users attempt to transmit to a single receiver. The value of a coalition of users is computed as the maximum sum-rate achievable by the coalition when the remaining players threaten to jam the network. The state of the system is represented by the channel coefficients, whose evolution over time follows a Markov chain, a classic assumption in wireless communications.

Our Markovian scenario can be seen as a natural extension to a dynamic context of static situations with some uncertainty in the model. E.g., let us consider games with agent failure (see e.g. [25,26,24]), in which each player may withdraw from the game with a certain probability. The dynamic version of this game can be modeled via a very simple Markov chain, where the probability of reaching a state where a certain subset Λ of players survives only depends on Λ and not on the current state. Interestingly, the approach utilized in [8] to approximate the Shapley value in static games has been adapted to a cooperative game with failures in [9]. In [12], a coalition formation scenario with uncertainty is considered, in which the state of the system accounts for the stochastic outcome of the collaboration among agents. Though our model does not consider coalition formation, a simple Markov chain can still be used to extend the scenario in [12] to a dynamic context, in which the transition probabilities still do not depend on the starting state.

It is also worth clarifying the meaning of the Shapley value ShM on Markovian games, defined as in (1). Classically, in static games, the Shapley value has a two-fold interpretation. It can be thought of either as a measure of agents' power or as a binding agreement the agents make regarding the sharing of the revenue earned by the grand coalition. The first interpretation still holds in Markovian games, where $\mathrm{ShM}_j(\Gamma_s)$ is the expected power of agent j in the long-run game Γ_s. The second interpretation is sensible only when the value of the grand coalition is deterministic; since $v(\mathscr{P},\Gamma_s)$ is an expected revenue, this second interpretation fails to hold in the Markovian game. Nevertheless, we can still view ShM under a revenue sharing perspective. Suppose indeed that the rewards at each state are deterministic. We see from (1) that $\mathrm{ShM}_j(\Gamma_s)$ equals the long-run expected payoff for player j, if in each state s the deterministic revenue $\mathrm{Sh}_j^{(s)}$ is assigned to player j. Therefore, $\{\mathrm{Sh}_j^{(s)}\}_{s\in\mathscr{S}}$ can be seen as the deterministic distribution of $\mathrm{ShM}(\Gamma_s)$ along the course of the dynamic game, for any initial state $s \in \mathscr{S}$. Moreover, it is straightforward to see that such distribution procedure is time consistent, i.e. if the state at time $t \geq 0$ is S_t, then $\beta^t \mathrm{ShM}_j(\Gamma_{S_t})$ is the long-run expected revenue for player j from time t onward. For a detailed discussion on this topic, in a more complex model in which the transition probabilities depend on the players' actions, we refer to [5].

4 Complexity of deterministic algorithms

Since the *exact* computation of the Shapley value, or equivalently of the Shapley–Shubik index, involves the calculation of the incremental asset brought by a player to each coalition, then its complexity is proportional to the number of such coalitions, i.e., 2^{P-1}, under oracle access to the characteristic function. In this section, we evaluate the complexity of any *deterministic* algorithm which *approximates* the Shapley–Shubik index in a simple Markovian game.

Before starting our analysis, let us introduce some ancillary concepts. We mean by *game instance* a specific Markovian game. In this paper, we implicitly assume that all the algorithms considered (deterministic or randomized) aim at approximating the Shapley value for player j, without loss of generality. Let us clarify our notion of "query."

Definition 3 A *query* of an algorithm (deterministic or randomized) - consists in the evaluation of the marginal contribution of player j to a coalition $\Lambda \subseteq \mathscr{P}\backslash\{i\}$, i.e., $v(\Lambda\cup\{i\}) - v(\Lambda)$.

Now we define the accuracy of a deterministic algorithm.

Definition 4 Let us assume that the Shapley–Shubik index for player j in the simple Markovian game Γ_s is $\mathrm{SSM}_j(\Gamma_s) = a$. Let ALG be a *deterministic* algorithm employing q queries. We say that ALG has an *accuracy* of at least $d > 0$ with q queries whenever, for all the game instances, ALG always answers $\mathrm{SSM}_j(\Gamma_s) \in [a-d; a+d]$.

We will first show that an exponential number of queries is necessary in order to achieve a polynomial accuracy for any deterministic algorithm aiming to approximate the Shapley–Shubik index in the static case. This is an extension of Theorem 3 in [8] to the Shapley–Shubik index, and its proof is in Appendix A.

Theorem 2 *Any deterministic algorithm computing one component of the Shapley–Shubik index in the simple static game in state s requires $\Omega(2^P/\sqrt{P})$ queries to achieve an accuracy of at least $1/(2P)$, for all $s \in S$.*

We remark that Theorem 2 does not apply to weighted voting games. There exist algorithms that exploit the weight/quota structure of a weighted voting game to decrease the complexity of the *exact* computation of the Shapley-Shubik index with respect to simple games. Algorithms based on generating functions are proposed in [33] and [11]. A pseudo-polynomial time algorithm based on a dynamic programming technique is described in [22]. In [18] the authors devise a fast deterministic algorithm whose time complexity is $\mathrm{O}(1.415^P P)$. The complexity of such algorithms is still exponential in the number of players. In [28] and [23] it is shown that the problem of determining whether a player of a weighted voting game is a dummy one is NP-complete. On the other hand, in the literature there does not exist - up to our knowledge - an *approximation* result analogous to Theorem 2 specifically for weighted voting games.

Finally, we are ready to derive a trade-off between the accuracy and the complexity of a deterministic algorithm approximating the Shapley–Shubik index in a simple Markovian game, as a function of the number of players P.

Corollary 1 *There exists $c > 0$ such that any deterministic algorithm approximating one component of the Shapley–Shubik index in the simple (but not weighted voting) Markovian game Γ_s requires $\Omega(2^P/\sqrt{P})$ queries to achieve an accuracy of at least c/P, for all $s \in S$.*

The results of the current section clearly discourage from computing exactly or even approximating SSM with a deterministic algorithm when the number of players P is large. Motivated by this, in the next sections we will direct our attention toward *randomized* approaches to construct confidence intervals for SSM, whose complexity does not even depend on P.

5 Randomized static approach

In this section, we will propose our first approach to compute a confidence interval for the Shapley value in Markovian games. The expression of the confidence interval that we will propose holds for the Shapley value of *any* Markovian game (ShM). Nevertheless, in the following sections, we will provide some results holding specifically for the Shapley–Shubik index in the particular case of simple Markovian games (SSM). Let us first define our performance evaluator for a randomized algorithm.

Definition 5 Let $1-\delta$ be the probability of confidence. The *accuracy of a randomized algorithm* is the length of the confidence interval produced by the randomized algorithm to approximate SSM.

In parallel, the reader learns the notion of accuracy of a deterministic algorithm from Definition 4. Throughout the paper, we suppose that the transition probability matrix $\mathbf{P}$ is known by the estimator agent. In this section, we also assume that the value of all coalitions in each single stage games are available *off-line* to the estimator agent.

Assumption 1 *The estimator agent has access to all the coalition values in each state:*

$$\{v^{(s)}(\Lambda),\ \forall \Lambda \subseteq \mathscr{P},\ s \in S\}$$

at the same time, before the Markovian game starts.

It is clear that, under Assumption 1, the estimator agent can perform an off-line randomized algorithm to approximate ShM.

Remark 1 Assumption 1 seems to be impractical for the intrinsic dynamics of the model we consider. Nevertheless, the randomized approach based on Assumption 1 that we propose next (SCI) will prove to be an insightful performance benchmark for two methods (DCI1 and DCI2) described in Sect. 6, based on a more realistic dynamic assumption.

First, let us find a formulation of the Shapley value in the Markovian game which is suitable for our purpose. Let X be the set of all the permutations of $\{1,\dots,P\}$. Let $\mathscr{C}_\chi(j)$ be the coalition of all the players whose index precedes j in the permutation $\chi \in X$, i.e.,

$$\mathscr{C}_\chi(j) \equiv \{i : \chi(i) < \chi(j)\}. \tag{2}$$

We can write the Shapley value of the Markovian game Γ_s, both for the discount and for the average criterion, as

$$\begin{aligned}
\mathrm{ShM}_j(\Gamma_s) &= \sum_{i=1}^{|S|} \sigma_i(s)\,\mathrm{Sh}_j^{(s_i)} \\
&= \frac{1}{P!}\sum_{\chi\in X}\sum_{i=1}^{|S|} \sigma_i(s)\left[v^{(s_i)}(\mathscr{C}_\chi(j)\cup\{j\}) - v^{(s_i)}(\mathscr{C}_\chi(j))\right] \\
&= \mathbb{E}_\chi\left[\sum_{i=1}^{|S|} \sigma_i(s)\left[v^{(s_i)}(\mathscr{C}_\chi(j)\cup\{j\}) - v^{(s_i)}(\mathscr{C}_\chi(j))\right]\right],
\end{aligned}$$

where $\mathbb{E}_\chi$ is the expectation over all the permutations $\chi \in X$, each having the same probability $1/P!$.

We now propose our first algorithm to compute a confidence interval for $\mathrm{ShM}_j(\Gamma_s)$, for each player j and initial state s. For each query, labeled by the index $k = 1,\dots,m$, let us select independently over a uniform distribution on X a permutation χ_k of $\{1,\dots,P\}$. Let us define $Z(j)$ as the random (over $\chi \in X$) variable

$$Z(j) \equiv \sum_{i=1}^{|S|} \sigma_i(s)\left[v^{(s_i)}(\mathscr{C}_\chi(j)\cup\{j\}) - v^{(s_i)}(\mathscr{C}_\chi(j))\right] \tag{3}$$
$$= v(\mathscr{C}_\chi(j)\cup\{j\},\Gamma_s) - v(\mathscr{C}_\chi(j),\Gamma_s)$$

and let $Z_k(j)$ be the kth realization of $Z(j)$. We remark that $Z(j)$ implies the computation of $|S|$ queries, one in each state. Thanks to Hoeffding's inequality, we can write that for all $\varepsilon > 0$,

$$\Pr\left(\left|\frac{1}{m}\sum_{k=1}^{m} Z_k(j) - \mathrm{ShM}_j(\Gamma_s)\right| \geq \varepsilon\right) \leq 2\exp\left(-\frac{2m\varepsilon^2}{[\overline{y}-\underline{y}]^2}\right)$$

where

$$\overline{y} = \max_{\mathscr{C}\subseteq\mathscr{P}} \sum_{i=1}^{|S|} \sigma_i(s)\left[v^{(s_i)}(\mathscr{C}\cup\{j\}) - v^{(s_i)}(\mathscr{C})\right],$$
$$\underline{y} = \min_{\mathscr{C}\subseteq\mathscr{P}} \sum_{i=1}^{|S|} \sigma_i(s)\left[v^{(s_i)}(\mathscr{C}\cup\{j\}) - v^{(s_i)}(\mathscr{C})\right].$$

We remark that, in the case of simple games, $\left[\overline{y}-\underline{y}\right]^2 \leq \left[\sum_{i=1}^{|S|}\sigma_i(s)\right]^2$. Now we are ready to propose our first confidence interval, based on Assumption 1.

Static Confidence Interval 1 (SCI) *Let $1 \leq j \leq P$, $s \in S$. Fix an integer n and set $\delta \in (0;1)$. Then, with probability of confidence $1-\delta$, $\mathrm{ShM}_j(\Gamma_s)$ belongs to the confidence interval*

$$\left[\frac{1}{m}\sum_{k=1}^{m} Z_k(j) - \varepsilon(m,\delta)\ ;\ \frac{1}{m}\sum_{k=1}^{n} Z_k(j) + \varepsilon(m,\delta)\right],$$

where

$$\varepsilon(m,\delta) = \sqrt{\frac{[\overline{y}-\underline{y}]^2\log(2/\delta)}{2m}}. \tag{4}$$

In the case of simple games, (4) becomes

$$\varepsilon(m,\delta) = \sqrt{\frac{\left[\sum_{i=1}^{|S|}\sigma_i(s)\right]^2\log(2/\delta)}{2m}}. \tag{5}$$

Under the average criterion, (5) can be written as $\varepsilon(m,\delta) = \sqrt{\log(2/\delta)/[2m]}$.

Not surprisingly, the confidence interval SCI is analogous to the one found in [8] for static games. Indeed, the intrinsic dynamics of the game is surpassed by Assumption 1, for which the estimator has global knowledge of all the coalition values, even before the Markov process initiates. Therefore, from the estimator agent's point of view, there exists no conceptual difference between the approach in [8] and SCI, except for the complexity, which increases by a factor $|S|$ in the dynamic game.

6 Randomized dynamic approaches

In this section we will propose two methods to compute a confidence interval for SSM, for which Assumption 1 on global knowledge of coalition values is no longer necessary. Indeed, the reader will notice that their conception naturally arises from the assumption that the estimator agent learns the coalition values in each single stage game while the Markov chain process unfolds, as formalized below.

Assumption 2 *The state in which the estimator agent finds itself at each time step follows the same Markov chain process of the Markovian game itself. The estimator agent has local knowledge of the game that is being played, i.e., at step $t \geq 0$ the estimator agent has access only to the coalition values associated to the static game in the current state S_t.*

Remark 2 The approaches described in this section can also be employed under Assumption 1. Indeed, any algorithm requiring the query on coalition values separately in each state can also be run under a static assumption.

In the following, we still assume that the transition probability matrix $\mathbf{P}$ is known by the estimator agent. As in Sect. 5, the randomized approaches that we are going to introduce hold for the Shapley value of *any* Markovian game.

6.1 First dynamic approach

We propose our first randomized approach to compute a confidence interval for ShM, holding both under the static Assumption 1 and under the dynamic Assumption 2. Let $\chi \in X$ be, as in Sect. 5, a random permutation uniformly distributed on the set $\{1,\dots,P\}$. Let us define $Y^{(s_i)}(j)$ as the random (over $\chi \in X$) variable associated to state s_i:

$$Y^{(s_i)}(j) \equiv \; v^{(s_i)}(\mathscr{C}_\chi(j) \cup \{j\}) - v^{(s_i)}(\mathscr{C}_\chi(j)). \tag{6}$$

Our dynamic approach suggests to sample the r.v. $Y^{(s_i)}(j)$ n_i times in state s_i. Let $n = \sum_{i=1}^{|S|} n_i$ be the total number of queries. We can still exploit Hoeffding's inequality to say that, for all $\varepsilon' > 0$,

$$\Pr\left(\left| \sum_{i=1}^{|S|} \frac{\sigma_i(s)}{n_i} \sum_{t=1}^{n_i} Y_t^{(s_i)}(j) - \mathrm{ShM}_j(\Gamma_s) \right| \geq n\varepsilon' \right) \leq 2\exp\left(- \frac{2[n\varepsilon']^2}{\sum_{i=1}^{|S|} \sigma_i^2(s)[\overline{x}(i) - \underline{x}(i)]^2/n_i} \right)$$

where, for all $i = 1,\dots,|S|$,

$$\overline{x}(i) = \max_{\mathscr{C} \subseteq \mathscr{P}} \; v^{(s_i)}(\mathscr{C} \cup \{j\}) - v^{(s_i)}(\mathscr{C})$$

$$\underline{x}(i) = \min_{\mathscr{C} \subseteq \mathscr{P}} \; v^{(s_i)}(\mathscr{C} \cup \{j\}) - v^{(s_i)}(\mathscr{C})$$

We notice that, in the case of simple games, $\overline{x}(i) = 1$ and $\underline{x}(i) = 0$ for all $i = 1, \ldots, |S|$. Now set $\widetilde{\varepsilon} = n\,\varepsilon'$. Now we are ready to propose our second confidence interval for $\mathrm{ShM}_j(\Gamma_s)$, the first one holding under Assumption 2.

Dynamic Confidence Interval 1 (DCI1) *Let $1 \le j \le P$, $s \in S$. Fix the number of queries n and set $\delta \in (0;1)$. Then, with probability of confidence $1-\delta$, $\mathrm{ShM}_j(\Gamma_s)$ belongs to the confidence interval*

$$\left[\sum_{i=1}^{|S|} \frac{\sigma_i(s)}{n_i} \sum_{t=1}^{n_i} Y_t^{(s_i)}(j) - \widetilde{\varepsilon}(n,\delta)\ ;\ \sum_{i=1}^{|S|} \frac{\sigma_i(s)}{n_i} \sum_{t=1}^{n_i} Y_t^{(s_i)}(j) + \widetilde{\varepsilon}(n,\delta)\right],$$

where

$$\widetilde{\varepsilon}(n,\delta) = \sqrt{\frac{\log(2/\delta)}{2} \sum_{i=1}^{|S|} \frac{\sigma_i^2(s)}{n_i} [\overline{x}(i) - \underline{x}(i)]^2}. \tag{7}$$

In the case of simple games, (7) becomes

$$\widetilde{\varepsilon}(n,\delta) = \sqrt{\frac{\log(2/\delta)}{2} \sum_{i=1}^{|S|} \frac{\sigma_i^2(s)}{n_i}}\,. \tag{8}$$

Optimal sampling strategy In this section we focus exclusively on *simple Markovian games*. It is interesting to investigate the optimum number of times n_i^* that the variable $Y^{(s_i)}(j)$ should be sampled in each state s_i, in order to minimize the length of the confidence interval DCI1, keeping the confidence probability fixed. We notice that, by fixing $1-\delta$, we can find the optimal values for $n_1, \ldots, n_{|S|}$ by setting up the following integer programming problem:

$$\begin{cases} \min\limits_{n_1,\ldots,n_{|S|}} \sum_{i=1}^{|S|} \sigma_i^2(s)[\overline{x}^2(i) - \underline{x}^2(i)]/n_i \\ \sum_{i=1}^{|S|} n_i = n, \qquad n_i \in \mathbb{N} \end{cases} \tag{9}$$

Remark 3 If the static Assumption 1 holds, then the computation of the optimum values $n_1^*, \ldots, n_{|S|}^*$ in (9) is the only information we need to maximize the accuracy of DCI1, since the sampling is done off-line. Otherwise, if Assumption 2 holds, the estimator does not know in advance the succession of states hit by the process, hence it is crucial to plan a sampling strategy of the variable $Y^{(s_i)}(j)$ along the Markov chain. Of course, a possible strategy would be, when n is fixed, to sample n_i^* times the variable $Y^{(s_i)}(j)$ only the first time the state s_i is hit, until all the states are hit. Nevertheless, this approach is clearly not efficient, since in several time steps the estimator is forced to remain idle.

Motivated by Remark 3, now we devise an efficient and straightforward sampling strategy, consisting in sampling $Y^{(s_i)}(j)$, *each* time the state s_i is hit, an equal number of times over all $i = 1, \ldots, |S|$. Let us first show a useful classical result for Markov chains. Let η be the number of steps performed by the Markov chain. Let η_i be the number of visits to state s_i, i.e.,

$$\eta_i = \sum_{t=0}^{\eta-1} \mathbb{I}(S_t = s_i).$$

Theorem 3 **([2])** *Let $\{S_t,\, t \geq 1\}$ be an ergodic Markov chain. Let $\hat{\pi}_i^{(\eta)} \equiv \eta_i/\eta$. Then, for any distribution on the initial state and for all $i = 1, \ldots, |S|$,*

$$\hat{\pi}_i^{(\eta)} \stackrel{\eta\uparrow\infty}{\longrightarrow} \pi_i \qquad \text{with probability } 1,$$

where π is the stationary distribution of the Markov chain.

It is evident from (7) that $\widetilde{\varepsilon}(n,\delta) \in \Theta(n^{-1/2})$. Now we will show under which conditions the straightforward sampling strategy described above allows to achieve asymptotically for $n \uparrow \infty$ the best rate of convergence of $\widetilde{\varepsilon}(n,\delta)$, for δ fixed. The reader can find the proof of the next theorem in Appendix D.

Theorem 4 *Suppose that Assumption 2 holds. Let the Markov chain of the simple Markovian game be ergodic. Fix the confidence probability $1-\delta$. Under the average criterion, if each time the state s_i is hit then the estimator agent samples the r.v. $Y^{(s_i)}(j)$ a constant number of times not depending on i (e.g., 1), then with probability 1:*

$$\sqrt{n}\,\widetilde{\varepsilon}(n,\delta) \stackrel{n\uparrow\infty}{\longrightarrow} \inf_{n\in\mathbb{N}} \min_{\substack{n_1,\ldots,n_{|S|}:\\ \sum_i n_i = n}} \sqrt{n}\,\widetilde{\varepsilon}(n,\delta) = \sqrt{\frac{\log(2/\delta)}{2}}.$$

6.2 Second dynamic approach

Since Hoeffding's inequality has a very general applicability and does not refer to any particular probability distribution of the random variables at issue, it is natural to look for confidence intervals especially suited to particular instances of games. In this section, we will show a third confidence interval for the Shapley value of the Markovian game Γ which is tighter *i*) the higher the confidence probability $1-\delta$ is and *ii*) the tighter the confidence intervals $[l_i; r_i]$ are. As an example, in Sect. 6.2, we will show a tight confidence interval for simple Markovian games.

We still assume that the estimator agent samples the r.v. $Y^{(s_i)}(j)$ n_i times, in each state s_i. Here, we suppose to know beforehand that $\mathrm{Sh}_j^{(s_i)}$ lies in the confidence interval $[l_i; r_i]$ with probability of at least $1-\delta_i$. In general, the extrema l_i and r_i may depend on n_i, $\sum_{t=1}^{n_i} Y_t^{(s_i)}(j)$, and δ_i.
As in the case of DCI1, the randomized approach proposed in this section also holds both under the static Assumption 1 and under the dynamic Assumption 2. It is based on the following lemma, whose proof is in Appendix C.

Lemma 1 *Let $A_1,\dots,A_k$ be k random variables such that* $\Pr(A_i \in [l_i;r_i]) \ge 1-\delta_i$*. Let $c_i \ge 0$, for $i=1,\dots,k$. Then*

$$\Pr\left(\sum_{i=1}^{k} c_i A_i \in \left[\sum_{i=1}^{k} c_i l_i \,;\, \sum_{i=1}^{k} c_i r_i\right]\right) \ge \prod_{i=1}^{k}[1-\delta_i]$$

The reader should keep in mind that, the smaller the single confidence levels $\delta_1,\dots,\delta_k$ are, the tighter the lower bound on the confidence probability $\prod_{i=1}^{k}(1-\delta_i)$ is. Now we are ready to present our second dynamic approach. Let the r.v. $Y^{(s_i)}(j)$ be defined as in (6).

Dynamic Confidence Interval 2 (DCI2) *Set $\delta_i \in (0;1)$, for all $i=1,\dots,|S|$. Let*

$$\left[l^{(s_i)}\left(n_i, \sum_{t=1}^{n} Y_t^{(s_i)}(j), \delta_i\right) \,;\, r^{(s_i)}\left(n_i, \sum_{t=1}^{n} Y_t^{(s_i)}(j), \delta_i\right)\right] \tag{10}$$

be the confidence interval for $\mathrm{Sh}^{(s_i)}$*, with probability of confidence* $1-\delta_i$*, for all $i=1,\dots,|S|$. Let $1\le j\le P$, $s\in S$. Then, with probability of confidence* $\prod_{i=1}^{|S|}(1-\delta_i)$*,* $\mathrm{ShM}_j(\Gamma_s)$ *belongs to the confidence interval*

$$\left[\sum_{i=1}^{|S|} \sigma_i(s)\, l^{(s_i)}\left(n_i, \sum_{t=1}^{n_i} Y_t^{(s_i)}(j), \delta_i\right) \,;\, \sum_{i=1}^{|S|} \sigma_i(s)\, r^{(s_i)}\left(n_i, \sum_{t=1}^{n_i} Y_t^{(s_i)}(j), \delta_i\right)\right].$$

We notice that the confidence interval DCI2 reveals the most natural connection between the issue of computing confidence intervals of the Shapley value in static games, already addressed in [8], and in Markovian games under the dynamic Assumption 2.

We already saw in Sect. 6.1 that the accuracy of DCI1 can be maximized by adjusting the number of queries $n_1,\dots,n_{|S|}$ in each state. Here, in addition, we could optimize DCI2 also over the set of confidence levels $\delta_1,\dots,\delta_{|S|}$, under the nonlinear constraint:

$$\prod_{i=1}^{|S|}[1-\delta_i] = 1-\delta\,.$$

Simple Markovian games The aim of this section is twofold. First, we suggest methods to compute a confidence interval for the Shapley–Shubik index in simple static games, as a complement of the study in [8]. Secondly, we stress that such methods can be utilized to compute efficiently the confidence interval DCI2 for SSM, as it is clear from the definition of DCI2 itself.

In [8], the authors derived a confidence interval for the Shapley value of a single stage game, based on Hoeffding's inequality. Nevertheless, for simple static games, a tighter confidence interval can be obtained, by applying the following approach. Let $\chi \in X$ be a random permutation of $\{1,\dots,P\}$. Let us assume that $\{\chi_k \in X\}$, $k\ge 1$, are uniform and independent. Let us define the Bernoulli variable $Y^{(s)}(j)$ as in (6). As pointed out in [8], we can interpret the Shapley–Shubik index $\mathrm{SS}_j^{(s)}$ as

$$\mathrm{SS}_j^{(s)} = \Pr\left(Y^{(s)}(j)=1\right).$$

Let $Y_1^{(s)}(j),\dots,Y_n^{(s)}(j)$ be independent realization of $Y^{(s)}(j)$. It is evident that

$$\sum_{k=1}^{n} Y_k^{(s)}(j) \sim \mathscr{B}(n, \mathrm{SS}_j^{(s)}),$$

where $\mathscr{B}(a,b)$ is the binomial distribution with parameters a,b. Hence, computing a confidence interval for $\mathrm{SS}_j^{(s)}$ boils down to the computation of confidence intervals of the probability of success of the Bernoulli variable $Y^{(s)}(j)$ given the proportion of successes $\sum_{k=1}^{n} Y_k^{(s)}(j)/n$, which is a well-know problem in literature. Of course, this might be accomplished by using the general Hoeffding's inequality as in [8], but over the last decades some more efficient methods have been proposed, like the Chernoff bound [13], the Wilson's score interval [36], the Wald interval [35], the adjusted Wald interval [1], and the "exact" Clopper–Pearson interval [14].

7 Comparison among the proposed approaches

In this section we focus on *simple Markovian games*, and we compare the accuracy of the proposed randomized approaches. We know that, under the static Assumption 1, we are allowed to use any of the three methods presented in this article, SCI, DCI1, and DCI2, to compute a confidence interval for the Shapley–Shubik index in simple Markovian games. In fact, DCI1 and DCI2 involve independent queries over the different states, and this can also be done under Assumption 1. Therefore, it makes sense to compare the tightness of the two confidence intervals SCI and DCI1.

Lemma 2 *Consider simple Markovian games. Let $2\varepsilon(n,\delta)$ be the accuracy of SCI (see Eq. (5)). Let $2\widetilde{\varepsilon}(n,\delta)$ be the accuracy of DCI1 (see Eq. (8)). Then, for any integer n and for any confidence probability $1-\delta$,*

$$\varepsilon(n,\delta) \le \widetilde{\varepsilon}(n,\delta).$$

An interested reader can find the proof of Lemma 2 in Appendix E.

Remark 4 The reader should not be misled by the result in Lemma 2. In fact, n being equal in the two cases, the number of queries needed for confidence interval SCI is $|S|$ times bigger than for DCI1, since each sampling of the variable $Z(j)$, defined in (3), requires $|S|$ queries, one per each state. The comparison between the two confidence intervals would be fair only if the estimator agent knew beforehand the coalition values of the long-run game $\{v(\Lambda,\Gamma_s)\}_{s,\Lambda}$.

According to Remark 4, we should compare the length of the confidence interval for the static case, $2\varepsilon(n,\delta)$, with the one for the dynamic case, $2\widetilde{\varepsilon}(|S|n,\delta)$, calculated with $|S|$ times many queries. Intriguingly, the relation between the tightness of SCI and DCI is now, for a suitable query strategy, reversed, as we show next.

Theorem 5 *In the case of simple Markovian games, for any integer n,*

$$\min_{\substack{n'_1,\dots,n'_{|S|}:\\ \sum_i n'_i = |S|n}} \widetilde{\varepsilon}(|S|n,\delta) \le \varepsilon(n,\delta).$$

Proof We can write

$$\min_{\substack{n'_1,\dots,n'_{|S|}:\\ \sum_i n'_i=|S|n}} \sum_{i=1}^{|S|} \frac{\sigma_i^2(s)}{n'_i} \le \sum_{i=1}^{|S|} \frac{\sigma_i^2(s)}{\sum_{k=1}^{|S|} n'_k/|S|} = \sum_{i=1}^{|S|} \frac{\sigma_i^2(s)}{n} \le \frac{\left[\sum_{i=1}^{|S|} \sigma_i(s)\right]^2}{n}, \qquad (11)$$

where the last inequality holds since $\sigma_i(s) \ge 0$. Hence, by inspection over the expressions (5) and (8), the thesis is proved.

Theorem 5 clarifies the relation between the confidence intervals SCI and DCI1, under the condition of simple Markovian games. We highlight its significance in the next two remarks.

Remark 5 Theorem 5 claims that the approach DCI1 is more accurate than SCI for a suitable choice of $n'_1,\dots,n'_{|S|}$, when the number of queries is equal for the two methods. In essence, this occurs because the dynamic approach allows us to tune the number of queries in the coalition values according to the weight $\sigma_i(s)$ of each state s_i in the long-run game. Moreover, the queries on coalition values are independent among the states, hence providing more diversity to the statistics.

Remark 6 As we already remarked, the dynamic Assumption 2 is more pragmatic and less restrictive than the static Assumption 1. Let us now give some insights on the accuracy that can be achieved by the approaches SCI and DCI1 under Assumptions 1 and 2. The approach DCI1 can be also utilized under static Assumption 1, and in finite time DCI1 is more accurate under Assumption 1 than under Assumption 2. Indeed, for a fixed n and under the static Assumption 1, the value of $n'_1,\dots,n'_{|S|}$ in (11) can always be set to the optimum value, since the algorithm DCI1 is run off-line. Instead, under the dynamic Assumption 2, the sequence of states over time $S_0,S_1,S_2,\dots$ is unknown *a priori* by the estimator agent, hence $n'_1,\dots,n'_{|S|}$ cannot be optimized for a finite n. Hence, in finite time, the static Assumption 1 has still an edge over the dynamic Assumption 2 for the implementation of DCI1.
Nevertheless, we know from Theorem 4 that, for the average criterion in ergodic Markov chains, there exists a query strategy enabling to achieve an optimum rate of convergence for DCI1's accuracy. Therefore, we can conclude with the following consideration. Under the average criterion, DCI1, when employed under the dynamic Assumption 2, can be *asymptotically* as accurate as DCI1 itself and more accurate than SCI, when both of these approaches are employed under the stronger static Assumption 1.

In addition to what has just been discussed, simulations showed that when the number of queries n and the confidence level δ are equal for the two methods, then the *effective* confidence probability for SCI is generally higher than for DCI1, i.e., the lower bound $1-\delta$ is loose. We explain this by reminding that the centers of the confidence intervals SCI and DC1, respectively,

$$\frac{1}{m}\sum_{k=1}^{m} Z_k(j) \quad , \quad \sum_{i=1}^{|S|} \frac{\sigma_i(s)}{n_i} \sum_{t=1}^{n_i} Y_t^{(s_i)}$$

are already two estimators for $\mathrm{SSM}(\Gamma_s)$, and the former possesses a smaller variance than the second one.

$1-\delta$	$a_{2\succ 1}$ (%)
.97	100
.95	99.9
.9	87.5
.8	57.7

Table 1 Percentage $a_{2\succ 1}$ of cases in which the confidence interval DCI2 is narrower than confidence interval DCI1, at different confidence probabilities. The Clopper–Pearson interval is considered for DCI2.

Regarding the performance of confidence interval DCI2, the simulations confirmed our intuitions. We utilized the Clopper–Pearson interval to compute a confidence interval for the Shapley–Shubik index in simple static games, and we saw that the tightness of DCI2 increases when the confidence probability approaches 1. Let $a_{2\succ 1}$ be the percentage of simple Markovian game instances, generated randomly, in which the confidence interval DCI2 is narrower than confidence interval DCI1. In Table 1, we show, for each value of confidence probability $1-\delta$, the values of $a_{2\succ 1}$ obtained from simulations. We see that for $1-\delta < 0.8$, the two confidence intervals have a comparable length. For $1-\delta \geq 0.8$, the confidence interval DCI2 is apparently tighter than DCI1 under these settings.

8 Complexity of confidence intervals

In Sect. 4, we motivated the importance of devising an algorithm that approximates SSM with a polynomial accuracy in the number of players P without the need of an exponential number of queries. In this section we show that the proposed randomized approaches SCI and DCI1 fulfill this requirement, since they only require a polynomial number of queries to reach an accuracy which is polynomial in P. Interestingly, the number of queries required by SCI and DCI1 does not even depend on the number of players P.

Proposition 2 *Fix the confidence level δ and the length of confidence interval 2ε. Then n queries are required to compute the confidence interval SCI, where*

$$n = \frac{\left[\overline{y} - \underline{y}\right]^2 \log(2/\delta)}{2\varepsilon^2}.$$

Proof The proof follows straightforward from the expression of confidence interval SCI.

Proposition 3 *Fix the confidence level δ and the length of confidence interval $2\widetilde{\varepsilon}$. Then, there exist values of $n_1,\ldots,n_{|S|}$, with $\sum_i n_i = n$, such that n queries are required to compute the confidence interval DCI1, where*

$$n \leq \frac{|S|\left[\overline{y} - \underline{y}\right]^2 \log(2/\delta)}{2\widetilde{\varepsilon}^2}.$$

Proof The proof follows straightforward from Theorem 5.

From Propositions 2 and 3, we derive the following fundamental result on the complexity of SCI and DCI1.

Theorem 6 *Let $p(P)$ be a polynomial in the variable P. The number of queries required to achieve an accuracy of $1/p(P)$ is $O(p^2(P))$, for both the confidence intervals SCI and DCI1.*

Since we did not provide an explicit expression for the confidence interval DCI2, then we cannot provide a result analogous to Theorem 6 for DCI2 likewise. Anyway, we notice that the expression (10) of confidence interval DCI2 does not depend on the number of players P. Moreover, if the Hoeffding's inequality is used to compute the confidence interval for the Shapley value in the static games, then a result similar to Theorem 6 can be derived for DCI2.

Remark 7 Corollary 1 and Theorem 6 explain in what sense the proposed randomized approaches SCI and DCI1 are better than any deterministic approach, according to Definitions 4 and 5 of "accuracy". For instance, in order to achieve an accuracy in the order of P^{-1}, for a number of players P sufficiently high, the number of queries needed by SCI and DCI1 is always smaller than the number of queries employed by any deterministic algorithm.

9 Conclusions

In Sect. 4, we proved that an exponential number of queries is necessary for any deterministic algorithm even to approximate SSM with polynomial accuracy. Hence, we directed our attention to randomized algorithms and we proposed three different methods to compute a confidence interval for SSM. The first one, described in Sect. 5 and called SCI, assumes that the coalition values in each state are available off-line to the estimator agent. SCI can be seen as a benchmark for the performance of the other two methods, DCI1 in Sects. 6.1 and DCI2 in Sect. 6.2. The last two methods can be utilized also if we pragmatically assume that the estimator learns the coalition values in each static game while the Markov chain process unfolds. DCI2 reveals the most natural connection between confidence intervals of the Shapley value in static games, presented in [8], and in Markovian games. As a by-product of the study of DCI2, we provided confidence intervals for the Shapley–Shubik index in static games, which are tighter than the one proposed in [8]. In Sect. 6.1, we proposed a straightforward way to optimize the tightness of DCI1. In Sect. 7, we compared the three proposed approaches in terms of tightness of the confidence interval. We proved that DCI1 is tighter than SCI, with an equal number of queries and for a suitable choice of the number of queries on coalition values in each state. This occurs essentially because DCI1 allows us to tune the number of samples according to the weight of the state. Hence, we showed that, *asymptotically*, the dynamic Assumption 2 is not restrictive with respect to the much stronger static Assumption 1, under the average criterion and for what concerns SCI and DCI1. The simulations confirmed that DCI2 is more accurate than the SCI and DCI1 when both the confidence probability is close to 1 and a tight confidence interval for the Shapley–Shubik index of static games is available, like the Clopper–Pearson interval. Finally, in Sect. 8, we showed that a polynomial number of queries is sufficient to achieve a polynomial accuracy for the proposed algorithms. Hence, in order to compute SSM, the proposed randomized approaches are more accurate than any deterministic approach for a number of players sufficiently high. The three proposed randomized approaches can be utilized to compute confidence intervals for the Shapley value in *any* cooperative Markovian game, as well. Our results on simple games, except the ones in Sect. 4, also apply to weighted voting games. In Table 2 we summarize the features of the three proposed confidence intervals: SCI, DCI1, and DCI2.

SCI	Confidence interval based on Hoeffding's inequality. Valid under static Assumption 1. Its general formulation in (4) holds for the Shapley value in any Markovian game, as well. The number of queries required to achieve an accuracy of $1/p(P)$ is $O(p^2(P))$ (Theorem 6).
DCI1	Confidence interval based on Hoeffding's inequality. Valid under both static Assumption 1 and dynamic Assumption 2. Its general formulation in (7) holds for the Shapley value in any Markovian game, as well. Theorem 4 provides a sampling strategy maximizing its accuracy, applicable under dynamic Assumption 2. The number of queries required to achieve an accuracy of $1/p(P)$ is $O(p^2(P))$ (Theorem 6).
DCI2	Confidence interval valid under both static Assumption 1 and dynamic Assumption 2. Its formulation holds for the Shapley value in any Markovian game, as well.
SCI vs. **DCI1**	Under the static Assumption 1, there exists a sampling strategy for which DCI1 is at least as tight as SCI, for any number of sampling n (see Theorem 5). Under dynamic Assumption 2 and average criterion, DCI1 can be made at least as tight as SCI asymptotically, for $n \uparrow \infty$ (see Theorem 4).
DCI1 vs. **DCI2**	By utilizing Clopper–Pearson intervals, DCI2 is tighter than DCI1 for all $1-\delta > 0.8$, simulations suggest (see Table 1).

Table 2 Summary of results for the three proposed approaches to compute confidence intervals for the Shapley–Shubik power index in Markovian games, i.e., SCI, DCI1, and DCI2.

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A Proof of Theorem 2

Proof We will prove that there exists a class $\mathscr{F}$ of game instances for which any deterministic algorithm computing $\mathrm{SS}_j^{(s)}$ with accuracy of at least $1/(2P)$ must utilize $\Omega(2^P/\sqrt{P})$ queries. Similarly to [8], let us construct $\mathscr{F}$ when P is odd. Let $\Lambda \subseteq \mathscr{P}\backslash\{j\}$. There exists a set D_o of $\binom{P-1}{[P-1]/2}/2$ coalitions of cardinality

$[P-1]/2$ such that player j is critical only for D_o. In particular, for $|\Lambda| \leq [P-1]/2$, $v^{(s)}(\Lambda) = 0$; if $|\Lambda| = [P-1]/2$, then, if $\Lambda \in D_o$, $v^{(s)}(\Lambda \cup \{j\}) = 1$, otherwise $v^{(s)}(\Lambda \cup \{j\}) = 0$. The values of the remaining coalitions are 1 if and only if they contain a winning coalition among the ones constructed so far. The Shapley value for player j is thus

$$\mathrm{SS}_j^{(s)} = \frac{([P-1]/2)!([P-1]/2)!}{2(P)!}\binom{P-1}{[P-1]/2} = \frac{1}{2P}$$

Hence, for any deterministic algorithm ALG_o employing a number of queries smaller than $\mu_o(P)$, where

$$\mu_o(P) = \frac{1}{2}\binom{P-1}{[P-1]/2},$$

there always exists an instance belonging to $\mathscr{F}$ for which ALG_o would answer $\mathrm{SS}_j^{(s)} = 0$. By Stirling's approximation, we can say that $\mu_o(P) \in \Omega(2^P/\sqrt{P})$. Let us now construct the class $\mathscr{F}$ of instances when P is even and $P > 2$. Let D_e be a set of $\binom{P-2}{[P-2]/2}$ coalitions of cardinality $[P-2]/2$, belonging to $\mathscr{C}\setminus\{j\}$, such that player j is critical only for D_e. Then

$$\mathrm{SS}_j^{(s)} = \frac{(P/2-1)!(P/2)!}{(P)!}\binom{P-2}{[P-2]/2} = \frac{1}{2[P-1]} > \frac{1}{2P}.$$

Similarly to before, for any deterministic algorithm ALG_e using a number of queries smaller than

$$\mu_e(P) = \binom{P-1}{[P-2]/2} - \binom{P-2}{[P-2]/2} = \frac{P-2}{P}\binom{P-2}{[P-2]/2},$$

there always exists an instance belonging to $\mathscr{F}$ for which ALG_e would answer $\mathrm{SS}_j^{(s)} = 0$. By Stirling approximation, we can say that $\mu_e(P) \in \Omega(2^P/\sqrt{P})$. Hence, a number of samples $\mu \in \Omega(2^P/\sqrt{P})$ is needed to achieve an accuracy of at least $1/(2P)$. Hence, the thesis is proved.

B Proof of Corollary 1

Proof Any deterministic algorithm employs a certain number of queries in each state s in order to compute $\mathrm{SSM}_j(\Gamma_s) = \sum_{i=1}^{|S|} \sigma_i(s)\mathrm{SS}_j^{(s_i)}$. Let I_0 be a game instance in which player j is a dummy player in all the single stage games $\{v^{(s)}\}_{s\in S}$, i.e., $\mathrm{SS}_j^{(s)} = 0$ for all $s \in S$. Let I_1 be a game instance such that $\mathrm{SS}_j^{(s)} = 0$ for all s except for s_k, for which $\sigma(s_k) \neq 0$, and such that the game $\Psi^{(s_k)}$ belongs to the class $\mathscr{F}$ of instances described in the proof of Theorem 2. Therefore,

$$\mathrm{SSM}_j(\Gamma_s) = \frac{\sigma_k(s)}{2P}$$

in the case that P is odd and

$$\mathrm{SSM}_j(\Gamma_s) = \frac{\sigma_k(s)}{2[P-1]}$$

if P is even. Hence, any deterministic algorithm needs $\Omega(2^P/\sqrt{P})$ queries in state s_k to achieve an accuracy better than $\sigma_k(s)/(2P)$. Set $c = \sigma_k(s)/2$. Hence, the thesis is proved.

C Proof of Lemma 1

Proof We will provide the proof for continuous random variables; the proof for the discrete case is totally similar. By induction, it is sufficient to prove that, if $\Pr(A_1 \in [l_1;r_1]) \geq 1-\delta_1$ and $\Pr(A_2 \in [l_2;r_2]) \geq 1-\delta_2$, then

$$\Pr(A_1 + A_2 \in [l_1 + l_2\ ;\ r_1;r_2]) \geq (1-\delta_1)(1-\delta_2).$$

Let f_A be the probability density function of the r.v. A. Let $\overline{f}_{A_i}(x) = f_{A_i}(x)\mathbf{1}(x \in [l_i; r_i])$, $i = 1, 2$. Then

$$\begin{aligned}
\Pr\left(A_1 + A_2 \in [l_1 + l_2; r_1; r_2]\right) &= \int_{l_1+l_2}^{r_1+r_2} f_{A_1+A_2}(x)dx \\
&= \int_{l_1+l_2}^{r_1+r_2} \int_{\mathbb{R}} f_{A_1}(x-\tau) f_{A_2}(\tau)\, d\tau\, dx \\
&\geq \int_{l_1+l_2}^{r_1+r_2} \int_{\mathbb{R}} \overline{f}_{A_1}(x-\tau) \overline{f}_{A_2}(\tau)\, d\tau\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f}_{A_1}(x-\tau) \overline{f}_{A_2}(\tau)\, d\tau\, dx \\
&= \int_{\mathbb{R}} \overline{f}_{A_1}(x)\, dx \int_{\mathbb{R}} \overline{f}_{A_2}(x)\, dx \\
&= \Pr(A_1 \in [l_1; r_1]) \Pr(A_2 \in [l_2; r_2]) \\
&\geq (1-\delta_1)(1-\delta_2).
\end{aligned}$$

Hence, the thesis is proved.

D Proof of Theorem 4

Proof Let us consider the following constrained minimization problem over the reals:

$$\begin{cases} \min\limits_{\omega_1,\ldots,\omega_{|S|}} \sum_{i=1}^{|S|} \sigma_i^2(s)/\omega_i \\ \sum_{i=1}^{|S|} \omega_i = n, \qquad \omega_i \in \mathbb{R}. \end{cases} \tag{12}$$

By using e.g. the Lagrangian multiplier technique, it is easy to see that the optimum value for ω_i is

$$\omega_i^* = \frac{\sigma_i(s)\, n}{\sum_{k=1}^{|S|} \sigma_k(s)}$$

and that the minimum value of the objective function is

$$\xi^* = \frac{\left[\sum_{i=1}^{|S|} \sigma_i(s)\right]^2}{n}. \tag{13}$$

The value ξ^* clearly represents a lower bound for the optimization problem over the integers in the case of simple games. Since we deal with the average criterion, let $\sigma_i(s) \equiv \pi_i$. Now we can find a lower bound for $\sqrt{n}\,\widetilde{\varepsilon}(n,\delta)$ over n that does not depend on the number of queries n:

$$\begin{aligned}
&\inf_{n\in\mathbb{N}} \min_{\substack{n_1,\ldots,n_{|S|}: \\ \sum_i n_i = n}} \sqrt{n}\,\widetilde{\varepsilon}(n,\delta) = \\
&= \inf_{n\in\mathbb{N}} \min_{\substack{n_1,\ldots,n_{|S|}\in\mathbb{N}: \\ \sum_i n_i = n}} \sqrt{\frac{n\log(2/\delta)}{2} \sum_{i=1}^{|S|} \frac{\pi_i^2}{n_i}} \\
&= \inf_{\substack{q_1,\ldots,q_{|S|}\in\mathbb{Q}^+: \\ \sum_i q_i = 1}} \sqrt{\frac{\log(2/\delta)}{2} \sum_{i=1}^{|S|} \frac{\pi_i^2}{q_i}} \\
&= \min_{\substack{x_1,\ldots,x_{|S|}\in\mathbb{R}^+: \\ \sum_i x_i = 1}} \sqrt{\frac{\log(2/\delta)}{2} \sum_{i=1}^{|S|} \frac{\pi_i^2}{x_i}} \\
&= \sqrt{\frac{\log(2/\delta)}{2}}
\end{aligned} \tag{14}$$

and the optimum value of x_i in (14) is

$$x_i^* = \frac{\pi_i}{\sum_{k=1}^{|S|} \pi_k} = \pi_i\,.$$

For Theorem 3,

$$n_i/n \xrightarrow{n\uparrow\infty} \pi_i \qquad \text{with probability 1.}$$

Hence, n_i/n converges with probability 1 to the optimum value x_i^* and, by continuity, the thesis is proved.

E Proof of Lemma 2

Proof In the case of simple Markovian games, the optimization problem (9) turns into

$$\begin{cases} \min\limits_{n_1,\dots,n_{|S|}} \sum_{i=1}^{|S|} \sigma_i^2(s)/n_i \\ \sum_{i=1}^{|S|} n_i = n, \qquad n_i \in \mathbb{N}. \end{cases} \tag{15}$$

Let us consider the constrained minimization problem over the reals in (12). Since evidently ξ^*, defined in (13), is not greater than the minimum value of the objective function in (15), then by straightforward inspection over the expressions (5) and (8) the thesis is proved.

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