1999 IEEE ITW, Kruger National Park, South Africa, June 20 - 25

Limiting Performance of Block-Fading Channels with Multiple Antennas

Giorgio Taricco* Politecnico di Torino (Italy)
Ezio Biglieri* Politecnico di Torino (Italy)
Giuseppe Caire Eurecom (France)

Abstract — **We study the performance limits of a radio system consisting of a transmitter with t antennas and a receiver with r antennas over a block-fading additive white Gaussian noise channel. We derive the optimal coding scheme minimizing the information outage probability and analyze its performance in terms of outage probability and delay-limited capacity. We show that, asymptotically, the delay-limited capacity grows linearly with $m \triangleq \min(t, r)$ and is almost independent of the number of fading blocks M which implies the existence of a trade-off between spatial (multiple antennas) and time diversity (interleaving).**

I. Channel model, capacity and outage probability

We consider a multiple-antenna system with t transmit and r receive antennas and a block-fading additive white Gaussian noise (BF-AWGN) channel with M blocks. We call it a $t \times r$ BF-AWGN channel. On each block the fading gain is constant. Code words of MNt complex symbols are transmitted as M blocks of Nt symbol. Each block is divided into t sub-blocks of N symbols. Finally, each sub-block is transmitted from a different antenna, so that approximately t symbols/s/Hz are transmitted. At the receiver, each antenna receives the superposition of the t transmitted symbols affected by fading and additive noise. The outputs of the r receiving antennas are processed jointly.

The discrete-time baseband equivalent channel model can be written as

$$\mathbf{y}_k[n] = \mathbf{A}_k \mathbf{x}_k[n] + \mathbf{z}_k[n] \tag{1}$$

for $k = 1, \ldots, M$ and $n = 1, \ldots, N$. Here, $\mathbf{A}_k \in \mathbb{C}^{r\times t}$ is the matrix of complex channel gains, $\mathbf{x}_k[n] \in \mathbb{C}^t$ and $\mathbf{y}_k[n], \mathbf{z}_k[n] \in \mathbb{C}^r$ are the n-th transmitted, received and noise vectors. The noise vector is circularly-symmetric complex Gaussian distributed $\sim \mathcal{N}_c(\mathbf{0}, \mathbf{I})$. The following transmit power constraint is assumed:

$$E[|\mathbf{x}_k[n]|^2] = \mathrm{Tr}(E[\mathbf{x}_k[n]\mathbf{x}_k[n]^\dagger]) \le \Gamma$$

Channel state information (CSI), i.e., knowledge of $\mathbf{A}_k$, is assumed to be available at the receiver (CSIR) and at the transmitter (CSIT).

*This work was supported by the Italian Space Agency (ASI).

Following a standard approach [6], we obtain a more convenient equivalent model for (1) from the SVD decomposition $\mathbf{A}_k = \mathbf{U}_k \mathbf{S}_k \mathbf{V}_k^\dagger$ [3] ($\mathbf{U}_k$ and $\mathbf{V}_k$ are unitary matrices and the main diagonal of $\mathbf{S}_k$ contains the singular values of $\mathbf{A}_k$, i.e., the square roots of the strictly positive eigenvalues of $\mathbf{A}_k\mathbf{A}_k^\dagger$, namely, $\lambda_{k,1}^{1/2} > \cdots > \lambda_{k,m}^{1/2} > 0$ with $m \triangleq \min\{r, t\}$). Defining $\widetilde{\mathbf{y}}_k[n] \triangleq \mathbf{U}_k^\dagger \mathbf{y}_k[n]$, $\widetilde{\mathbf{z}}_k[n] \triangleq \mathbf{U}_k^\dagger \mathbf{z}_k[n]$, and $\widetilde{\mathbf{x}}_k[n] \triangleq \mathbf{V}_k^\dagger \mathbf{x}_k[n]$, we obtain

$$\widetilde{\mathbf{y}}_k[n] = \mathbf{S}_k \widetilde{\mathbf{x}}_k[n] + \widetilde{\mathbf{z}}_k[n] \tag{2}$$

This model is equivalent to (1) since $\widetilde{\mathbf{z}}_k[n] \sim \mathcal{N}_c(\mathbf{0}, \mathbf{I})$ and $E[|\widetilde{\mathbf{x}}_k[n]|^2] = E[|\mathbf{x}_k[n]|^2$, so that any constraint on $\widetilde{\mathbf{x}}_k[n]$ is mapped to the same constraint on $\mathbf{x}_k[n]$.

Ergodic capacity. We assume finite block length N and $M \to \infty$, $\{\mathbf{A}_k\}_{k=1}^M$ is an asymptotically ergodic process and, with perfect CSIR, the capacity is independent of N [4] and given by [6]:

Proposition 1. *The capacity of a $t \times r$ BF-AWGN channel, under the power constraint $E[|\mathbf{x}_k[n]|^2] \le \Gamma$, is given by:*

i) With perfect CSIR and no CSIT,

$$C_{\mathrm{CSIR}}(\Gamma) = mE\left[\log(1 + \Gamma\lambda_1/t)\right]^{1} \tag{3}$$

ii) With perfect CSIR and CSIT,

$$C_{\mathrm{CSIR,CSIT}}(\Gamma) = \sum_{i=1}^{m} E[(\log(\xi\lambda_i))_+] \tag{4}$$

where λ_i is the i-th (decreasingly ordered) singular value of $\mathbf{A}$, (which is a random matrix distributed as any $\mathbf{A}_k$'s), $\gamma_i = (\xi - 1/\lambda_i)_+$, $(x)_+ \triangleq \max\{0, x\}$, and ξ is the solution to the nonlinear equation $\sum_{i=1}^m \gamma_i = \Gamma$. □

Delay-limited performance. Here we assume a fixed transmission rate R, finite M, and $N \to \infty$, and apply the coding theorem [1, Prop. 2] to the vector channel (2). First, let us define $\boldsymbol{\lambda}_k \triangleq (\lambda_{k,1}, \ldots, \lambda_{k,m})$, $\boldsymbol{\Lambda} \triangleq (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_M)$, $\boldsymbol{\gamma}_k$ as the row-vector of diagonal elements of $E[\widetilde{\mathbf{x}}_k[n]\widetilde{\mathbf{x}}_k[n]^\dagger]^2$, and $\boldsymbol{\Gamma} \triangleq (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_M)$. Next, we define the *instantaneous mutual information* $I_M(\boldsymbol{\Lambda}, \boldsymbol{\Gamma})$ as

$$I_M(\boldsymbol{\Lambda}, \boldsymbol{\Gamma}) \triangleq \frac{1}{M} \sum_{k=1}^{M} \sum_{i=1}^{m} \log(1 + \lambda_{k,i}\gamma_{k,i}) \tag{5}$$

[1] Hereafter, we consider base-2 logarithms.

[2] We skip the proof that the instantaneous mutual information is maximum when $\boldsymbol{\gamma}_k$ is independent of n.

With perfect CSIT, the transmitter can distribute the available power in order to minimize the *information outage probability* [5]

$$P_{\text{out}}(R,\Gamma) \triangleq \Pr(I_M(\mathbf{\Lambda},\mathbf{\Gamma}) < R) \tag{6}$$

This is obtained under a constraint on the transmitted power which may be a *short-term* or *long-term* constraint. The former is the mean SNR over a frame of M blocks:

$$\text{(short-term)} \qquad \frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m}\gamma_{k,i} \le \Gamma \tag{7}$$

The latter is given by

$$\text{(long-term)} \quad E\left[\frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m}\gamma_{k,i}\right] \le \Gamma \tag{8}$$

and allows the transmitter to allocate more or less power to different code words to compensate channel fading while keeping the mean value bounded in the long term (with ergodic fading). Plainly, the long-term constraint is weaker than the short-term constraint since (7) implies (8).

The minimum outage probability is closely related to the capacity of the BF-AWGN channel. This capacity, under a delay constraint, has been called *delay-limited capacity* in [7]. The delay-limited capacity of an M-block $t \times r$ BF-AWGN channel is given by

$$C_{\text{delay}}(\Gamma) \triangleq \inf_{\mathbf{\Lambda}} \sup_{\mathbf{\Gamma}=\mathbf{\Gamma}(\mathbf{\Lambda})} I_M(\mathbf{\Lambda},\mathbf{\Gamma}) \tag{9}$$

where the supremum is over all $\mathbf{\Gamma}(\mathbf{\Lambda})$ satisfying the power constraint (7) or (8) and the infimum is over all non-negative $\mathbf{\Lambda}$. An equivalent definition is given by the following Proposition. This is a coding theorem deriving from [1, Prop. 2] by observing that the equivalent vector channel (2) can be seen as a scalar channel with Mm fading blocks.

Proposition 2. *The maximum ϵ-achievable rate of a $t \times r$ BF-AWGN channel subject to a short-term (resp., long-term) power constraint is given by*

$$C_\epsilon(\Gamma) = \sup_{\mathbf{\Gamma}=\mathbf{\Gamma}(\mathbf{\Lambda})} \sup\{R : P_{\text{out}}(R,\Gamma) \le \epsilon\} \tag{10}$$

where the supremum is, again, over all $\mathbf{\Gamma}(\mathbf{\Lambda})$ satisfying the power constraint (7) or (8). The delay-limited capacity (9) is given by $C_{\text{delay}}(\Gamma) = \lim_{\epsilon\downarrow 0} C_\epsilon(\Gamma)$.

II. Minimum outage probability

Since the vector BF-AWGN channel is equivalent to a scalar BF-AWGN channel as described by (2) with Mm blocks and fading power gains $\lambda_{k,i}$, Propositions 3 and 4 of [1] apply almost unchanged and solve the problem of minimizing the information outage probability with a short-term (resp., long-term) power constraint. They are given as follows.

Proposition 3. *The outage probability (6) under the constraint (7) is minimized for*

$$\mathbf{\Gamma} = \widehat{\mathbf{\Gamma}}(\mathbf{\Lambda}) = \begin{cases} \mathbf{\Gamma}^{\text{st}}(\mathbf{\Lambda}) & \text{if } \mathbf{\Lambda} \in \mathcal{R}_{\text{on}}(R,\Gamma) \\ \mathbf{G}(\mathbf{\Lambda}) & \text{if } \mathbf{\Lambda} \notin \mathcal{R}_{\text{on}}(R,\Gamma) \end{cases} \tag{11}$$

i) $\mathbf{G}(\mathbf{\Lambda})$ can be any function from $\mathbb{R}_+^{Mm}$ to $\mathbb{R}_+^{Mt}$ satisfying the constraint (7).

ii) $\mathbf{\Gamma}^{\text{st}}(\mathbf{\Lambda})$ maximizes $I_M(\mathbf{\Lambda},\mathbf{\Gamma})$ under the constraint (7). The (k,i)-th component of $\mathbf{\Gamma}^{\text{st}}(\mathbf{\Lambda})$ is given by

$$\gamma_{k,i}^{\text{st}} = \left(\xi^{\text{st}}(\mathbf{\Lambda}) - \frac{1}{\lambda_{k,i}}\right)_+ \tag{12}$$

where $\xi^{\text{st}}(\mathbf{\Lambda})$ satisfies (7) with equality.

iii) $\mathcal{R}_{\text{on}}(R,\Gamma)$ is the power-on *region defined as*

$$\mathcal{R}_{\text{on}}(R,\Gamma) = \left\{\mathbf{\Lambda} : I_M(\mathbf{\Lambda},\mathbf{\Gamma}^{\text{st}}(\mathbf{\Lambda})) > R\right\} \tag{13}$$

□

Proposition 4. *The outage probability (6) under the constraint (8) is minimized for*

$$\mathbf{\Gamma} = \widehat{\mathbf{\Gamma}}(\mathbf{\Lambda}) = \begin{cases} \mathbf{\Gamma}^{\text{lt}}(\mathbf{\Lambda}) & \text{if } \mathbf{\Lambda} \in \mathcal{R}_{\text{on}}(R,s^*) \\ \mathbf{0} & \text{if } \mathbf{\Lambda} \notin \mathcal{R}_{\text{on}}(R,s^*) \end{cases} \tag{14}$$

where

i) $\mathbf{\Gamma}^{\text{lt}}(\mathbf{\Lambda})$ minimizes $\frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m}\gamma_{k,i}$ under the constraint $I_M(\mathbf{\Lambda},\mathbf{\Gamma}) \ge R$. The (k,i)-th component of $\mathbf{\Gamma}^{\text{lt}}(\mathbf{\Lambda})$ is given by

$$\gamma_{k,i}^{\text{lt}} = \left(\xi^{\text{lt}}(\mathbf{\Lambda}) - \frac{1}{\lambda_{k,i}}\right)_+ \tag{15}$$

where $\xi^{\text{lt}}(\mathbf{\Lambda})$ satisfies (8) with equality.

ii) $\mathcal{R}_{\text{on}}(R,s^*)$ is the power-on region *defined as*

$$\mathcal{R}_{\text{on}}(R,s^*) = \{\mathbf{\Lambda} : \frac{1}{M}\sum_{k=1}^{M}\sum_{i=1}^{m}\gamma_{k,i}^{\text{lt}} < s^*\} \tag{16}$$

where $s^* > 0$ is the value of Γ satisfying (8) with equality. □

Remark. Proposition 4 aplies only when the fading distribution is continuous. Otherwise, the optimal Γ is a random function of $\mathbf{\Lambda}$ (see [1] for further details). The shape of the power-on region is illustrated in Figure 1 for the case $r = t = 1$, $M = 2$, $R = 2$ bit/s/Hz, $s^* = 1$, 3, and 5 dB.

Optimal beamforming. The optimal transmission scheme (based on the optimal power allocation scheme $\widehat{\mathbf{\Gamma}}(\mathbf{\Lambda})$) can be obtained as the concatenation of a standard Gaussian code of length MNt with i.i.d. components distributed as $\mathcal{N}_c(0,1)$ and an optimal time-varying *beamformer*. Dividing each code word into MN segments

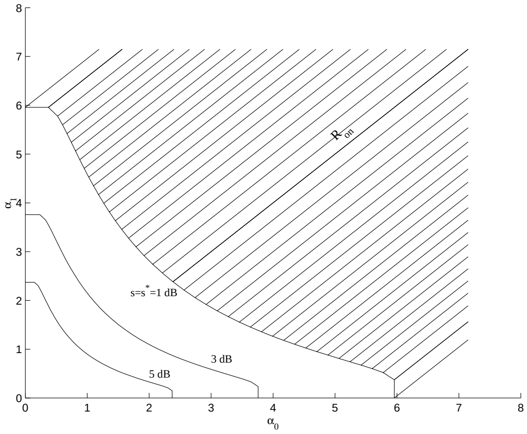

Fig. 1: Power-on region (dashed) $\mathcal{R}_{\text{on}}$ for a system with $t = r = 1$, $M = 2$, $R = 2$ bit/s/Hz, and $s^* = 1$ dB. The fading gains α_1 and α_2 are Rayleigh-distributed with unit second moment.

$\widetilde{\mathbf{x}}_k[n]$, the optimal codeword is given by the segments $\mathbf{x}_k[n] = \mathbf{W}_k \widetilde{\mathbf{x}}_k[n]$ with

$$\mathbf{W}_k = \mathbf{V}_k \operatorname{diag}(\gamma_{k,1}^{1/2}, \dots, \gamma_{k,m}^{1/2}, \overbrace{0, \dots, 0}^{t-m}) \qquad (17)$$

where $\gamma_{k,i}$ are the components of $\widehat{\mathbf{\Gamma}}(\mathbf{\Lambda})$.

III. Asymptotic results

The following asymptotic result is stated here without proof for the sake of brevity.

Proposition 5. *The delay-limited capacity of a $t \times r$ BF-AWGN channel with optimal power control under a long-term power constraint (Proposition 4) is given by*

$$C_{\text{delay}}(\Gamma) \approx m \log[\Gamma/(m\, E[1/\bar{\lambda}])]$$

asymptotically as $\Gamma \to \infty$, where $\bar{\lambda} \triangleq \prod_{k,i} \lambda_{k,i}^{1/(Mm)}$ □

Numerical results are reported in Figures 2 and 3. These figures plot the delay-limited capacity versus SNR (Γ) of the $t \times t$ independent Rayleigh BF-AWGN channel for $t = 2$, 4, 8, and 16 with $M = 1$ and 4, respectively, optimal power control and a long-term constraint. For comparison, the capacity of the $t \times t$ AWGN channel $C_{\text{AWGN}} = \log_2(1 + t^2\Gamma)$ [6] is reported as well. It can be noted that the capacity curves of the BF-AWGN channel with optimal power control have a higher slope (by a factor of t) than the curves corresponding to the AWGN channel, confirming the statement in Proposition 5. Moreover, there is no noticeable difference between the case $M = 1$ and $M = 4$ which confirms the expected trade-off between spatial and time diversity.

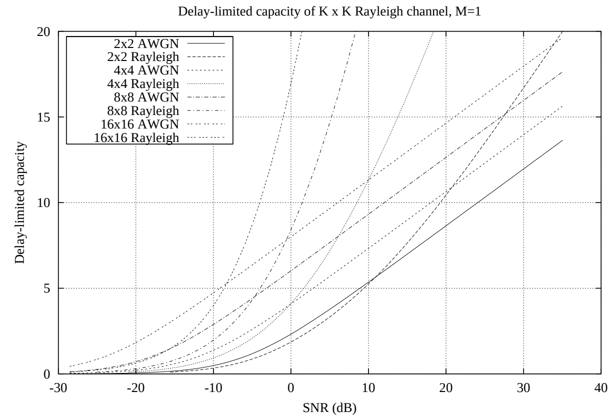

Fig. 2: Delay-limited capacity for the independent Rayleigh $t \times t$ BF-AWGN for $M = 1$ and $t = 2, 3, 4, 8$, and 16.

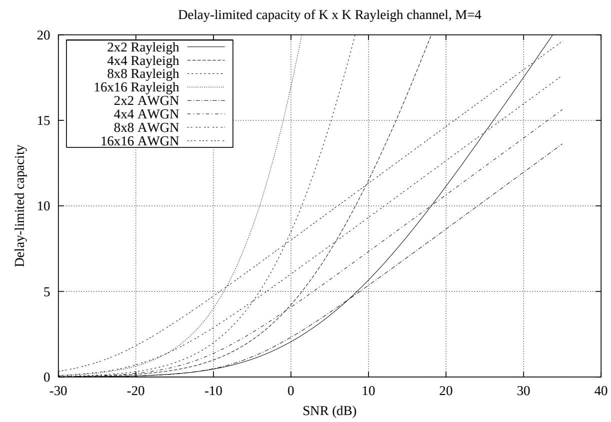

Fig. 3: Delay-limited capacity for the independent Rayleigh $t \times t$ BF-AWGN for $M = 4$ and $t = 2, 3, 4, 8$.

References

[1] G. Caire, G. Taricco and E. Biglieri, "Optimal power control for the fading channel," submitted to *IEEE Trans. on Inform. Theory,* 1997.

[2] R. Gallager, *Information Theory and Reliable Communication.* New York: Wiley, 1968.

[3] R. Horn and C. Johnson, *Matrix Analysis.* New York: Cambridge University Press, 1985.

[4] R. McEliece and W. Stark, "Channels with block interference," *IEEE Trans. Inform. Theory,* Vol. 30, No. 1, pp. 44–53, Jan. 1984.

[5] L. Ozarow, S. Shamai, and A. D. Wyner, "Information theoretic considerations for cellular mobile radio," *IEEE Trans. Vehic. Tech.,* Vol. 43, No. 2, May 1994.

[6] İ. E. Telatar, "Capacity of multi-antenna Gaussian channels," submitted to *IEEE Trans. Inf. Theory*, 1997.

[7] V. Hanly and D. Tse, "Multi-access Fading Channels: Part II, Delay-Limited Capacities," *IEEE Trans. Inform. Theory,* Vol. 44, No. 7, pp. 2816–31, Nov. 1998.