Distortion bounds and a Protocol for One-Shot Transmission of Correlated Random Variables on a Non-Coherent Multiple-Access Channel

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***Abstract*—Bounds on the distortion derived in [1] are adapted to the case where two continuous random variables are sent over a multiple access channel with phase shifts using the help of two feedback channels. The first source is defined to be uniformly distributed and the second source is defined as the sum of the first source and an auxiliary random variable which is also uniform. Additionally, using the same definition of the two sources, the two-round protocol introduced in [2] is studied in detail and a comparison is made in order to discuss the tightness of the information-theoretic bounds.**

I. Introduction

We consider simple transmission strategies for sensor networks that are able to measure a physical phenomenon from different locations. Under the condition of energy-limited sensors and considering also a wireless transmission medium, a low-complexity one-shot coding method is needed. The key issue is that digital transmission for small amounts of (typically) analog data will induce overhead which is wasteful, especially for massive networks of simple nodes. Joint source-channel coding (JSCC), which combines the efforts of the channel and source code, addresses such problems.In this paper, we consider JSCC for transmission of multiple spatially-distributed samples of a slowly time-varying random field.

A source is represented by a random variable U of zero mean and variance $\sigma_u^2 = 1$. The sensor is in general a tiny device with strict energy constraints. The communication channel between the sender and the receiver is an additive white Gaussian noise channel. The adressed problem is how to efficiently encode the random source, and what to expect in terms of performance. We focus our attention on the case where unitary samples of the source are transmitted sporadically due to slow time-variation, and consequently we cannot perform sequence coding. Each realization of the source is mapped into $\mathbf{X} \triangleq (X_1, \dots, X_N)$. Here, the dimension of the channel input is denoted by N and can be assumed to be large. $\mathbf{X}$ is then sent across the channel corrupted by a white Gaussian noise sequence $\mathbf{Z}$, and is received as the output signal $\mathbf{Y}$. The receiver constructs an estimate $\widehat{U}$ of U given $\mathbf{Y}$. The chosen criteria to measure the 'goodness' of estimation is the mean square error distortion $D \triangleq \mathbb{E}[(U-\widehat{U})^2]$, which is desired to be minimized under the mean energy constraint $\mathbb{E}[||\mathbf{X}||^2] \leq E$. The linear encoder $X = \sqrt{E}U$ achieves the best performance under the mean energy constraint for the special case $N = 1$, (see [3], [4], [5]). A lower bound on the distortion over all possible encoders and decoders is easily derived in [3] using classical information theory, and given by

$$D \geq e^{-2E/N_0} \tag{1}$$

where $N_0/2$ is the variance of the channel noise per dimension. This case can be extended to multi-user systems again with continuous sources, where the distortion is caused by the quantization applied to the sources and the channel itself. Thus the derivation of a bound on the performance is directly related to optimizing the number of the quantization bits to reach to the minimum distortion as it was done in [6] and [7]. Another classical example of a joint-source channel mapping is the coherent PPM scheme with ML detection [8, pp. 623], which gives a $\mathrm{e}^{-E/3N_0}$ behaviour for distortion. A simple scheme described in [9] combines a scalar quantizer with an orthogonal modulation and MAP receiver or an MMSE estimator. Such a scheme achieves the same $\mathrm{e}^{-E/3N_0}$ behaviour, both for coherent or non-coherent detection, which is significantly worse than the lower bound in (1).

In Section II, we provide lower bounds on the distortion level, for both of the sources and their product when the phase shifts are assumed to be perfectly known by the receiver. In Section III, upper bounds are derived for the system defined in II with non-coherent reception. Finally, in Section IV a comparison of the obtained results is made.

II. Bounds on distortion for uniformly correlated sources

Bounds for the distortion derived in [1] are observed for arbitrarily correlated continuous sources sent over a Gaussian multiple access channel with phase shifts in the presence of feedback. Through a different approach, i.e. without splitting the noise as done in [1], tighter bounds are achieved on distortion level. The described system is as given in Figure 1. Source vectors $\mathbf{U_1}, \mathbf{U_2}$ have a dimension of K identically independent distributed samples of the correlated sources U_1, U_2.The correlational relationship between the sources U_1, U_2 is defined as

$$U_2 = \rho U_1 + \sqrt{1-\rho^2}U_2' \tag{2}$$

based on U_1 and U_2' which is also uniform on $(-\sqrt{3}, \sqrt{3})$. The first source U_1 is defined to be uniformly distributed over $(-\sqrt{3}, \sqrt{3})$ and the second source U_2 is defined to have a

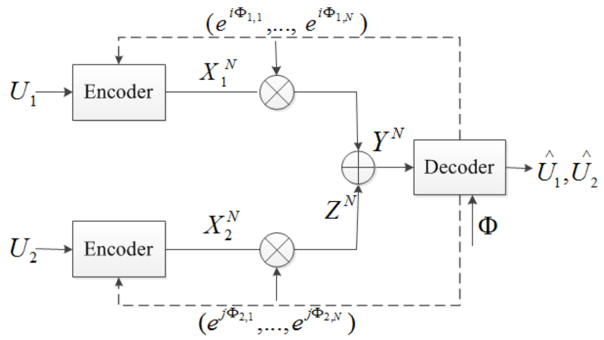

Fig. 1. Correlated sources over GMAC with feedback.

contaminated uniform distribution, so we have one uniform and one almost uniform source having covariance equal to the correlation coefficient ρ between them. The received signal $\mathbf{Y} = \{Y_i; i = 1, ..., N\}$ and the power constraints are given as

$$Y_i = X_{1,i}e^{j\Phi_{1,i}} + X_{2,i}e^{j\Phi_{2,i}} + Z_{1,i} + Z_{2,i}, \tag{3}$$

$$\frac{1}{K}\sum_{i=1}^{N} E[|X_{m,i}|^2] \leq E_m, \tag{4}$$

for $m = 1, 2$ denoting the sources and $i = 1, ..., N$, respectively. The criteria to satisfy is chosen as the squared-error distortion measure, which is $d(u_i, \hat{u}_i) = (u_i - \hat{u}_i)^2$ for $i = 1, 2$. $\mathbf{\Phi}_m = \{\Phi_{m,i}; i = 1, ..., N\}$ denotes the random phases which are assumed to be unknown to the transmitter but known to the receiver. After a detailed description of the model, in the following we derive a relatively simple mutual information between the m^{th} source U_m and the output signal Y through two different expansions considering the case where the sources are highly correlated, i.e. the correlation coefficient ρ has a value close to 1. We will use the notation m to denote one of the sources and m' will be used to indicate the other source, explicitly m and m' cannot be equal to 1 or 2 at the same time, when m equals 1 then m' has to be equal to 2 or vice versa. To obtain a lower bound on the distortion level, we will use two different expansions of $I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'})$ considering the extreme case when the sources are highly correlated. First expansion of the desired mutual information based on the output signal is given by

$$I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'}) \leq N \log\left(1 + \frac{K(E_m + E_{m'})}{NN_0}\right). \tag{5}$$

Same mutual information was derived through a different expansion and given by

$$I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'}) \geq h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m) \tag{6}$$

Its derivation can be found in Appendix V-A together with the required entropies for $m = 1$ and $m = 2$. Substituting (43) and (45) with $m = 1$ into (42), provides the second expansion of the desired mutual information for the first source. In the same way, (43) and (45) with $m = 2$ is substituted into (42). Equating the outcomes to (5) provides the below given bound on distortion level for the m^{th} source.

$$D_m \geq C_m \left(1 + \frac{K(E_m + E_{m'})}{NN_0}\right)^{-\frac{2N}{K}} \tag{7}$$

where C_m is a constant defined as

$$C_m = \begin{cases} \frac{6}{\pi e} & \text{if } m = 1, \\ \frac{6\rho^2 + 6(1-\rho^2) + 6\rho\sqrt{1-\rho^2}}{\pi e} & \text{if } m = 2. \end{cases}$$

The asymptotic of (7) is obtained as

$$D_m \geq C_m e^{-\frac{2(E_m + E_{m'})}{N_0}}. \tag{8}$$

Note that, both bounds given above have the same asymptotic behaviour and bring out the correlation benefit by using the sum energy of the two sources. Next, we will observe the change on this behaviour based on the decrease in the correlation coefficient by deriving the mutual information $I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'})$ through two different expansions of it. We have the following expansions and the derivations are given in the Appendix V-B.

$$I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \leq N \log\left(1 + \frac{KE_m}{NN_0}\right), \tag{9}$$

$$I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \geq h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m). \tag{10}$$

Using the above derived expressions for the adequate source, the mutual information (47) and through equating to the first expansion of it given by (51), we obtain the general expression for the distortion bound $D_m \geq C_m \left(1 + \frac{KE_m}{NN_0}\right)$ and its asymptotic is as follows

$$D_m \geq C_m e^{-\frac{2E_m}{N_0}}, \tag{11}$$

where C_m is a constant varying based on m given by

$$C_m = \begin{cases} \frac{36(1-\rho^2)}{\pi^2 e^2} & \text{if } m = 1, \\ \frac{6(1-\rho^2)}{\pi e} & \text{if } m = 2. \end{cases}$$

Due to the channel construction, the sum channel does not give the exponential behaviour for two extreme cases of high and low correlation through one single bound on distortion. For that reason, we will give the bound on the distortion level of the second source as

$$D_2 \geq \max(D_{2,low}(E_2), D_{2,high}(E_1 + E_2)) \tag{12}$$

where we denote (11) for $m = 2$ by $D_{2,low}(E_2)$ and (8) by $D_{2,high}(E_1 + E_2)$. In the same way with the second source, the general bound on D_1 will be given as a maximum function of the two bounds (11) and (8) when $m = 1$

$$D_1 \geq \max(D_{1,low}(E_1), D_{1,high}(E_1 + E_2)) \tag{13}$$

where we denote (11) by $D_{1,low}(E_1)$ and (8) by $D_{1,high}(E_1 + E_2)$.

The asymptotic of the bound on product distortion $D_1 D_2$ is obtained as

$$D_1 D_2 \geq \frac{36\left(1 - \rho^2\right)}{\pi^2} \exp\left(-\frac{2(E_1 + E_2)}{N_0} - 2\right) \tag{14}$$

which is not a useful bound, since it goes to zero for a very high correlation between the sources. The derivation of the bound (14) is given in Appendix V-C.

A. Discussion

In order to discuss the tightness of the bounds (13) and (12), we will consider another case with a single source $\mathbf{U}$, whose message is sent by being split into two branches through two different codebooks. Let us call the encoded parts of $\mathbf{U}$ as $\mathbf{X}_1$ and $\mathbf{X}_2$. The estimate $\hat{\mathbf{U}}$ is received after $\mathbf{X}_1$ and $\mathbf{X}_2$ is merged before being decoded. In the following, $I(\mathbf{U};\hat{\mathbf{U}})$ is derived using two different ways and the distortion D is lower bounded.

$$\begin{aligned} I(\mathbf{U};\hat{\mathbf{U}}) &\leq I(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}|\mathbf{\Phi}) \\ &= h(\mathbf{Y}|\mathbf{\Phi}) - h(\mathbf{Y}|\mathbf{X}_1,\mathbf{X}_2,\mathbf{\Phi}) \\ &\leq \sum_{i=1}^{N} h(Y_i|\mathbf{\Phi}) - h(\mathbf{Z}) \\ &\leq N\left(\sum_{i=1}^{N}\log(\mathbb{E}[Y_i^2]) - \log(NN_0)\right) \\ &= N\log\left(1+\frac{KE}{NN_0}\right) \end{aligned} \tag{15}$$

and also

$$\begin{aligned} I(\mathbf{U};\hat{\mathbf{U}}) &= h(\mathbf{U}) - h(\mathbf{U}|\hat{\mathbf{U}}) \\ &\geq h(\mathbf{U}) - h(\mathbf{U}-\hat{\mathbf{U}}) \\ &\geq K\log(2\sqrt{3}) - \sum_{i=1}^{K} h(U_i-\hat{U}_i) \\ &\geq \frac{K}{2}\log(\frac{6}{\pi e D}) \end{aligned} \tag{16}$$

Combining (16) and (15), we get the distortion bound as given by

$$D \geq \frac{6}{\pi e(1+\frac{KE}{NN_0})^{2N/K}} \tag{17}$$

and letting N go to infinity we get $D \geq \frac{6}{\pi e}e^{-\frac{2E}{N_0}}$.

III. Asymptotic Optimality of Two-Way Protocol with Dual Uniform Sources

This section includes a more detailed version of the achievable scheme introduced in [2] in order to make a better comparison with the distortion-bounds derived in Section II. As in [10] and its non-coherent version analyzed in[2, Section II] the protocol consists of two phases which composes one round. First phase is called the data phase, in which the messages of both sources are transmitted and in return feedback of the messages are received from the decoder. Protocol uses fixed total energy which is denoted by $\mathcal{E}_{\mathrm{D},i,j}$ in the data phase and $\mathcal{E}_{\mathrm{C},i,j}$ in the control phase on the i^{th} round by the j^{th} source, where $i,j=1,2$. The quantized source samples of the sources are encoded into 2^B messages per source with dimension N. Basically after quantization the first source sends its message $m_1(U_1)$ to the receiver with energy $\mathcal{E}_{\mathrm{D},1,1}$. After detection and feedback of the first message, the second source sends $m_2(\hat{m}_1,U_2)$ with energy $\mathcal{E}_{\mathrm{D},1,2}$. Data phase on the first round ends after $m_2(\hat{m}_1,U_2)$ is fed back by the receiver. Second phase is the control phase and in this phase the sources send ACK/NACK regarding their own messages

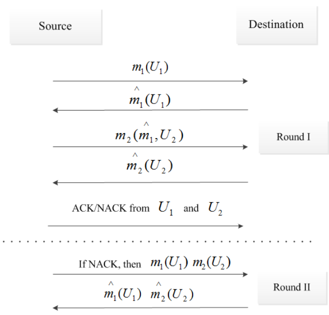

Fig. 2. Two-round protocol

to the decoder using the energy $\mathcal{E}_{\mathrm{C},i,j}$. If the decoder receives ACK from both, then the protocol halts at the end of the first round. Otherwise the encoders enter the data phase of the second round for retransmission. Proceeding of the protocol is illustrated in Figure(2). The correlational relationship between the two sources together with their statistics are kept in the same way as given by (2). We denote the error events by $E_{1,j}$ for the first round and the j^{th}source and by E_2 for the second round. $e_{1,j}$ and $c_{1,j}$ denote erroneous and correct decoding on U_j, respectively. Accordingly $E_{c\to e,1}$ denotes a misdetected acknowledged error and its probability is considered as the sum of the two sources on one round as

$$\Pr(E_{c\to e,1}) = \Pr(E_{c\to e,1,1}) + \Pr(E_{c\to e,1,2}). \tag{18}$$

$E_{e\to c,1}$ denotes an uncorrectable error, which means it is an irreversible error since it acknowledges an error as correct decoding. Its probability is taken as the sum of both sources in the same way as

$$\Pr(E_{e\to c,1}) = \Pr(E_{e\to c,1,1}) + \Pr(E_{e\to c,1,2}). \tag{19}$$

The probability of the error is bounded in [2], eq:15, by

$$P_e \leq \Pr(E_{1,1}\cup E_{1,2})\Pr(E_{e\to c,1}) + \Pr(E_2). \tag{20}$$

We give the average energy used by the protocol with the following equality

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_{\mathrm{D},1,1} + E(\mathcal{E}_{\mathrm{D},1,2}(\hat{m}_1,U_2)) + \mathcal{E}_{\mathrm{C},1,1}\Pr(E_{1,1}) \\ &+ \mathcal{E}_{\mathrm{C},1,2}\Pr(E_{1,2}) + \mathcal{E}_{\mathrm{D},2}[\Pr(E_{1,1})(1-\Pr(E_{e\to c,1,1})) \\ &+ \Pr(E_{1,2})(1-\Pr(E_{e\to c,1,2})) + (1-\Pr(E_{1,1})) \\ &\Pr(E_{c\to e,1,1}) + (1-\Pr(E_{1,2}))\Pr(E_{c\to e,1,2})] \end{aligned} \tag{21}$$

and bound it by

$$\begin{aligned} \mathcal{E} &\leq \mathcal{E}_{\mathrm{D},1} + \Pr(E_{1,1}\cup E_{1,2})\mathcal{E}_{\mathrm{C},1} + \mathcal{E}_{\mathrm{D},2}[\Pr(E_{1,1}\cup E_{1,2}) \\ &(1-\Pr(E_{e\to c,1})) + (1-\Pr(E_{1,1}\cup E_{1,2}))\Pr(E_{c\to e,1})] \end{aligned} \tag{22}$$

where $E(\mathcal{E}_{\mathrm{D},1,2}(\hat{m}_1,U_2))$ is the expected energy to be used in the data phase of the first round by the second source. The total energy for a certain phase and round is obtained by taking the sum over the both sources, i.e. the energy in the control phase of the i^{th} round is defined as $\mathcal{E}_{\mathrm{C},i} = \sum_j^2 \mathcal{E}_{\mathrm{C},i,j}$ and the total energy in the data phase of the first round is $\mathcal{E}_{\mathrm{D},i} = \sum_j^2 \mathcal{E}_{\mathrm{D},1,j}$. The output signal based on the N dimensional observation of the j^{th} source is

$$\mathbf{Y}_d = \sqrt{\mathcal{E}_{\mathrm{D},1,j}}\mathrm{e}^{j\Phi_j}\mathbf{S}_{m_j} + \mathbf{Z}_j. \tag{23}$$

We assume the random phases Φ_j to be distributed uniformly on $[0, 2\pi)$, the channel noise $\mathbf{Z}_j$ to have zero mean and equal autocorrelation $N_0\mathbf{I}_{N\times N}$ for $j = 1, 2$ and $\mathbf{S}_{m_j}$ are the N-dimensional messages, with $m = 1, 2, \cdots, 2^B$ and $j = 1, 2$. $e_j = I\left(|y_{c,j}|^2 > \lambda\mathcal{E}_{\mathrm{C},1,j}\right)$ is the detector for the j^{th} source with $y_{c,j} = \mathbf{Y_{c,j}}^H\mathbf{S}_{c,j}$. λ is a threshold value to be optimized and included within the interval $[0, 1)$.The probability of error of an uncorrectable error $E_{e\to c}$ for U_j is given by

$$\begin{aligned}\Pr(E_{e\to c,1,j}) &= \Pr(|\sqrt{\mathcal{E}_{\mathrm{C},1,j}} + z_{c,j}|^2 \le \lambda\mathcal{E}_{\mathrm{C},1,j}) \\ &= 1 - Q_1(\sqrt{\frac{\mathcal{E}_{\mathrm{C},1,j}}{N_0/2}}, \sqrt{\frac{\lambda\mathcal{E}_{\mathrm{C},1,j}}{N_0/2}}) \\ &\le 1/2\exp(-\frac{(\sqrt{\lambda}-1)^2\mathcal{E}_{\mathrm{C},1,j}}{N_0}). \end{aligned} \tag{24}$$

obtained through using the bound on the $Q_1(\alpha, \beta)$ given in [11, eq:4]. And the probability of a misdetected acknowledged error $E_{c\to e}$for U_j is

$$\Pr(E_{c\to e,1,j}) \le \exp\{-\frac{\lambda\mathcal{E}_{\mathrm{C},1,j}}{N_0}\}. \tag{25}$$

The detection rule is defined by

$$U_{k,l} = | < \mathbf{Y}_1, \mathbf{S}_{m_k} > |^2 + | < \mathbf{Y}_2, \mathbf{S}_{m_l} > |^2 \tag{26}$$

using [12, Chapter 12] considering the following 4 possible decision variables in the first round under the assumption of (k, l) is transmitted.

$$U_{k,l} = |\sqrt{\mathcal{E}_{D,1,1}} + N_k|^2 + |\sqrt{\mathcal{E}_{D,1,2}} + N_l|^2 \tag{27}$$

$$U_{k',l} = |N_{k'}|^2 + |\sqrt{\mathcal{E}_{D,1,2}} + N_l|^2 \tag{28}$$

$$U_{k,l'} = |\sqrt{\mathcal{E}_{D,1,1}} + N_k|^2 + |N_{l'}|^2 \tag{29}$$

$$U_{k',l'} = |N_{k'}|^2 + |N_{l'}|^2 \tag{30}$$

An error is committed if any of the $U_{k',l}$, $U_{k,l'}$ and $U_{k',l'}$ is greater than $U_{k,l}$. The union bound on $P_e(k, l)$ is defined as

$$P_e(k, l) \le \sum_{(k',l')\neq(k,l)} \Pr\left(u_{k,l} < u_{k',l'}|(k, l)\right) \tag{31}$$

For the derivation of the union probability of $E_{1,1}, E_{1,2}$, the decision variables given from (27) to (30) are used and combined through the bound (31). In the following, we give the expression for each conditional probability considering each decision variable given (k, l) is transmitted.

$$\begin{aligned}&\Pr(U_{k,l} < U_{k',l'}|(k, l)) \\ &= \Pr(|\sqrt{\mathcal{E}_{D,1,1}}+N_k|^2+|\sqrt{\mathcal{E}_{D,1,2}}+N_l|^2 < |N_{k'}|^2+|N_{l'}|^2)\end{aligned} \tag{32}$$

$$\Pr(U_{k,l} < U_{k',l}|(k, l)) = \Pr(|\sqrt{\mathcal{E}_{D,1,1}} + N_k|^2 < |N_{k'}|^2) \tag{33}$$

$$\Pr(U_{k,l} < U_{k,l'}|(k, l)) = \Pr(|\sqrt{\mathcal{E}_{D,1,2}}+N_l|^2 < |N_{l'}|^2) \tag{34}$$

In [12, p. 686], $P_2(L)$ is defined as the probability of error in choosing between $U_{k,l}$ and any other decision variable $U_{k',l}$, $U_{k,l'}$ or $U_{k',l'}$. Here to bound the union probability of $E_{1,1}, E_{1,2}$, we set $P_2(2)$ for (32), and $P_2(1)$ for (33) and (34) and give the probability of error by

$$\begin{aligned}&\Pr(E_{1,1} \cup E_{1,2}) \\ &\le \left\lceil 2^B\sqrt{\frac{1-\rho^2}{3}}\right\rceil 2^{B-3}\exp\left\{-\frac{\mathcal{E}_{\mathrm{D},1}}{2N_0}\right\}\left(4 + \frac{\mathcal{E}_{D,1}}{N_0}\right) \\ &+ \left\lceil 2^B\sqrt{\frac{1-\rho^2}{3}}\right\rceil 2^{-1}\left(\exp\left\{-\frac{\mathcal{E}_{D,1,1}}{2N_0}\right\} + \exp\left\{-\frac{\mathcal{E}_{D,1,2}}{2N_0}\right\}\right)\end{aligned} \tag{35}$$

To obtain $Pr(E_2)$, decision variables from (27) to (30) will have some additional terms for the second round since the second round includes energy and/or noise terms of both rounds cumulatively. The resulting expression (36) is given on the top of the next page. At the end of the second round, the protocol is terminated with distortion bounded as

$$D(\mathcal{E}, N_0, 2, \lambda) \le 2^{-2B}(1 + \rho^2) + \left(\frac{1-\rho^2}{3}P_{e,1} + 4P_{e,2}\right). \tag{37}$$

which is obtained through

$$D = D_q(1 - P_e) + D_eP_e \le D_q + D_{e,1}P_{e,1} + D_{e,2}P_{e,2} \tag{38}$$

where P_e given by (20) is the total probability of error which consists of $P_{e,1}$ and $P_{e,2}$ indicating the probability of error on one of the sources and both sources, respectively. In (35) and (36), the first terms with the sum energies brings out together $P_{e,2}$, and the ones with individual energies shape $P_{e,1}$. D_q represents the distortion caused by the quantization process and D_e corresponds to the MSE distortion for the case where an error was made. Splitting the distortion for error case, where $D_{e,1}$ denotes the distortion for one source in error and in the same way $D_{e,2}$ denotes the case when both in error.

Through combining (20), (24),(35), (36) and (37), we get the following bound on distortion as

$$\begin{aligned}&D(\mathcal{E}, N_0, 2, \lambda) \le K_1e^{-2B\ln 2+\ln(1+\rho^2)} \\ &+\left(K_2\sqrt{\frac{1-\rho^2}{3}}e^{B\ln 2} + K_3\epsilon(\rho)\right)e^{(B-1)\ln 2-\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{C},1}(\sqrt{\lambda}-1)^2}{2N_0}} \\ &+\left(K_4\sqrt{\frac{1-\rho^2}{3}}e^{B\ln 2} + K_5\epsilon(\rho)\right)e^{\ln\left(\frac{1-\rho^2}{6}\right)-\frac{\mathcal{E}_{\mathrm{D},1}+2\mathcal{E}_{\mathrm{C},1}(\sqrt{\lambda}-1)^2}{4N_0}} \\ &+\left(K_6\sqrt{\frac{1-\rho^2}{3}}e^{B\ln 2} + K_7\epsilon(\rho)\right)e^{(B-5)\ln 2-\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2N_0}} \\ &+\left(K_8\sqrt{\frac{1-\rho^2}{3}}e^{B\ln 2} + K_9\epsilon(\rho)\right)e^{\ln\left(\frac{1-\rho^2}{12}\right)-\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{4N_0}}\end{aligned} \tag{39}$$

where K_1, K_4, K_5 are $O(1)$, K_2, K_3 are $O(\mathcal{E}_{\mathrm{D},1})$, K_6, K_7, K_8, K_9 are $O((\mathcal{E}_{\mathrm{D},1} + \mathcal{E}_{\mathrm{D},2})^3)$ with $\epsilon(\rho) \in [0, 1)$ which arose from the ceiling functions in (35) and (36). For a very weak correlation between U_1 and U_2 the terms with $\epsilon(\rho)$ in (39) become insignificant. Equating the order of the exponentials in the first case, we can set the relations of the energies as $\mathcal{E}_{\mathrm{C},1} = \frac{\mathcal{E}_{\mathrm{D},2}}{2(\sqrt{\lambda}-1)^2}$ and $\mathcal{E}_{D,2} = (2 - \mu)\mathcal{E}_{D,1}$ where

$$\Pr(E_2) \leq \left\lceil 2^B \sqrt{\frac{1-\rho^2}{3}} \right\rceil 2^{B-7} \exp\left\{ -\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2N_0} \right\} \left(64 + 29\left(\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{N_0}\right) + 8\left(\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{N_0}\right)^2 + \left(\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{N_0}\right)^3 \right)$$
$$+ \left\lceil 2^B \sqrt{\frac{1-\rho^2}{3}} \right\rceil 2^{-3} \left(4 + \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2N_0}\right) \left(\exp\left\{ -\frac{\mathcal{E}_{\mathrm{D},1,1}+\mathcal{E}_{\mathrm{D},2,1}}{2N_0} \right\} + \exp\left\{ -\frac{\mathcal{E}_{\mathrm{D},1,2}+\mathcal{E}_{\mathrm{D},2,2}}{2N_0} \right\} \right) \quad (36)$$

μ is an arbitrary constant within the interval $(0, 2)$. Thus the bound on distortion for constant correlation level becomes

$$D_{low}(\mathcal{E}, N_0, 2) \leq K_{low,1}\sqrt{2(1+\rho^2)}e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/2)}{2N_0}} + K_{low,2}(1+\rho^2)^{1/3}\left(\frac{1-\rho^2}{6}\right)^{2/3} e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/3)}{2N_0}} + K_{low,3}\sqrt{2(1+\rho^2)}e^{-\frac{\mathcal{E}_{\mathrm{D},1}(3-\mu)}{4N_0}} \quad (40)$$

where $K_{low,1} = K_2 + K_3$, $K_{low,2} = K_4 + K_5 + K_8 + K_9$ and $K_{low,3} = K_6 + K_7$.

On the other hand, with highly correlated sources, i.e. when $\sqrt{\frac{1-\rho^2}{3}} < \theta 2^{-B}$, we set the relations of the energies as $\mathcal{E}_{C,1} = \frac{\mathcal{E}_{D,2}}{(1-\sqrt{\lambda})^2}$ and $\mathcal{E}_{D,2} = (2-\mu)\mathcal{E}_{D,1}$ where μ is an arbitrary constant satisfying $\mu \in (0, 2)$. And the bound in case of a high correlation becomes

$$D_{high}(\mathcal{E}, N_0, 2) \leq K_{high}(1+\rho^2)^{2/3}e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/3)}{N_0}} \quad (41)$$

where $K_{high} = K_3 + K_7$. To simplify the calculations the energy used by a source on a particular phase is assumed to be half of the energy on the corresponding round, for example $\mathcal{E}_{D,1} = 2\mathcal{E}_{D,1,1} = 2\mathcal{E}_{D,1,2}$. Note that the exponential behaviour observed in (41), is the same with a single source yields in [2]. Furthermore, there is a difference of factor $1/2$ in the exponentials of (41) and the information theoretic bounds (8), (11) and (17).

IV. Conclusion

Lower bounds on the distortion level, which is caused by quantization applied to the sources and by the channel itself, are derived for correlated analog dual-sources in the presence of causal feedback. An improvement respect to the performance achieved in [1] is obtained in terms of the asymptotic behaviour of the derived bounds on distortion with additional feedback. The information theoretic bounds introduced in Section II are provided when the random phases are assumed to be perfectly known at the receiver, which is not a realistic assumption for achievable schemes. On the other hand, the bounds provided in Section III were derived through non-coherent detection. Another point worths mentioning is the discussion made in Section II-A regarding to the comparison between the performance of a single source and two highly correlated sources. It was shown that, highly correlated dual-sources can achieve the performance of a single source in terms of distortion level. Lastly, it is seen through a comparison between the outer bounds and the bounds obtained for the achievable scheme that the performance of the theoretic bounds can be reached by the protocol repeating more than two rounds.

V. Appendix

A. Appendix I

The mutual information $I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'})$ is derived through two different expansion as given in the following respectively.

$$\begin{aligned} I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'}) &= I(\mathbf{U}_m; \mathbf{Y}|\Phi_m, \Phi_{m'}) + I(\mathbf{U}_m; \Phi_m, \Phi_{m'}) \\ &= h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m|\mathbf{Y}) \\ &\geq h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m). \end{aligned} \quad (42)$$

$h(\mathbf{U}_m)$ differs based on m as given in the following.

$$h(\mathbf{U}_1) = K \log 2\sqrt{3}, \quad (43)$$

$$\begin{aligned} h(\mathbf{U}_2) &= h(\rho\mathbf{U}_1 + \sqrt{1-\rho^2}\mathbf{U}_2') \\ &\geq K \log\left(2^{\frac{1}{K}(K\log|\rho| + h(\mathbf{U}_1))} + 2^{\frac{1}{K}h(\sqrt{1-\rho^2}\mathbf{U}_2')}\right) \\ &= K\log\left(2^{\frac{1}{K}K\log|\rho|2\sqrt{3}} + 2^{\frac{1}{K}K\log 2\sqrt{3|1-\rho^2|}}\right) \\ &= K\log\left(|\rho|2\sqrt{3} + 2\sqrt{3|1-\rho^2|}\right). \end{aligned} \quad (44)$$

The final term required to derive (42) is given by

$$\begin{aligned} h(\mathbf{U}_m - \hat{\mathbf{U}}_m) &\leq \sum_{j=1}^{K} h(U_{m,j} - \hat{U}_{m,j}) \\ &\leq \frac{K}{2}\log\left(2\pi e \frac{1}{K}\sum_{j=1}^{K}\mathbb{E}[(U_{m,j} - \hat{U}_{m,j})^2]\right) \\ &\leq K\log\left(\sqrt{2\pi e D_m}\right) \end{aligned} \quad (45)$$

which is the same for both values of m. On the other hand, we have

$$\begin{aligned} & I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'}) = h(\mathbf{Y}|\Phi_m, \Phi_{m'}) - h(\mathbf{Y}|\mathbf{U}_m, \Phi_m, \Phi_{m'}) \\ &= \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \Phi_m, \Phi_{m'}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_m, \Phi_m, \Phi_{m'}) \\ &\leq \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \Phi_m, \Phi_{m'}) \\ &- \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_m, \mathbf{X}_m e^{j\phi_m}, \mathbf{X}_{m'} e^{i\phi_{m'}}, \Phi_{m'}, \Phi_m) \\ &= \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \Phi_m, \Phi_{m'}) - \sum_{i=1}^{N} h(Z_i) \\ &\leq \sum_{i=1}^{N} \log\left(1 + \frac{E_{m,i} + E_{m',i}}{NN_0}\right) \\ &\leq N\log\left(1 + \frac{\sum_{i=1}^{N}(E_{m,i} + E_{m',i})}{NN_0}\right) \\ &\leq N\log\left(\frac{\sum_{i=1}^{N} Var(A_i(\phi))}{NN_0}\right) \end{aligned} \quad (46)$$

where $A_i(\phi)$ is defined as $A_i(\phi) \triangleq X_{m,i}e^{i\phi_m} + X_{m',i}e^{i\phi_{m'}} + Z_i$ and its variance is obtained as $\sum_{i=1}^{N} Var(A_i(\phi)) = K(E_m + E_{m'}) + NN_0$. Consequently, the desired mutual information is obtained as given by (5). Equating two expansions of $I(\mathbf{U}_m; \mathbf{Y}, \Phi_m, \Phi_{m'})$, we obtain the bound (7).

B. Appendix II

The first expansion of $I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'})$ is as follows

$$\begin{aligned} I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) &= h(\mathbf{U}_m|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \\ &\quad - h(\mathbf{U}_m|\mathbf{U}_{m'}, \mathbf{Y}, \Phi_m, \Phi_{m'}) \\ &= h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m|\mathbf{U}_{m'}, \mathbf{Y}) \\ &\geq h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m). \end{aligned} \tag{47}$$

The entropy $h(\mathbf{U}_m|\mathbf{U}_{m'})$ is obtained for $m = 1$ and $m = 2$ as

$$\begin{aligned} h(\mathbf{U}_1|\mathbf{U}_2) &= -I(\mathbf{U}_1; \mathbf{U}_2) + h(\mathbf{U}_1) \\ &= -h(\mathbf{U}_2) + h(\mathbf{U}_2|\mathbf{U}_1) + h(\mathbf{U}_1) \\ &\overset{(a)}{\geq} -\frac{K}{2}\log 2\pi e + K \log 2\sqrt{3|1-\rho^2|} + K \log 2\sqrt{3} \\ &= K \log\left(\frac{12\sqrt{|1-\rho^2|}}{\sqrt{2\pi e}}\right) \end{aligned} \tag{48}$$

$$\begin{aligned} h(\mathbf{U}_2|\mathbf{U}_1) &= h(\rho\mathbf{U}_1 + \sqrt{1-\rho^2}\mathbf{U}_{2'}|\mathbf{U}_1) \\ &= h(\sqrt{1-\rho^2}\mathbf{U}_{2'}) \\ &= K \log 2\sqrt{3|1-\rho^2|} \end{aligned} \tag{49}$$

,respectively. In step (a) of (48), the entropy of U_2 is bounded by the entropy of a standard gaussian random variable. And we also have

$$I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \tag{50}$$

$$\begin{aligned} &= h(\mathbf{Y}|\mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) - h(\mathbf{Y}|\mathbf{U}_m, \mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \\ &= \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_m, \mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) \\ &= \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_{m'}, \mathbf{X}_{m'}e^{i\phi_{m'}}, \Phi_m, \Phi_{m'}) \\ &- \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_m, \mathbf{U}_{m'}, \mathbf{X}_m e^{j\phi_m}, \mathbf{X}_{m'}e^{i\phi_{m'}}, \Phi_m, \Phi_{m'}) \\ &\overset{(a)}{=} \sum_{i=1}^{N} h(X_{m,i}e^{j\phi_{m,i}} + Z_i|Y^{i-1}, \mathbf{U}_{m'}, \Phi_m, \Phi_{m'}) - \sum_{i=1}^{N} h(Z_i) \\ &\leq \sum_{i=1}^{N} h(X_{m,i}e^{j\phi_{m,i}} + Z_i) - \sum_{i=1}^{N} h(Z_i) \\ &= N \log\left(1 + \frac{KE_m}{NN_0}\right) \end{aligned} \tag{51}$$

C. Appendix III

The mutual information $I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}|\mathbf{\Phi})$ is obtained as

$$\begin{aligned} I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}|\mathbf{\Phi}) &= h(\mathbf{Y}|\mathbf{\Phi}) - h(\mathbf{Y}|\mathbf{U}_1, \mathbf{U}_2, \mathbf{\Phi}) \\ &= h(\mathbf{Y}|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_1, \mathbf{U}_2, \mathbf{\Phi}) \\ &\leq \sum_{i=1}^{N} h(Y_i|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1}, \mathbf{U}_1, \mathbf{X}_1, \mathbf{U}_2, \mathbf{X}_2, \mathbf{\Phi}) \\ &= \sum_{i=1}^{N} h(Y_i|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Z_i). \end{aligned} \tag{52}$$

The variance of the received signal Y_i becomes $\sum_{i=1}^{N} Var(Y_i) = K(E_1 + E_2) + NN_0$ and the desired mutual information is obtained as

$$I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}|\mathbf{\Phi}) \leq N \log(1 + \frac{K(E_1 + E_2)}{NN_0}). \tag{53}$$

We also have

$$\begin{aligned} I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}|\mathbf{\Phi}) &= I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}, \mathbf{\Phi}) \\ &\geq I(\mathbf{U}_1, \mathbf{U}_2; \hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2) \\ &\geq h(\mathbf{U}_1, \mathbf{U}_2) - h(\mathbf{U}_1 - \hat{\mathbf{U}}_1) - h(\mathbf{U}_2 - \hat{\mathbf{U}}_2) \\ &\geq \frac{K}{2}\log 144(1-\rho^2) - \frac{K}{2}\log(2\pi e)^2 D_1 D_2 \\ &= \frac{K}{2}\log(\frac{36(1-\rho^2)}{\pi^2 e^2 D_1 D_2}). \end{aligned} \tag{54}$$

Two expressions ((53) and (54)) of the same mutual information are equalized to obtain (14).

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