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Dynamic Rate Allocation in Markovian Quasi-Static Multiple Access Channels

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Abstract

We deal with multiple access channels in which the channel coefficients follow a quasi-static Markov process on a finite set of states. We address the issue of allocating the rate to the users in each time interval, such that the optimality and the fairness of the allocation are preserved throughout the communication, and moreover all the users are consistently satisfied with it. We first show how to allocate the rates in a global optimal fashion. We give a sufficient condition for the optimal rates to fulfil some fairness criteria in a time consistent way. We then utilize the game-theoretical concepts of time consistent Core and Cooperation Maintenance and we show that in our model the sets of rates fulfilling these properties coincide, and they also coincide with the set of global optimal rate allocations. The relevance of our dynamic rate allocation to LTE systems is also shown.

Index Terms

Quasi-static fading, fair rate allocation, Dynamic Cooperative Game theory.

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1 Introduction

In the last few years, the concepts of user fairness and satisfaction have received significant attention. These notions will play an increasingly crucial role in future networks, due to the paradigm shift that we are witnessing, from fully centralized with dumb terminals to distributed networks with rational users able to pool resources with each other.
In the literature, the notion of fair and satisfactory rate allocation has been dealt with under manifold perspectives in static Gaussian or ergodically fading Multiple Access Channels (MAC). In [1], the fairness of a rate allocation in a Gaussian MAC is related to the economical concept of Lorenz order, used for measuring disparity in income distributions. Such fair allocation always exists, it is Pareto optimal, and also solution of a Nash bargaining problem with zero disagreement payoff allocation. In the following [2], the authors show the existence of a unique rate allocation which is max-min and proportional fair. The results in [1, 2] are extended to the general framework of α-fairness [3] in [4]. For MAC's with polymatroid regions, all α-fair rate allocations collapse into a single point, which is max-min and proportional fair, too. An analysis of rate allocations in the context of constrained games points out that the normal Nash equilibrium [5] also coincides with the α-fair and Pareto optimal allocations.
Furthermore, the issue of users satisfaction is addressed by Cooperative Game Theory (CGT) with non-transferable utility (NTU) (see [6] for an overview), which provides powerful tools to derive efficient and stable allocations in a setting in which the users can cooperate to reach a common goal. In [7], the capacity of the Gaussian MAC is studied with a game-theoretical approach. In [8], the authors expressed the rate allocation problem in static Gaussian MAC with jamming in a cooperative game-theoretical setting. They found a satisfactory rate allocation fulfilling the newly introduced concept of envy-free. The envy-free allocation exists, is unique and Pareto optimal, but in general it does not coincide with the α-fair solution.
In this contribution we study and extend for the first time the concepts of optimal, fair, and satisfactory rate allocations to a *dynamic* scenario, described by a Gaussian MAC where the channel evolves quasi-statically, according to a Homogeneous Markov Chain (HMC) on a finite state space.
We stress the scenario that we consider is relevant for the modern LTE systems. In fact, in LTE, the average channel state information is estimated by the receiver and fed back to each transmitter at regular intervals. Hence, in each of these intervals, a different rate for each user needs to be allocated and it is desirable that fairness and users' satisfaction is guaranteed along the course of the communication.

The paper is structured into two main sections. The former is Sect. 3, in which we discuss the *design of optimal and fair allocations* in a dynamic process. The latter is Sect. 4, in which we *characterize the optimal rate allocations as the allocations which are also satisfactory throughout the communication*, according to a

Dynamic Cooperative Game Theory (DCGT) formulation. We study a bottom-up (Sect. 3.1) and a top-down procedure (Sect. 3.2) to allocate a global optimal rate in each state of the HMC. The former prescribes to allocate first the static allocations and derive next the long-run ones; conversely, the latter suggests to select first the long-run rate allocations. Though the top-down procedure would be more useful since the user have a long-run perspective, it is not always feasible since it is described by a non bijective mapping. We then suggest a procedure to overcome this problem. In Sect. 4 we provide a sufficient condition under which there exists a rate allocation which is fair, i.e. max-min, proportional, and α-fair, both state-wisely and in the long-run process. Most importantly, the fairness property of such allocation is time consistent, i.e. it is fair throughout the process, from any intermediate step onwards. Conversely, a fair allocation always exists in the static case [4], [9]. We remark that all our results in Sect. 3 apply to any communication system characterized by a polymatroid capacity structure (see [4] for some examples).

In Sect. 4 we introduce a game formulation with jamming users similar to the one in [8], but in a dynamic scenario. We then characterize the set of global optimal allocations as satisfactory too, since it coincides with the set of rates for which two crucial DCGT properties hold. These properties are the (time consistent) Core, introduced in [10], and the Cooperation Maintenance property [11]. Such properties formulate the concept of acceptable allocations throughout a dynamic process for all users in two different, but equally appealing, manners.

We refer the reader to [12] for the proofs of all our results, omitted here to comply with the space constraint.

2 System Model

We consider a wireless system in which K terminals attempt to send information to a single receiver or base station. Let $\mathcal{K} = \{1, \ldots, K\}$ be the set of all users. Each user k has a power constraint P_k. We assume a quasi-static channel, i.e. the channel coefficients can be considered constant for the whole duration of a codeword. Thus, the t-th signal block received by the unique receiver, for $t \in \mathbb{N}_0$, can be written as

$$\mathbf{y}[t] = \sum_{k=1}^{K} h^{(k)}[t]\, \mathbf{x}^{(k)}[t] + \mathbf{w}[t],$$

where $\mathbf{x}^{(k)}[t]$ is the codeword of user k, $h^{(k)}[t]$ is the complex channel coefficient for user k at time step t, and $\mathbf{w}[t]$ is zero mean white Gaussian noise with variance N_0. We assume that the set of channel coefficients $\{h^{(1)}, h^{(2)}, \ldots, h^{(K)}\}$ is finite and it follows a discrete time HMC, which can change state at every new codeword. In other words, if S_t is the channel state at time step t, where

$$S_t := \left[h^{(1)}[t], \ldots, h^{(K)}[t]\right],$$

then the random process $\{S_t,\ t \geq 0\}$ is a HMC. We define $\mathcal{S}$ as the set of all the N possible states of the HMC. Let $\mathbf{P}$ be its N-by-N transition probability matrix, such that $P_{i,j}$ is the probability of transition from state s_i to state s_j.

We point out that the codeword length is supposed to be very long, such that the conditions of applicability of the Shannon Capacity (i.e. infinite codeword) are practically satisfied. This assumption is widely applied in quasi-static channels (see e.g. [13]).

2.1 Markovian feasibility region

In each channel state, we consider a Gaussian MAC scenario, in which K users communicate with a single receiver. By relying on the classic quasi-static approximation assumption (see e.g. [13]), we can compute the capacity rate region for all users in state s as the polymatroid $\mathcal{R}(\mathcal{K}, s)$ with rank function $g_{(\mathcal{K})}$ [14]:

$$\mathcal{R}(\mathcal{K}, s) = \left\{ \mathbf{r} \in \mathbb{R}^K : \sum_{k \in \mathcal{T}} r_k \leq g_{(\mathcal{K})}(\mathcal{T}, s),\ \forall\, \mathcal{T} \subseteq \mathcal{K} \right\}$$
$$g_{(\mathcal{K})}(\mathcal{T}, s) := C\left(\sum_{k \in \mathcal{T}} |h^{(k)}(s)|^2 P_k, N_0 \right), \quad \forall\, \mathcal{T} \subseteq \mathcal{K}, \tag{1}$$

where $C(a, b) = \log_2(1 + a/b)$. When considering the channel dynamics, an HMC evolves on a finite set of channel states $\mathcal{S} = \{s_1, \ldots, s_N\}$. Since we consider the channel to be constant during a codeword, the transition among states occurs at the end of each coherence period of the channel.

We allocate a rate to each user in each of the state of the Markov chain. We assume that the rate assigned in state $S_t \in \mathcal{S}$ at time t depends only on the value of S_t, and not on the past history of state/allocations up to time t. In this sense, we say that the dynamic allocation is *stationary*, and we call $r_k(s)$ the rate assigned to user k in state s. In our model the users prefer the current rate allocation over the future ones, which are discounted by a factor $\beta \in [0; 1)$. This assumption has been widely adopted in the literature on game theory for networks (see e.g. [15]). In this case, the *utility* for user k over the whole stream of state-wise rate allocations equals

$$r_k(\Gamma_s) = \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t\, r_k(S_t) \right), \tag{2}$$

where Γ_s is the Markov process starting at time 0 in state s. An alternative interpretation of (2) is the actual expected long-run rate when the length of the communication is finite, but of unknown duration; $1 - \beta$ is the probability that, at any time step, the communication terminates. In the literature on dynamic games it is common to multiply expression (2) by the normalization factor $(1 - \beta)$. We anticipate that both the normalization factor and the choice of β are irrelevant to all our results. By recalling the relation $\sum_{t \geq 0} \beta^t \mathbf{P}^t = (\mathbf{I} - \beta \mathbf{P})^{-1}$, we can write (2) in the following matricial form:

$$\begin{bmatrix} \mathbf{r}(\Gamma_{s_1}) \\ \vdots \\ \mathbf{r}(\Gamma_{s_N}) \end{bmatrix} = (\mathbf{I} - \beta \mathbf{P})^{-1} \begin{bmatrix} \mathbf{r}(s_1) \\ \vdots \\ \mathbf{r}(s_N) \end{bmatrix}, \tag{3}$$

where $\mathbf{r}(s) := [r_1(s), r_2(s), \ldots, r_K(s)]$ and $\mathbf{r}(\Gamma_s)$ is defined similarly. By defining $\Phi := (\mathbf{I} - \beta\mathbf{P})^{-1}$ and utilizing a compact matrix notation, we rewrite (3) as

$$[\mathbf{r}(\Gamma_s)]_{s \in \mathcal{S}} = \Phi \, [\mathbf{r}(s)]_{s \in \mathcal{S}} \tag{4}$$

Remark 1. *Expression (4) defines an application from the set of stationary state-wise rate allocations to the set of feasible long-run rates. In Sect. 3.2 we will show that, in general, the application is* not invertible, *since multiplying a set of long-run allocations by Φ^{-1} does not always produce feasible state-wise allocations.* □

It is natural to define the long-run rate region $\mathcal{R}(\mathcal{K}, \Gamma_s)$ as the set of all rates $\mathbf{r}(\Gamma_s)$ that can be written as the long-run expected sum of stationary state-wise rate allocations, as in (3). We now give a convenient expression for $\mathcal{R}(\mathcal{K}, \Gamma_s)$, which follows from [16], p. 241, Theorem 12.1.5, claiming that the sum of polymatroids is still a polymatroids whose rank function is the sum of the rank functions of the summands.

Lemma 21. *For any $s_j \in \mathcal{S}$, the long-run rate feasibility region $\mathcal{R}(\mathcal{K}, \Gamma_{s_j})$ is a polymatroid with rank function:*

$$g_{(\mathcal{K})}(\mathcal{T}, \Gamma_{s_j}) = \sum_{n=1}^{N} \nu_n(s_j)\, g_{(\mathcal{K})}(\mathcal{T}, s_n), \quad \forall \mathcal{T} \subseteq \mathcal{K},$$

where $\boldsymbol{\nu}(s_j)$ is the j-th row of the matrix Φ. □

2.2 Relevance to LTE systems

In LTE systems, the statistics of the channel are estimated at regular intervals and used for resource allocation. Under the common assumption of fast fading Gaussian channel in additive Gaussian noise, in each period t the state of the HMC is given by the channel distribution, completely characterized by its second-order statistics. The rate region in absence of instantaneous knowledge of the channel at the transmitter is still a polymatroid, with rank function $\mathbb{E}_h[g_{(\mathcal{K})}(\mathcal{T}, s)]$, as shown in [17]. Since the results presented in the following strongly rely on the polymatroid structure of the rate region in each state of the HMC, then they also hold for LTE systems. Hence, our general results in particular address the issue of allocating the rate to users in a MAC LTE system at each feed-back time interval, so that optimality, fairness, and the users' satisfaction is preserved throughout the communication.

3 Optimal and fair rate allocation design

In this section we address the issue of allocating the rate to all users during the transmission process, in each state of the channel Markov chain. We stress that all the results in this section apply to any communication system in which the capacity region in the single channel state has a polymatroid structure (see [4] for a list of such systems).
For a classic result on polymatroids (see e.g. [16]), we know that the dominant facet, or simply facet, $\mathcal{M}(\mathcal{R}(\mathcal{K},s))$ of the rate region $\mathcal{R}(\mathcal{K},s)$ is maximum sum-rate, i.e.

$$\mathcal{M}(\mathcal{K},s) := \mathcal{M}(\mathcal{R}(\mathcal{K},s)) = \underset{\mathbf{r}\in\mathcal{R}(\mathcal{K},s)}{\operatorname{argmax}} \sum_{k\in\mathcal{K}} r_k. \tag{5}$$

Similarly, the facet $\mathcal{M}(\mathcal{K},\Gamma_s)$ is maximum sum-rate in the long-run process Γ_s. Hence, the global optimum rate design solution would be that both the state-wise and the long-run rate allocations belong to the facets $\mathcal{M}(\mathcal{K},s)$ and $\mathcal{M}(\mathcal{K},\Gamma_s)$, for all $s \in \mathcal{S}$. Hence, we will restrict our focus on the allocations inside $\mathcal{M}$, defined as in the following.

Definition 1 ($\mathcal{M}$)**.** *$\mathcal{M}$ is the set of stationary state-wise allocations belonging to the dominant facets of both state-wise and long-run feasibility regions, i.e.*

$$\mathcal{M} := \Big\{ \{\mathbf{r}(s)\}_{s\in\mathcal{S}} : \mathbf{r}(s) \in \mathcal{M}(\mathcal{K},s), \quad \mathbf{r}(\Gamma_s) \in \mathcal{M}(\mathcal{K},\Gamma_s), \, \forall\, s \in \mathcal{S} \Big\},$$
$$\text{where } [\mathbf{r}(\Gamma_s)]_{s\in\mathcal{S}} = \Phi\, [\mathbf{r}(s)]_{s\in\mathcal{S}}. \qquad \square$$

Now, we will investigate two different approaches to select an allocation in $\mathcal{M}$. The first, called bottom-up procedure (Sect. 3.1), is the most natural one, and it prescribes to select a set of state-wise allocations in $\mathcal{M}(\mathcal{K},s)$, for all $s \in \mathcal{S}$, and then to derive the set of associated long-run allocations via multiplication by Φ. Conversely, the second approach, dubbed top-down (Sect. 3.2), would be more useful, but unfortunately it is not always feasible. It suggests to select first the long-run allocations, in $\mathcal{M}(\mathcal{K},\Gamma_s)$, for all $s \in \mathcal{S}$, and then to multiply by Φ^{-1} to obtain the state-wise allocations. Clearly, the choice over the adopted procedure depends on the priority that the designer gives to the state-wise/long-run allocation. By adopting the top-down procedure, one embraces a long-run perspective of the process, by preferring to adhere to a specific fairness selection criterion in the long-run process, rather than in the state-wise one. We anticipate from Sect. 3.3 that one can select the unique allocation point in the long-run process, that is α-fair, proportional fair, and max-min fair, simultaneously. Clearly, the best scenario would consist in being fair in each state, in the long-run process, and from each intermediate step onwards. A sufficient condition to attain this will be provided in Sect. 3.3.

3.1 BOTTOM-UP DESIGN: From single-stage to long-run allocations

In this section we investigate the feasibility of our first procedure to select an allocation in $\mathcal{M}$. It is called *bottom-up* rate allocation approach, and it consists in selecting a set of stage-wise allocations belonging to the dominant facet of each state-wise feasibility region. Then, we need to compute the respective long-run allocations and check whether they belong to the dominant facets of the feasibility region of the respective long-run processes. By a linearity argument, it is easy to see that the facet $\mathcal{M}(\mathcal{K}, \Gamma_s)$ is obtained as the Minkowski sum $\sum_{n=1}^{N} \nu_n(s)\mathcal{M}(\mathcal{K}, s_n)$. Therefore, if the state-wise allocations all belong to the dominant facet in the respective states, then their expected long-run sum also lies in the dominant facet of the long-run process. Then, the bottom-up procedure always produces stationary allocations belonging to $\mathcal{M}$.

Proposition 31 (Bottom-up allocation procedure)**.** *Select a set of state-wise rate allocations* $\{\mathbf{r}(s) \in \mathcal{M}(\mathcal{K}, s)\}_{s\in\mathcal{S}}$. *Then, their associated long-run allocations* $[\mathbf{r}(\Gamma_s)]_{s\in\mathcal{S}} = \Phi[\mathbf{r}(s)]_{s\in\mathcal{S}}$ *belong to the respective long-run dominant facets, i.e.* $\mathbf{r}(\Gamma_s) \in \mathcal{M}(\mathcal{K}, \Gamma_s)$*, for all* $s \in \mathcal{S}$. □

Then, the first positive result of Proposition 31 is that there exist allocations belonging to the dominant facet of both state-wise and long-run processes, jointly, i.e. $\mathcal{M}$ is non-empty. Secondly, it is easy to find them, since it suffices to select a rate allocation on the dominant facet of $\mathcal{R}(\mathcal{K}, s)$, for all $s \in \mathcal{S}$. Finally, as a by-product of Proposition 31, we are allowed to simplify the definition of $\mathcal{M}$ as:

$$\mathcal{M} \equiv \Big\{\{\mathbf{r}(s)\}_{s\in\mathcal{S}} \text{ s.t. } \mathbf{r}(s) \in \mathcal{M}(\mathcal{K}, s),\ \forall\, s \in \mathcal{S}\Big\}.$$

Proof. If $\mathbf{r}(n) \in \mathcal{M}(\mathcal{R}(n))$, for all $n = 1, \ldots, N$, then trivially $\sum_{n=1}^{N} \mathbf{r}(n) \in \mathcal{M}(\mathcal{R})$. Conversely, fix $\mathbf{r} \in \mathcal{M}(\mathcal{R})$. We know from [16], p. 241, Theorem 12.1.5, that there exist $\{\mathbf{r}(n) \in \mathcal{R}(n)\}_{n=1,\ldots,N}$ such that $\mathbf{r} = \sum_{n=1}^{N} \mathbf{r}(n)$. If $\mathbf{r}(n) \notin \mathcal{M}(\mathcal{R}(n))$ for some n, then there would exist $\mathbf{r}' \in \mathcal{M}(\mathcal{R})$ such that $\sum_{k=1}^{K} r'_k > \sum_{k=1}^{K} r_k$, which is impossible. Hence, the thesis is proven. □

3.2 TOP-DOWN DESIGN: From long-run to single-stage allocations

The bottom-up procedure always produces feasible allocations, but it is not what really concerns us. Indeed, the users are endowed with a long-term perspective of the communication process, hence one may wish to select first a set of long-run allocations in $\{\mathcal{M}(\mathcal{K}, \Gamma_s)\}_{s\in\mathcal{S}}$ which adhere to a certain criterion in the respective long-run processes (e.g. a fairness criterion, as in Sect. 3.3). Then, the state-wise rate allocations $\{\mathbf{r}(s)\}_{s\in\mathcal{S}}$ are obtained via multiplication by Φ^{-1}. Unfortunately this method, dubbed *top-down*, does not always produces feasible stationary state-wise allocations. We interpret this fact by saying that the linear application defined by Φ in (4) is not always invertible in the space of feasible stationary allocations. In Example 31 we show an instance of the described scenario.

Example 31. *Set* $\beta = 0.8$, $N_0 = 0.1\,W$. *Consider two users, with power constraints* $P_1 = P_2 = 2\,W$. *Consider two states. In* s_1, $|h^{(1)}(s_1)|^2 = 0.1$, $|h^{(2)}(s_1)|^2 = 0.2$. *In* s_2, $|h^{(1)}(s_2)|^2 = 0.15$, $|h^{(2)}(s_2)|^2 = 0.15$. *The transition probability matrix is* $\mathbf{P} = [0.8\ 0.2;\ 0.3\ 0.7]$. *Choose the optimal allocations in the long-run process*

$$\mathbf{r}(\Gamma_{s_1}) = [0.5843;\ 1.1109] \in \mathcal{M}(\mathcal{K}, \Gamma_{s_1})\ \text{bits/s/Hz}$$
$$\mathbf{r}(\Gamma_{s_2}) = [0.8270;\ 0.8682] \in \mathcal{M}(\mathcal{K}, \Gamma_{s_2})\ \text{bits/s/Hz.}$$

The corresponding state-wise allocations, through Φ^{-1}, *are both not feasible, because*

$$\mathbf{r}(s_1) \cong [0.0780;\ 0.2610] \notin \mathcal{R}(\mathcal{K}, s_1)$$
$$\mathbf{r}(s_2) \cong [0.2236;\ 0.1154] \notin \mathcal{R}(\mathcal{K}, s_2).$$
□

Remark 2. *One may argue that there is no need to select the whole set of long-run allocations* $\{\mathbf{r}(\Gamma_s)\}_{s\in\mathcal{S}}$, *but only the one corresponding to the actual initial state. Indeed, since the channel state* S_0 *at time 0 is known, one could select* $\mathbf{r}(\Gamma_{S_0})$ *according to the desired criterion and then compute the state-wise allocations by choosing one solutions among the infinite possible of the equation*

$$\mathbf{r}(\Gamma_{S_0}) = \sum_{n=1}^{N} \nu_n(S_0)\,\mathbf{r}(s_n).$$

Finally, the remaining long-run allocations are automatically computed by re-inverting the relation, as $\Phi[\mathbf{r}(s)]_{s\in\mathcal{S}}$. *Of course, in this way there is no control over the long-run allocations* $\mathbf{r}(\Gamma_s)$, *with* $s \neq S_0$.
On the other hand, thanks to the stationarity of the payoff allocation, the long-run sub-process starting at time $T > 0$ *is precisely the* β^T*-scaled version of* Γ_{S_T}, *i.e.*

$$\mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}(S_t)\ \middle|\ \mathbf{h}(T)\right) = \beta^T \mathbf{r}(\Gamma_{S_T}),$$

where $\mathbf{h}(T)$ *is the history of state/allocations from time 0 up to time* T. *Therefore, jointly choosing the long-run allocations* $\mathbf{r}(\Gamma_s)$ *for* all *states* $s \in \mathcal{S}$ *is equivalent to assign the long-run allocations that each user obtains in each sub-process from any intermediate time step* $T \geq 0$ *onwards.* □

Example 31 seems to discourage a top-down allocation procedure. Indeed in general, if one chooses a set of long-run allocations, there is no guarantee that the allocation is actually feasible, since the associated stationary state-wise allocation might be not feasible. Of course, this does not rule out the possibility to carry out a

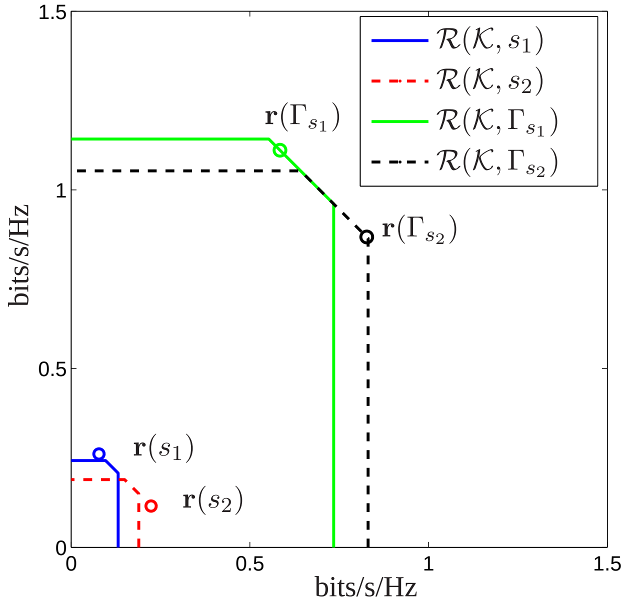

Figure 1: *Example 31.* $\mathbf{r}(\Gamma_s) \in \mathcal{M}(\mathcal{K}, \Gamma_s)$*, for* $s = s_1, s_2$*, but* $\mathbf{r}(s) \notin \mathcal{R}(\mathcal{K}, s)$*, for* $s = s_1, s_2$*, where* $[\mathbf{r}(s)]_{s\in\mathcal{S}} = \Phi^{-1}[\mathbf{r}(\Gamma_s)]_{s\in\mathcal{S}}$*.*

top-down allocation procedure successfully. Indeed, in Theorem 33 we will present a top-down procedure guaranteeing the feasibility of the associated state-wise rate allocations. Before, let us introduce a classic result on polymatroids (see [18]). Let $\mathcal{R}$ be a polymatroid on the ground set $\{1, \dots, K\}$, with rank function g. Let $\Pi(K)$ be the set of permutations of $\{1, \dots, K\}$. The facet $\mathcal{M}(\mathcal{R})$ has at most $K!$ extreme points, and each of them has an explicit characterization as a function of the rank function g. Indeed, $\mathbf{w}$ is a vertex of $\mathcal{M}(\mathcal{R})$ if and only if there exists a permutation π of $\{1, \dots, K\}$ such that, for all $k = 1, \dots, K$,

$$\mathbf{w}_k = g(\{\pi_1, \dots, \pi_{k-1}, \pi_k\}) - g(\{\pi_1, \dots, \pi_{k-1}\}) := \mathbf{w}_k(\pi).$$

Proposition 32. *Let* $a_n \geq 0$*, for* $n = 1, \dots, N$*. Let* $\mathcal{R}_1, \dots, \mathcal{R}_N$ *be* N *polymatroids on the ground set* $\{1, \dots, K\}$*. Let* $\mathcal{R} = \sum_{n=1}^{N} a_n \mathcal{R}_n$*. Let* $\mathbf{w}(\pi)(n)$ *be the vertex of the facet* $\mathcal{M}(\mathcal{R}_n)$ *associated to the permutation* $\pi \in \Pi(K)$*. Let* $\mathbf{w}(\pi)$ *be a vertex of* $\mathcal{M}(\mathcal{R})$*. Then,*

$$\mathbf{w}(\pi) = \sum_{n=1}^{N} a_n \, \mathbf{w}(\pi)(n), \quad \forall \pi \in \Pi(N). \qquad \square$$

Proposition 32 claims that the vertex of the facet $\mathcal{M}(\mathcal{R})$ associated to the permutation π can be decomposed into the sum of the vertices associated to the same π of each facet $\mathcal{M}(\mathcal{R}_n)$, $n = 1, \dots, N$. Then, our idea is to choose *one* set of convex coefficients, valid for any $s \in \mathcal{S}$, and to define the set of long-run allocations

$\{\mathbf{r}(\Gamma_s) \in \mathcal{M}(\mathcal{K}, \Gamma_s)\}_{s\in\mathcal{S}}$ as the same convex combination of the vertices of the respective dominant facets. The associated state-wise allocations are then obtained as the *same* convex combination of the vertices of the respective state-wise dominant facets, hence they are feasible and optimal.

Theorem 33 (Top-down allocation procedure)**.** *Choose a set of convex coefficients* $\{c(\pi)\}_{\pi\in\Pi(K)}$*, such that* $c(\pi) \geq 0$ *and* $\sum_{\pi\in\Pi(K)} c(\pi) = 1$*. Let* $\mathbf{w}(\pi)(\Gamma_s)$ *be the vertex of* $\mathcal{M}(\mathcal{K}, \Gamma_s)$ *associated to the permutation* π*. Compute the set of long-run allocations as*

$$\mathbf{r}(\Gamma_s) = \sum_{\pi\in\Pi(K)} c(\pi)\, \mathbf{w}(\pi)(\Gamma_s), \quad \forall\, s \in S.$$

Then,

$$[\mathbf{r}(s)]_{s\in S} = \ \Phi^{-1}\, [\mathbf{r}(\Gamma_s)]_{s\in S}$$

is a set of feasible state-wise rate allocations, and moreover $\mathbf{r}(s) \in \mathcal{M}(\mathcal{K}, s)$*, for all* $s \in S$*.* □

Proof. Let us write

$$\begin{bmatrix}\mathbf{r}(s_1)\\ \vdots \\ \mathbf{r}(s_N)\end{bmatrix} = \Phi^{-1} \begin{bmatrix}\sum_{\pi\in\Pi(K)} c(\pi)\mathbf{w}(\pi)(\Gamma_{s_1})\\ \vdots \\ \sum_{\pi\in\Pi(K)} c(\pi)\mathbf{w}(\pi)(\Gamma_{s_N})\end{bmatrix}$$
$$= \sum_{\pi\in\Pi(K)} c(\pi)\Phi^{-1} \begin{bmatrix}\mathbf{w}(\pi)(\Gamma_{s_1})\\ \vdots \\ \mathbf{w}(\pi)(\Gamma_{s_N})\end{bmatrix}.$$

For Proposition 32, we can say that

$$\begin{bmatrix}\mathbf{r}(s_1)\\ \vdots \\ \mathbf{r}(s_N)\end{bmatrix} = \sum_{\pi\in\Pi(K)} c(\pi) \begin{bmatrix}\mathbf{w}(\pi)(s_1)\\ \vdots \\ \mathbf{w}(\pi)(s_N)\end{bmatrix}.$$

Hence, the thesis is proven. □

The top-down allocation procedure provided in Theorem 33 is not the only possible of course, but it leads to an intuitive remark. Each vertex $\mathbf{w}(\pi)(s)$ can be achieved by letting the receiver decode sequentially, in the reverse order of π, the signals coming from each user in channel state $s \in S$, and by considering the signals not decoded yet as Gaussian noise (e.g. see [14]). Therefore, any rate allocation on $\mathcal{M}(\mathcal{K}, s)$ can be achieved by time sharing such decoding configurations, and *the time-sharing procedure is independent of the state* s.
We suggest an interesting future research, which may study how to optimize the convex coefficients $c(\pi)$ to make the resulting long-run allocations globally close

to the set of long-run allocations fulfilling a certain criterion, e.g. the fairness criterion that we will present in the next section.

3.3 FAIR ALLOCATION DESIGN: being fair throughout the process

In this section we deal with a fairness criterion to select an allocation rate inside $\mathcal{M}$. In the static channel case, it is possible to find rate allocations which are fair, under plenty of different criteria (see [4]). In the dynamic case, the definition of fairness is much more demanding, and not always there exist allocations fulfilling it. Firstly, we demand an allocation to be fair in the long-run process, since users are endowed with a long-term perspective of the transmission process. Then, the top-down procedure would be best, because it would guarantee the rate allocations to be fair in the long-run. However, in Sect. 3.2 we showed that this approach not always produces feasible stationary rate allocations. Secondly, we demand that an allocation respects the fairness criterion not only from the beginning of the transmission onwards, but throughout it, i.e. it should be time consistent. Thirdly, we wish that the rate allocation is also fair in each state of the HMC. We will see that these three conditions are not generally satisfied, however we provide a sufficient condition for them to hold.

3.3.1 Fairness criteria: A review

Let us first introduce the fairness criteria that we will utilize in the next section. In the literature, three fair allocations have been extensively studied: α-fair, max-min fair, and proportional fair allocations. We now provide their general definition, by considering a general rate feasibility region $\mathcal{R}$.

Definition 2 (max-min fairness)**.** *An allocation $\mathbf{r}^{(\mathrm{MM})}$ is max-min fair whenever no user j with rate $\mathbf{r}_j^{(\mathrm{MM})}$ can yield resources to a user i with $\mathbf{r}_i^{(\mathrm{MM})} < \mathbf{r}_j^{(\mathrm{MM})}$ without violating feasibility in $\mathcal{R}$.* □

Definition 3 (α-fairness)**.** *Let $u^{(\alpha)}(r_k) = r_k^{1-\alpha}/[1-\alpha]$ be the utility function for user k. The α-fair allocation $\mathbf{r}^{(\alpha\mathrm{F})}$, with $\alpha \geq 0$, is defined as*

$$\mathbf{r}^{(\alpha\mathrm{F})} = \underset{\mathbf{r}\in\mathcal{R}}{\mathrm{argmax}} \sum_{k=1}^{K} u^{(\alpha)}(r_k). \qquad \square$$

Definition 4 (proportional fairness)**.** *The proportional fair allocation $\mathbf{r}^{(\mathrm{PF})}$ coincides with the α-fair allocation when $\alpha \to 1$, i.e.*

$$\mathbf{r}^{(\mathrm{PF})} = \underset{\mathbf{r}\in\mathcal{R}}{\mathrm{argmax}} \prod_{k=1}^{K} r_k. \qquad \square$$

We point out that, in general, the α-fair allocation is also max-min fair for $\alpha \uparrow \infty$ and proportional fair for $\alpha \to 1$.
If we consider the long-run process Γ_s, then in Definitions 2, 3, and 4 we should interpret $\mathcal{R} \equiv \mathcal{R}(\mathcal{K}, \Gamma_s)$, while in channel state s, $\mathcal{R} \equiv \mathcal{R}(\mathcal{K}, s)$.
In the special case in which the feasibility region is a polymatroid, as for $\mathcal{R}(\mathcal{K}, s)$ and $\mathcal{R}(\mathcal{K}, \Gamma_s)$, for all $s \in \mathcal{S}$, then the three fair allocations coincide.

Theorem 34 ([4]). *If the feasibility region is a polymatroid $\mathcal{R}$, then max-min, proportional, and α-fair allocations coincide for all $\alpha \geq 0$, and moreover belong to the facet $\mathcal{M}(\mathcal{R})$ i.e.*

$$\mathbf{r}^{(\mathrm{MM})} = \mathbf{r}^{(\mathrm{PF})} = \mathbf{r}^{(\alpha\mathrm{F})} := \mathbf{r}^{(\mathrm{F})} \in \mathcal{M}(\mathcal{R}). \qquad \square$$

For Theorem 34, the three mentioned fair solutions coincide both in the long-run process Γ_s and in state s, for all $s \in \mathcal{S}$. Therefore, we can generally refer to them as *fair allocations*, and we call $\mathbf{r}^{(\mathrm{F})}(\Gamma_s)$ the fair allocation in the long-run process Γ_s, and $\mathbf{r}^{(\mathrm{F})}(s)$ the fair allocation in state s. Moreover, a fair allocation belongs to the dominant facet of the associated feasibility region, hence it is a proper criterion to select a set of allocations in $\mathcal{M}$.

3.3.2 Fair allocation design

Finally, we are ready to deal with the design of fair rate allocations on quasi-static channels. We will show under which conditions it is possible to allocate a rate which is *fair* (i.e. max-min, proportional, and α-fair at the same time) both in each state and in the long-run process, and which is fair throughout the game, from each intermediate step, i.e. it is time consistent. More formally, we look for a sufficient condition for which the following holds:

$$\begin{cases} \Phi^{-1}\,[\mathbf{r}^{(\mathrm{F})}(\Gamma_s)]_{s\in\mathcal{S}} = [\mathbf{r}^{(\mathrm{F})}(s)]_{s\in\mathcal{S}} \\ \Phi\,[\mathbf{r}^{(\mathrm{F})}(s)]_{s\in\mathcal{S}} = [\mathbf{r}^{(\mathrm{F})}(\Gamma_s)]_{s\in\mathcal{S}}. \end{cases} \tag{6}$$

We stress that property (6) is crucial, mainly for three reasons, that we list below.

- The top-down procedure may fail, hence if we choose $\{\mathbf{r}^{(\mathrm{F})}(\Gamma_s)\}_{s\in\mathcal{S}}$, not necessarily it is feasible among the stationary allocations, i.e. in general it may happen that

$$\exists\, s \in \mathcal{S} \,:\, \mathbf{r}(s) \notin \mathcal{R}(\mathcal{K}, s),$$
$$\text{with } [\mathbf{r}(s)]_{s\in\mathcal{S}} = \Phi^{-1}\,[\mathbf{r}^{(\mathrm{F})}(\Gamma_s)]_{s\in\mathcal{S}}.$$

- Though the bottom-up procedure always produces feasible allocations, if the allocation is fair in each state, then not necessarily it is also fair in the long-run processes. Indeed, it may happen that

$$\exists\, s \in \mathcal{S} \,:\, \mathbf{r}(\Gamma_s) \neq \mathbf{r}^{(\mathrm{F})}(\Gamma_s), \\ \text{with } [\mathbf{r}(\Gamma_s)]_{s\in\mathcal{S}} = \Phi\,[\mathbf{r}^{(\mathrm{F})}(s)]_{s\in\mathcal{S}} \tag{7}$$

  As an example, in Figure 2 we show an instance in which (7) is verified.

- Most importantly, if relation (6) holds, then the fairness property of the rate allocation is *time consistent* (see Theorem 35).

The time consistency of fair allocations claims that the fairness criteria that induces to enforce a certain rate allocation at time 0 should be consistent in time, at steps $T > 0$ as well. More formally, at each time step T, the β-discounted sum of allocations that each user obtains from time T onwards should be fair in the long-run process Γ_{S_T}.

Theorem 35. *If condition (6) holds, then the fairness of the stationary rate allocation* $\{\mathbf{r}^{(\mathrm{F})}(s)\}_{s\in\mathcal{S}}$ *is* time consistent, *i.e. for all* $T \in \mathbb{N}_0$,

$$\mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}^{(\mathrm{F})}(S_t) \;\Big|\; \mathbf{h}(T)\right) = \beta^T \mathbf{r}^{(\mathrm{F})}(\Gamma_{S_T}),$$

where $\mathbf{h}(T)$ *is the history of states/rate allocations up to time* T. □

Proof. Thanks to the stationarity of the rate allocations, we claim

$$\begin{aligned}\mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}^{(\mathrm{F})}(S_t) \;\Big|\; \mathbf{h}(T)\right) &= \mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}^{(\mathrm{F})}(S_t) \;\Big|\; S_T\right) \\ &= \beta^T\, \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t \mathbf{r}^{(\mathrm{F})}(S_{t+T}) \;\Big|\; S_T\right) \\ &= \beta^T\, \mathbf{r}^{(\mathrm{F})}(\Gamma_{S_T}). \end{aligned} \tag{8}$$

where (8) comes from condition (6). Hence, the thesis is proven. □

After presenting the appealing properties of condition (6), we wish to find a sufficient condition for (6) to hold. For this purpose, it is useful to present first an algorithm, first studied in [9], that produces the fair allocation in a general polymatroid $\mathcal{R}$ with rank function g. Of course, it can be utilized to compute the fair allocation in any state-wise and long-run process.

Algorithm 36 ([9])**.** *Set* $q := 1$. *Set* $\mathcal{K}' := \mathcal{K}$, $g' := g$.

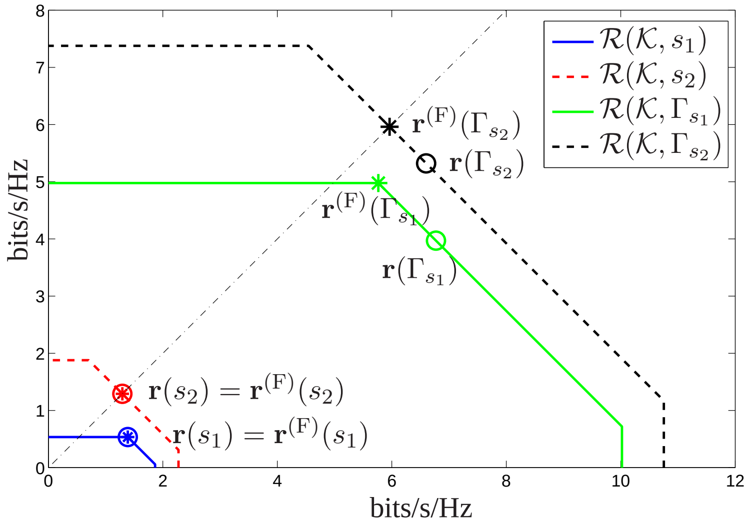

Figure 2: *Example of situation in (7) with two users and two states, in which the state-wise allocations are fair in the respective channel states but the relative long-run allocations are not fair in the respective long-run processes. The allocations indicated with the asterisk are fair, while the circle describes the actual computed allocations.*

1) *Compute*

$$\mathcal{T}^*_{(q)} = \underset{\mathcal{T}\subseteq\mathcal{K}'}{\operatorname{argmin}} \frac{g'(\mathcal{T})}{|\mathcal{T}|}, \quad r_k^{(\mathrm{F})} = \frac{g'(\mathcal{T}^*_{(q)})}{|\mathcal{T}^*_{(q)}|}, \quad \forall\, k \in \mathcal{T}^*_{(q)}.$$

2) *If* $\mathcal{T}^*_{(q)} = \mathcal{K}'$*, then stop. The rate allocation* $\mathbf{r}^F$ *is fair for* $\mathcal{R}$*. Otherwise, set* $q := q+1$, $\mathcal{K}' := \mathcal{K}'\backslash\mathcal{T}^*_{(q)}$,

$$g'(\mathcal{T}) := g'(\mathcal{T}\cup\mathcal{T}^*_{(q)}) - g'(\mathcal{T}^*_{(q)}), \quad \forall\,\mathcal{T}\subseteq\mathcal{K}',$$

and return to step 1). □

Finally, we are ready to provide a condition that ensures the existence of a rate allocation design which is fair both in each state and in every long-run process, as described in (6), and for which the fairness criterion is time consistent, as shown in Theorem 35.

Theorem 37 (SC existence fair allocations)**.** *Let* $\overline{\mathcal{T}}(s) = [\mathcal{T}^*_{(1)}(s), \ldots, \mathcal{T}^*_{(q(s))}(s)]$ *be the sequence computed in the iterations of step 1, Algorithm 36, applied to channel state* s*. Suppose that*

$$\exists\,\overline{\mathcal{T}} = \overline{\mathcal{T}}(s), \quad \forall\, s \in \mathcal{S},$$

i.e. $\overline{\mathcal{T}}(s)$ *does not depend on* s*. Then, condition (6) holds.* □

Proof. At step 1 of the first iteration of Algorithm 36 applied to the process Γ_s, we obtain

$$\mathcal{T}^*_{(1)}(\Gamma_s) = \underset{\mathcal{T}\subseteq\mathcal{K}}{\operatorname{argmin}} \frac{\sum_{n=1}^{N} \nu_n(s)\, g_{(\mathcal{K})}(\mathcal{T}, s_n)}{|\mathcal{T}|} = \mathcal{T}^*_{(1)}.$$

Hence, we can compute the fair allocation for the set of users $\mathcal{T}^*_{(1)}$ as $r^{\mathrm{F}}_k(\Gamma_s) = \sum_{n=1}^{N} \nu_n(s) r^{\mathrm{F}}_k(s_n)$, for all $k \in \mathcal{T}^*_{(1)}$. Then, at step 2, the update of the rank function:

$$g'_{(\mathcal{K})}(\mathcal{T}, \Gamma_s) = \sum_{n=1}^{N} \nu_n(s)\, g'_{(\mathcal{K})}(\mathcal{T}, s_n), \quad \forall\, \mathcal{T} \subseteq \mathcal{K} \backslash \mathcal{T}^*_{(1)}$$

preserves the linearity property of the rank function also in the next iteration. Hence, by induction, the thesis is proven. □

4 Optimal and Satisfactory allocations: A game-theoretical approach

Sect. 3 dealt with the *design* of the rate allocation in each channel state for each user. We restricted our focus solely on the set of global optimum rate region $\mathcal{M}$ (5), i.e. the set of stationary state-wise which are optimal both in each state and in the long-run process.

We now start the second part of the paper by turning our attention towards the *characterization* of the set of rates $\mathcal{M}$ in game theoretical terms. We will show indeed that $\mathcal{M}$, besides being global optimum, also "satisfies" all the users throughout the game, according to two important properties specific for dynamic CGT, namely the time consistent Core and the Cooperation Maintenance property.

4.1 CORE characterization of $\mathcal{M}$

Generally speaking, Static Cooperative Game Theory (SCGT) with non-transferable utility (NTU) studies one-shot interactions among different players who can collaborate with each other by coordinating the respective strategies. It is assumed that grand coalition $\mathcal{K}$, composed by all the players, is formed, and the main challenge consists in devising a payoff allocation for each player, according to some pre-defined criteria. To this aim, the typical procedure in SCGT consists in investigating the *potential* scenario in which a sub-coalition (or simply, coalition) $\mathcal{A} \subset \mathcal{K}$ of players withdraws from the grand coalition and no longer coordinates its actions with the excluded players; then, the set of feasible payoffs that $\mathcal{A}$ can earn on its own is computed (see [6] for a thorough survey). The payoff allocation is finally a function of such feasible sets.

Let us then translate these preliminary few concepts into our scenario. We first consider the static process in state s, that we call *static game*. For the static game case we adopt the same model as in [8]. In our situation, the players are the users, and the grand coalition is the set of transmitting users $\mathcal{K}$. We say that a coalition of users $\mathcal{A}_{\mathcal{J}} := \mathcal{K}\backslash\mathcal{J} \subset \mathcal{K}$ forms when its members share the respective codes with the receiver, which can then decode the signals transmitted by $\mathcal{A}_{\mathcal{J}}$. For us, the payoff for a player is the assigned transmission rate. The SCGT literature provides several ways to compute the set of rate allocations achievable by each subset of users $\mathcal{A}_{\mathcal{J}}$. One of the most utilized is the max-min method, originally introduced by von Neumann and Morgenstern in [19], suggesting that the set of feasible allocations $\mathcal{R}(\mathcal{A}_{\mathcal{J}}, s)$ should be defined as the *set of rate allocations that* $\mathcal{A}_{\mathcal{J}}$ *can achieve whatever is the transmission strategy employed by the remaining user* $\mathcal{J}$. Then, we need to take into account the *worst* possible scenario for $\mathcal{A}_{\mathcal{J}}$, i.e. when the users in $\mathcal{J}$ do not allow joint decoding and jam the network, and investigate the set of rates $\mathcal{R}(\mathcal{A}_{\mathcal{J}}, s)$ that the users in $\mathcal{A}_{\mathcal{J}}$ can achieve in this hypothetical worst-case scenario. When the users in $\mathcal{J}$ jam, they sum coherently the respective signals and transmit with an overall power:

$$\Lambda(\mathcal{J}, s) = \left(\sum_{k\in\mathcal{J}} |h^{(k)}(s)|\sqrt{P_k}\right)^2 .$$

In this worst-case scenario, in [8] it is shown that, among $\mathcal{A}_{\mathcal{J}}$, only the users $\widehat{\mathcal{A}}_{\mathcal{J}}$ whose associated received power level is high enough to overwhelm the jamming signal can communicate, i.e.

$$\widehat{\mathcal{A}}_{\mathcal{J}}(s) := \left\{k \in \mathcal{A}_{\mathcal{J}} :\ |h^{(k)}(s)|^2 P_k > \Lambda(\mathcal{J}, s)\right\}.$$

Then, $\mathcal{R}(\mathcal{A}_{\mathcal{J}}, s)$ is a polymatroid with rank function [8]:

$$g_{(\mathcal{A}_{\mathcal{J}})}(\mathcal{T}, s) := C\left(\sum_{k\in\mathcal{T}} |h^{(k)}(s)|^2 \widetilde{P}_k, \Lambda(\mathcal{J}, s) + N_0\right), \tag{9}$$

where $\widetilde{P}_k = P_k$ for $k \in \widehat{\mathcal{A}}_{\mathcal{J}}(s)$ and $\widetilde{P}_k = 0$ for all $k \in \mathcal{A}_{\mathcal{J}}\backslash\widehat{\mathcal{A}}_{\mathcal{J}}(s)$. Please note that, when $\mathcal{J} = \emptyset$, (9) boils down to expression (1).

Now, let us consider the feasibility region $\mathcal{R}(\mathcal{A}_{\mathcal{J}}, \Gamma_s)$ for a coalition $\mathcal{A}_{\mathcal{J}}$ in the long-run process (or game) Γ_s. Similarly to the static case, it is still defined in the max-min fashion, as the set of long-run rate allocations that the users $\mathcal{A}_{\mathcal{J}}$ can guarantee, whatever is the transmission strategy adopted by $\mathcal{J}$, throughout the process. Therefore, we have to consider the worst-case scenario in which $\mathcal{J}$ jams during the whole process Γ_s and, analogously to Lemma 21, we claim that $\mathcal{R}(\mathcal{A}_{\mathcal{J}}, \Gamma_s)$ is a polymatroid with rank function:

$$g_{(\mathcal{A}_{\mathcal{J}})}(\mathcal{T}, \Gamma_{s_j}) = \sum_{n=1}^{N} \nu_n(s_j)\, g_{(\mathcal{A}_{\mathcal{J}})}(\mathcal{T}, s_n), \quad \forall\, \mathcal{T} \subseteq \mathcal{A}_{\mathcal{J}}.$$

Our goal is now to further characterize $\mathcal{M}$, and we achieve this via the definition of the Core set for NTU cooperative games. The Core is the set of rate allocations that no coalition $\mathcal{A}_{\mathcal{J}} \subset \mathcal{K}$ can improve upon when the remaining users $\mathcal{J}$ jam. Let us define formally the Core of the static game in state s. We say that a rate allocation for the grand coalition $\mathbf{r} \in \mathcal{R}(\mathcal{K}, s)$ is *blocked* by the coalition $\mathcal{A}_{\mathcal{J}} \subseteq \mathcal{K}$ whenever there exists $\mathbf{r}' \in \mathcal{R}(\mathcal{A}_{\mathcal{J}}, s)$ such that $r'_k > r_k$ for all $k \in \mathcal{A}_{\mathcal{J}}$. In other words, the rate allocation $\mathbf{r}$ is unacceptable by the set of users in $\mathcal{A}_{\mathcal{J}}$.

Definition 5. *The Core $\mathcal{C}o(s)$ is the set of unblocked rate allocations in $\mathcal{R}(\mathcal{K}, s)$.*

Remark 3. *We can intuitively define the* Core as the set of all "acceptable" rates for all users*: indeed, if an allocation does not belong to the Core, at least a subset of users is dissatisfied with it, because they can all attain a better rate allocation even when the remaining users do not participate to the transmission and jam.* □

Additionally, an allocation in $\mathcal{C}o(s)$ is also not blocked by the grand coalition $\mathcal{K}$. Since $\mathcal{R}(\mathcal{K}, s)$ is a polymatroid, it follows that it is a region with maximum sum-rate, i.e. $\mathcal{C}o(s) \subseteq \mathcal{M}(\mathcal{K}, s)$, for all $s \in \mathcal{S}$.

The Core $\mathcal{C}o(\Gamma_s)$ in the long-run game Γ_s is defined analogously to the static case. We remark that it coincides with the set of long-run allocations that are acceptable for each subset of users *at the beginning of the long-run game*. This definition of $\mathcal{C}o(\Gamma_s)$ relates to static CGT, in which the coalition structure holds steady throughout the game and players do not change their preference over the rate allocations over time. This is a naïve perspective though, since the channel is dynamic. Hence, we demand that a stationary rate allocations is not only "acceptable" for each coalition at the beginning of the game, but also throughout the game. This property is called, in dynamic CGT, *time consistency* of the Core [10]. The philosophy behind this definition is analogous to the time consistency of fair allocations, in Theorem 35. Hence, if the Core property of an allocation is time consistent, then at *each* time step, if any coalition faces the dilemma "*do we withdraw now or we cooperate forever?*", it always prefers the second option. Therefore, we will focus our attention on the allocations in $\mathcal{C}o$, defined as follows, and we will prove that $\mathcal{C}o = \mathcal{M}$.

Definition 6 ($\mathcal{C}o$)**.** *$\mathcal{C}o$ is the set of stationary state-wise allocations belonging to the Core of each static game, and that belong to the Core of long-run games in a time consistent fashion throughout the game, i.e.*

$$\mathcal{C}o := \Bigg\{ \{\mathbf{r}(s)\}_{s\in\mathcal{S}} \colon\ \mathbf{r}(s) \in \mathcal{C}o(s), \\ \mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}(S_t) \,\Big|\, \mathbf{h}(T) \right) \in\ \beta^T \mathcal{C}o(\Gamma_{S_T}),\ \forall T \in \mathbb{N}_0 \Bigg\},$$

where $\mathbf{h}(T)$ be the history of states/rate allocations up to time T. □

Hence, $\mathcal{C}o$ is the set of stationary allocations that are maximum sum-rate, hence optimum for the global network, and that are "acceptable" for each subset of users, in both static and long-run games, throughout the game. Hence, we can already claim that $\mathcal{C}o \subseteq \mathcal{M}$. Let us show that $\mathcal{C}o = \mathcal{M}$.

In [8], La and Anatharam computed the Core of the static game by relying on SCGT with transferable utilities (TU). Their approach is not completely rigorous, since the rate cannot be shared in any manner among the users, but only within the capacity region. Nevertheless, NTU cooperative game theory yields the same result as [8], as we show next.

Theorem 41. *The Core $\mathcal{C}o(s)$ coincides with the facet $\mathcal{M}(\mathcal{K}, s)$ of the feasibility region $\mathcal{R}(\mathcal{K}, s)$ for the grand coalition.* □

Proof. Is is known (e.g. [18]) that all the points in $\mathcal{M}(\mathcal{K}, s)$ solve the linear program $\max_{\mathbf{r}\in\mathcal{R}(\mathcal{K},s)} \sum_{k\in\mathcal{K}} r_k$. Hence, all the points in $\mathcal{M}(\mathcal{K}, s)$ are efficient for $\mathcal{K}$. Moreover, in [8] it is shown that, for all $\mathbf{r} \in \mathcal{M}(\mathcal{K}, s)$,

$$\sum_{k\in\mathcal{A}_\mathcal{J}} r_k \geq g_{(\mathcal{A}_\mathcal{J})}(\mathcal{A}_\mathcal{J}, s), \quad \forall\, \mathcal{A}_\mathcal{J} \subset \mathcal{K}.$$

Hence, we can say that, for all $\mathbf{r} \in \mathcal{M}(\mathcal{K}, s)$, there exists no allocation belonging to $\mathcal{M}(\mathcal{A}_\mathcal{J}, s)$ that dominates $\mathbf{r}$ for coalition $\mathcal{A}_\mathcal{J}$. Since any rate allocations belonging to $\mathcal{R}(\mathcal{A}_\mathcal{J}, s)$ is dominated by a rate allocation in $\mathcal{M}(\mathcal{A}_\mathcal{J}, s)$, then $\mathcal{M}(\mathcal{K}, s) \subseteq \mathcal{C}o(s)$. If $\mathbf{r} \notin \mathcal{M}(\mathcal{K}, s)$, either it is not feasible or it is not efficient for $\mathcal{K}$. Then, $\mathcal{M}(\mathcal{K}, s) = \mathcal{C}o(s)$. □

In the light of Theorem 41 and Lemma 21, we can easily provide an expression for $\mathcal{C}o(\Gamma_s)$ as well.

Corollary 42. *The Core $\mathcal{C}o(\Gamma_s)$ of the long-run game Γ_s coincides with the facet $\mathcal{M}(\mathcal{K}, \Gamma_s)$.* □

Now, we are ready to claim that $\mathcal{M} = \mathcal{C}o$.

Theorem 43. *The set of stationary state-wise rate allocations $\mathcal{M}$ coincides with $\mathcal{C}o$, i.e. $\mathcal{M} = \mathcal{C}o$.* □

Proof. We know that $\mathcal{C}o \subseteq \mathcal{M}$. We have to prove that $\mathcal{M} \subseteq \mathcal{C}o$. For Theorem 41, if $\{\mathbf{r}(s)\}_{s\in\mathcal{S}} \in \mathcal{M}$, then $\{\mathbf{r}(s) \in \mathcal{C}o(s)\}_{s\in\mathcal{S}}$. Then, we just need to prove that, if $\{\mathbf{r}(s)\}_{s\in\mathcal{S}} \in \mathcal{M}$, then the Core is time consistent in the long-run game. Similarly to the proof of Theorem 35, we claim that for all $T \in \mathbb{N}_0$,

$$\mathbb{E}\left(\sum_{t=T}^{\infty} \beta^t \mathbf{r}(S_t) \,\middle|\, \mathbf{h}(T)\right) = \beta^T\, \mathbf{r}(\Gamma_{S_T}),$$

where $[\mathbf{r}(\Gamma_s)]_{s\in\mathcal{S}} = \Phi[\mathbf{r}(s)]_{s\in\mathcal{S}}$ is the set of the associated long-run allocations. Thanks to Proposition 31, $\mathbf{r}(\Gamma_{S_T}) \in \mathcal{C}o(\Gamma_{S_T})$. Hence, the thesis is proven. $\square$

Thanks to Theorem 43, the set of stationary state-wise rate allocations $\mathcal{M}$ gains further significance. Not only $\mathcal{M}$ is the maximal sum-rate region, but it also coincides with the set of rates which are "acceptable" both in the long-run and in the static games, under the definition of Core. Moreover, the Core criterion is time consistent, hence such rates are acceptable throughout the game.
In the next section we provide a second characterization of $\mathcal{M}$, based on a Cooperation Maintenance property.

4.2 COOPERATION MAINTENANCE characterization of $\mathcal{M}$

In this section we show that, by exploiting a crucial concept in DCGT, called Cooperation Maintenance property, we are able to provide a further characterization to the set $\mathcal{M}$ of the maximum sum-rate stationary state-wise allocations. The property that we are going to define is an adaptation to our NTU scenario of the Cooperation Maintenance property defined in [11], [20]. It claims that, at each time step, the maximum sum-rate that coalition $\mathcal{A}_\mathcal{J}$ expects to obtain if it withdraws (without any chance of joining back) from the grand coalition in one step should be not smaller than what $\mathcal{A}_\mathcal{J}$ obtains if it withdraws (still, without a second thought) at the current step.

Remark 4. *When we say, in a game-theoretical jargon, that a coalition $\mathcal{A}_\mathcal{J}$ is enticed to* withdraw *from the grand coalition, we actually mean that it is* dissatisfied *with its assigned rate, because, even in the* worst-case *scenario in which $\mathcal{J}$ jams, $\mathcal{A}_\mathcal{J}$ could achieve a better allocation. Hence, like in Sect. 4.1, we will utilize Game Theory as a tool to measure users' satisfaction with the assigned rate.* $\square$

The set of allocations for which the Cooperation Maintenance property holds is called $\mathcal{CM}$.

Definition 7 ($\mathcal{CM}$)**.** *The set of (first step) Cooperation Maintaining allocations $\mathcal{CM}$ is the set of stationary state-wise rate allocations $\{\mathbf{r}(s) \in \mathcal{M}(\mathcal{K}, s)\}_{s\in\mathcal{S}}$ such that, for all coalitions $\mathcal{A}_\mathcal{J} \subseteq \mathcal{K}$ and at each time step $T \in \mathbb{N}_0$,*

$$\sum_{k\in\mathcal{A}_\mathcal{J}} r_k(S_T) + \beta \sum_{s'\in\mathcal{S}} p(s'|S_T) \Big[\max_{\mathbf{r}(\Gamma_{s'})\in\mathcal{R}(\mathcal{A}_\mathcal{J},\Gamma_{s'})} \sum_{k\in\mathcal{A}_\mathcal{J}} r_k(\Gamma_{s'}) \Big] \geq \max_{\mathbf{r}(\Gamma_{S_T})\in\mathcal{R}(\mathcal{A}_\mathcal{J},\Gamma_{S_T})} \sum_{k\in\mathcal{A}_\mathcal{J}} r_k(\Gamma_{S_T}). \quad (10)$$

$\square$

The intuition behind the definition of $\mathcal{CM}$ is that, if a coalition faces the dilemma "*do we withdraw now or in one step?*", it should prefer the second option, at any instant. In this way, by induction, no coalition is ever enticed to withdraw and the grand coalition is cohesive throughout the game.

It follows from Definition 7 that $\mathcal{CM} \subseteq \mathcal{C}o$. Also, it is not difficult to show that, if the (first step) Cooperation Maintenance property holds, then the n-tuple step Cooperation Maintenance property also holds (see [20] for a more general case), i.e. if a coalition faces the dilemma "*do we withdraw now or in n steps?*", it prefers the second option. For $n \uparrow \infty$, such property suggests that whenever a coalition faces the dilemma "*do we withdraw now or cooperate forever?*", then it prefers to stick with the grand coalition forever. Not surprisingly, this notion coincides with the time consistency property of the Core that any allocation in $\mathcal{M}$ possesses, as illustrated in Theorem 43.
We remark that, in more general settings, $\mathcal{CM}$ is smaller than the set of the stationary distributions belonging to the Core of long-run games (see [20]). Hence, the definition of $\mathcal{CM}$ requires a "higher level of satisfaction" for the players than the Core. We now state that actually, in our scenario, $\mathcal{M} = \mathcal{CM}$. Through this result, we provide a second dynamic characterization of the set $\mathcal{M}$.

Theorem 44. *The maximum sum-rate set of stationary state-wise allocations $\mathcal{M}$ coincides with the Cooperation Maintaining set $\mathcal{CM}$, i.e. $\mathcal{M} = \mathcal{CM}$.* □

Proof. For Proposition 31, $\mathcal{CM} \subseteq \mathcal{M}$. Conversely, if an allocation $\{\mathbf{r}(s)\}_{s \in \mathcal{S}} \in \mathcal{M}$, then it also belongs to $\mathcal{C}o$. So, $\sum_{k \in \mathcal{A}_\mathcal{J}} r_k(s) \geq g_{(\mathcal{A}_\mathcal{J})}(\mathcal{A}_\mathcal{J}, s)$, for all $\mathcal{A}_\mathcal{J} \subseteq \mathcal{K}$, $s \in \mathcal{S}$. Then, thanks to Lemma 21, we can say that for all $\mathcal{A}_\mathcal{J} \subseteq \mathcal{K}$, $s \in \mathcal{S}$:

$$\sum_{k \in \mathcal{A}_\mathcal{J}} \begin{bmatrix} r_k(s_1) \\ \vdots \\ r_k(s_N) \end{bmatrix} \geq \Phi^{-1} \begin{bmatrix} g_{(\mathcal{A}_\mathcal{J})}(\mathcal{A}_\mathcal{J}, \Gamma_{s_1}) \\ \vdots \\ g_{(\mathcal{A}_\mathcal{J})}(\mathcal{A}_\mathcal{J}, \Gamma_{s_N}) \end{bmatrix},$$

which is an expression equivalent to (10). Hence, $\mathcal{M} \subseteq \mathcal{CM}$ and the thesis is proven. □

Therefore, in this section we have provided two game-theoretical characterizations for the global optimal set of allocations $\mathcal{M}$, i.e.

$$\mathcal{M} = \mathcal{C}o = \mathcal{CM}.$$

Hence, $\mathcal{M}$ coincides with the set of rates $\mathcal{C}o$ which are acceptable for all coalitions throughout the game, and with the set of rates $\mathcal{CM}$ that make the grand coalition cohesive at every step of the game.

5 Conclusions

In this paper we considered a quasi-static Markovian multiple access channel. We studied how to allocate the rate for each user in each channel state. Our work is motivated by the fact that, in the LTE technology, the statistics of the channel are estimated by the receiver and used at regular intervals to perform rate allocation. Hence, in each possible channel state, a different rate for each user needs to be allocated. We focused on the set $\mathcal{M}$ of allocations which are maximum sum-rate, both in each state and in the long run process. In Sect. 3 we investigated two rate allocation procedures, namely bottom-up and top-down. Though the latter is more useful under a long-run perspective, it does not always produce feasible allocations. Theorem 33 offers a remedy for this. In Sect. 3.3 we demanded the existence of an allocation which is fair both in each state and in the long-run process. Moreover, we demanded the fairness property to be time consistent. Theorem 37 provides a sufficient condition for this.

While in Sect. 3 we dealt with the issue of selecting a rate allocation inside the optimal set $\mathcal{M}$, in Sect. 4 we turned our attention towards a characterization of the set $\mathcal{M}$ in dynamic game-theoretical terms. Firstly, in Theorem 43 we claim that $\mathcal{M}$ coincides with the Core set $\mathcal{C}o$ of allocations which are, in a sense, "acceptable" for all the users, both in the static and in the long run game, in a time consistency fashion. Secondly, in Theorem 44 we state that $\mathcal{M}$ also coincides with the set of Cooperation Maintaining allocations $\mathcal{CM}$ that makes the coalition of all players cohesive throughout the game. Therefore, all allocations in $\mathcal{CM}$ are both global optimal and satisfy the users throughout the process, according to the criteria defined by $\mathcal{C}o$ and $\mathcal{CM}$.

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