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**Coding**  
**for bidimensional constellations**  
**based on the  $\mu$ -law**

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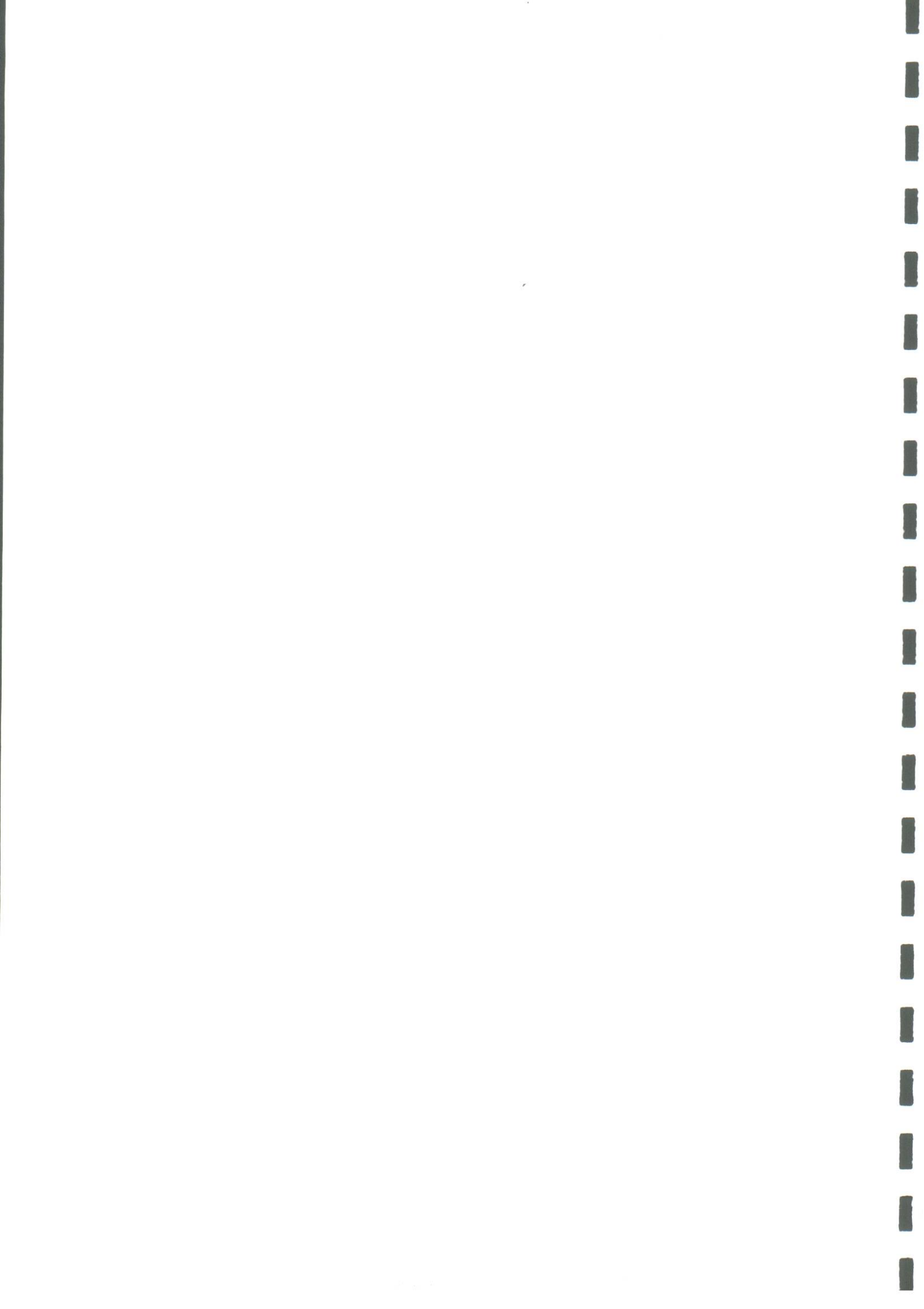
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## 1 Position of the problem

We want to transmit at a rate of 6 bits per symbol using a constellation which is in fact the so-called  $\mu$ -law.

Recall that the  $\mu$ -law is the set of signal values used in voice coding in the analog loop of the telephone network. (In some countries, it is made use of the  $A$ -law rather than the  $\mu$ -law. This does not change anything). The  $\mu$ -law consists of 8 “segments” and of their symmetrical counterparts. Each segment contains 16 values. In table 1, we indicate respectively the first value, the range between two successive values and the last value of each segment.

segment	first value	range	last value
1	0	2	30
2	33	4	93
3	99	8	219
4	231	16	471
5	495	32	975
6	1023	64	1983
7	2079	128	3999
8	4191	256	8031

Table 1: signal values of the  $\mu$ -law.

We want to evaluate the performances of coding with a trellis using bidimensional signals, i.e. the constellation is made up by the cartesian square of the  $\mu$ -law.

## 2 Principles of coding

We consider a trellis on this constellation with four states and rate 1/2. The constellation is then divided in four subsets which may overlap.

The sets  $A$  and  $D$ , respectively  $B$  and  $C$ , may have points in common. Overlapping has the advantage of reducing the average energy and the drawback of permitting quasi-catastrophic encoding. We want to evaluate its effect precisely.

Subsets are made so as to respect Ungerboeck’s rules :

- parallel transitions are associated with signal with maximum distance between them;
- transitions originating from or merging in one state are associated with the next larger distance.

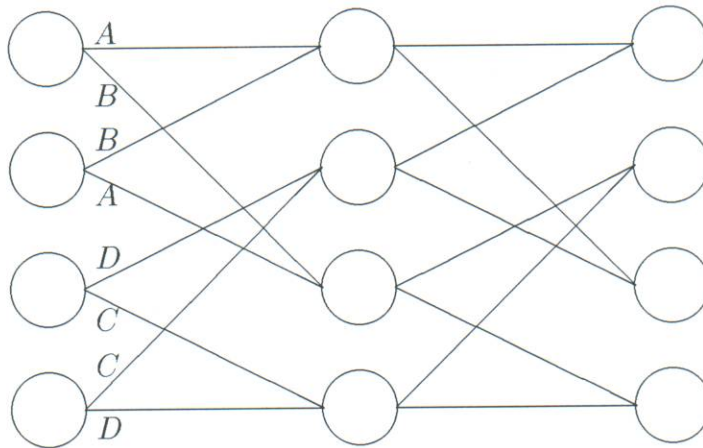


Figure 1: trellis with four states and rate 1/2

Therefore points are assigned to subsets according to the “checkerboard principle” represented hereafter.

$$\begin{array}{ccc} A & C & A \\ D & B & D \\ A & C & A \\ \dots & & \end{array}$$

Given a set, each second point in both directions belongs to this set, and the other one does not belong to it.

### 3 Error probability

In a first step, we think it worth deriving the formula in the simplified case of unbounded QAM modulation. That is, the distance profile does not depend on the point chosen.

We use the formula

$$P(\text{error}) \leq \sum_k \sum_{C \in P_k} P(C) \sum_d \sum_{c \in C} \sum_{\substack{(c, f) \in C \times F_C \\ d(c, f) = d}} w(c, f) P(c|C) Q\left(\frac{d}{2\sigma}\right) \quad (1)$$

where

- $k$  is the length of the error event;
- $P_k$  is the set of all length  $k$  sequences of subsets of the constellation;
- $d$  is a distance;
- $F_C$  is the set all incorrect subset sequences starting from the same state as  $C$  and ending in the same state;
- $d(c, f)$  is the distance between  $c$  and  $f$ .

The upper bound is generally considered as a good approximation.

We make the assumption that all the states and all the transitions are equiprobable. Then the different subset sequences have the same probability and the sum over  $C \in P_k$  disappears.

We now discuss on the distance. Since the error function  $Q(\cdot)$  decreases very quickly, we only have to evaluate the first terms. The very first distance to arise,  $d_0$  say, is the minimum distance in every subset and corresponds to parallel transitions or, in other words, to length 1 error events. For such an error event, an error on the symbol is four ways (the so-called error coefficient) yielding four times the same term in the error probability, with  $w = 1$  and  $P(c|C) = 1$ .

The second distance to arise,  $d_1$  say, corresponds to an error event which typically starts in the first state, has correct set path  $A \dots A$  and incorrect path  $BDADA \dots B$ . The first symbol is not the same in each path, all intermediate symbols are the same (thus symbols in  $D$  belong precisely to  $A \cap D$ ) and the last symbol differs again. Finally the total distance accumulated is  $2d(A, B)^2 = d_1^2$ . Remark that :

- such error events necessarily have an even length;
- in QAM with the  $Z^2$  lattice as constellation,  $d_0 = d_1$ .

For a typical error event of distance  $d_1$ , an error on the first  $A$  symbol is four ways and similarly for the last symbol. For such a term,  $w = 2$ .

A particular instance for a typical correct path is a sequence of symbols  $c = c_1 c_2 \dots c_k$  where  $c_1 \in A$   $c_2 \in A \cap D$   $c_3 \in A$   $c_4 \in A \cap D \dots c_k \in A$ , all those symbols being independent. Then

$$P(c|C) = \left( \frac{\#A \cap D}{\#A} \right)^{\frac{k-1}{2}} = P^{\frac{k-1}{2}}$$

where we set  $P = \frac{\#A \cap D}{\#A}$ .

Summing over all  $k$  provides a term

$$4 \times 4 \times 2 \times \frac{P}{1-P} Q \left( \frac{d_1}{2\sigma} \right). \quad (2)$$

The third distance,  $d_2$  say, correspond to typical error events with the same set path as above but where one of the intermediate symbols differs in each path. Then  $d_2^2 = d_1^2 + \Delta$  with  $\Delta = d(D - A, A - D)^2$ ,  $w = 3$ , the error on the first symbol is four ways, the one on the intermediate symbol, two ways and the one on the final symbol, four ways. There are  $(k-1)/2$  places for the intermediate symbol hence

$$P(c|C) = \frac{k-1}{2} P^{\frac{k-3}{2}} (1-P).$$

When summing over all the  $k \geq 3$ , we get

$$(1-P) \sum_{n \geq 1} n P^{n-1} = (1-P) \frac{d}{dP} \frac{1}{1-P} = \frac{1}{1-P}.$$

Hence in the error probability a term

$$\frac{4 \times 2 \times 4 \times 3}{1-P} Q \left( \frac{d_2}{2\sigma} \right). \quad (3)$$

More generally, we define a family of distances  $d_{n+1}$  such that  $d_{n+1}^2 = d_1^2 + n\Delta$ ,  $w = 2+n$ , the first and the last errors are four ways, the  $n$  intermediate errors are two ways and may be at  $\binom{\frac{k-1}{2}}{n}$  places then

$$P(c|C) = \binom{\frac{k-1}{2}}{n} P^{\frac{k-1}{2}} (1-P)^n.$$

For example, for  $n = 2$ , this yields in the error probability a term

$$4 \times 2 \times 2 \times 4 \times 4 \frac{2P}{(1-P)^3} Q \left( \frac{d_3}{2\sigma} \right). \quad (4)$$

The distance  $d_3$  corresponds also to error events with typical correct path  $AAAA$  and incorrect path  $BCCB$ . Indeed,

$$d_2^2 = d^2(A, B) + d^2(A, C) + d^2(A, C) + d^2(A, B)$$

For such a path, the error coefficients are succesively 4,2,2,4,  $w = 4$  and  $P(c|C) = 1$ .

We do not think it worth evaluating the contribution of greater distances.

Hence finally

$$\begin{aligned}
 P(\text{error}) \approx & 4Q\left(\frac{d_0}{2\sigma}\right) + \frac{32P}{1-P}Q\left(\frac{d_1}{2\sigma}\right) + \frac{96}{1-P}Q\left(\frac{d_2}{2\sigma}\right) \\
 & + 256\frac{2P}{(1-P)^3}Q\left(\frac{d_3}{2\sigma}\right) + 256Q\left(\frac{d_3}{2\sigma}\right)
 \end{aligned} \tag{5}$$

Coding is indeed catastrophic if overlapping is too pronounced that is,  $P \approx 1$ . If so, the upper bound is no more an approximation of the error probability since it can exceeds 1.

Consider the effective error distance  $d_{\text{eff}}$  defined by

$$Q\left(\frac{d_{\text{eff}}}{2\sigma}\right) = P(\text{error}). \tag{6}$$

It is now possible to evaluate the error coding gain by

$$G = 10 \log\left(\frac{d_{\text{eff}}^2}{d_u^2}\right) - 10 \log\left(\frac{E_c}{E_u}\right) \tag{7}$$

where the subscript  $u$  and  $c$  mean uncoded and coded respectively and  $E$  denotes the average energy of the constellation.

When using the  $\mu$  law rather than QAM, the distance profile depends on the point chosen. For a given distance  $d$ , the error coefficient becomes the mean value of the discrete random law “number of neighbours of the current symbol at distance  $d$ ”

$$\prod_{e=1}^w (p_e(1) + 2p_e(2) + 3p_e(3) + 4p_e(4)). \tag{8}$$

$p_e(i)$  is the proportion of points with  $i$  neighbours in the set of the current symbol. The last terms may be null.

## 4 Simulations

### 4.1 Algorithm

We construct a wide family of codes on the constellation  $\mu^2$  and investigate their properties.

The input of the main program is the square minimum distance. In the following, we will omit to precise “square”. The outputs are the average energy of the constellation, the probability  $P$ , an approximation of the error probability (the first two terms of equation 5 with  $d_0 = d_1$ ), the effective minimum distance, the coding gain for various bit rates and maximal energies. The outputs are put together into files whose name is on the model `d32_rate_9.res`.



About the approximation of the error probability, note that an exact formula, taking into account equation 8, would yield a lower value than the one which is computed. Therefore, it would enhance the coding gain. Second, setting  $d_0 = d_1$  is right if and only if subsets overlap. Otherwise, any path of length greater than 1 would yield a distance greater than the internal distance.

The program generates other interesting results like the intersection of the subsets which can become readable with a few modifications.

The main program is as follows (program `coding.m`):

### Algorithm 1

1. *enter the minimum distance;*
2. *checkerboard;*
3. *for various bit rates :*
  - (a) *shaping for uncoded transmission;*
  - (b) *shaping;*
  - (c) *overlapping;*

Let us now precise the contents of every subroutine.

**Checkerboard** (subroutine `checkerboard_*.m` where \* denotes the minimum distance).

Extract a subconstellation with the desired distance and label the points.

We erase some points in  $\mu^2$  in order to increase the minimum distance. Our choice is dictated by the region where the points are the tightest i.e. the square of the first segment of the  $\mu$  law. In this region, the points are arranged like in the lattice  $Z^2$ . We retain the points of the sublattice belonging to the chain

$$Z^2 \supset RZ^2 \supset 2Z^2 \supset 2RZ^2 \dots$$

which has the desired minimum distance. ( $R$  is the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Remark that  $R^2 = 2I$ ,  $RZ^2 = D_2$  and that the distance increases of a factor 2 between two successive lattices).

Practically, we must number the signal levels of the  $\mu$  law in a first step and work on their number.

Many variants are possible. We can extend more or less the region on which we perform the algorithm, we can combine the two algorithms and so on. . . .

Then we label the points chosen (in other words, we assign each point to some subset), according to the *checkerboard principle* explained in section 2.

Whichever lattice is chosen, it is similar to  $Z^2$  and the subsets appear to be (similar to) the four cosets of  $2Z^2$  in  $Z^2$ . They may be represented as  $2Z^2$ ,  $2Z^2 + (1, 0)$ ,  $2Z^2 + (0, 1)$ ,  $2Z^2 + (1, 1)$  that we label  $A$ ,  $C$ ,  $D$  and  $B$  respectively.

For example, when the lattice is  $D_2$ , we proceed as follows: consider  $(x, y) \in D_2$ , consider the mapping from  $D_2$  to  $Z^2$   $(x, y) \mapsto ((x+y)/2, (x-y)/2)$ . Label  $(x, y)$  according to the coset of its image in  $2Z^2$ .

We give on figure 2 an example of labelling yielding a distance of  $d^2 = 32$ . We have used the lattice  $D_2$  in the cross whose branches meet each other in the square of the first segment and the lattice  $Z^2$  otherwise.

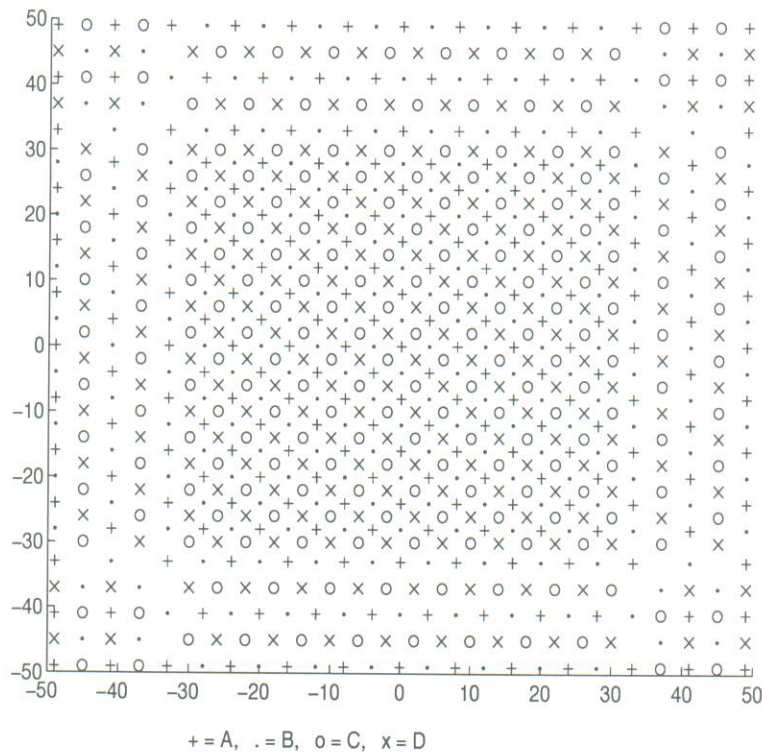


Figure 2: example of labelling with  $d^2 = 32$ .

**Shaping for uncoded transmission** (subroutine `shap_uncoded.m`). We are given a bit rate, for example, 6 bits/linear symbol. The rate is of 12 bits per planar symbol. The shaping for uncoded transmission consists in taking

out  $2^{12}$  of the innermost point from the constellation. Practically, we remove the outermost points until we reach the desired number. We compute the average energy  $E_u$  of the resulting constellation.

**Shaping** (subroutine `shaping1.m`). Among the 12 bits, one bit is for coding, for the choice of the set, the remaining 11 bits are for the choice of the symbol in the set. The shaping consists in taking out  $2^{11}$  of the innermost point from each subset of the constellation. Practically, we remove the outermost points until we reach the desired number.

**Overlapping** (subroutine `shaping2_*.m` where \* denotes the minimum distance).

The sets  $A$  and  $D$ , on one hand, and  $B$  and  $C$  on the other hand may overlap. We construct the intersection so as to retain the intraset and intersets minimum distances. Practically, we are given an energy, we remove the points in  $A$  (resp.  $B$ ) which exceed this energy, we declare the same number of points in  $D$  (resp.  $C$ ) as common to both the sets, and conversely. We do so for a decreasing sequence of energies while the distance constraints are met.

## 4.2 Results

We have made simulations for the (square) distances 16, 32 and 64 (of course, our program is able to run for many other distances). We have taken respectively  $\sigma = 0.7, 1, 1.5$ . The value of  $\sigma$  (the signal to noise ratio) is chosen so as to yield error probabilities between  $10^{-6}$  and  $10^{-2}$ .

Figure 3 represents the bit rate, in bits/symbol, versus the average energy, in dBm0. There are two curves per distance : the dashed one is for the case of partitionning, the solid one for the case of maximal attainable overlapping. We remark that the gain in energy provided by overlapping is very light. The curves for  $d^2 = 32$  and  $d^2 = 64$  coincide for bit rates greater than 12 bits per planar symbol. This suggests that increasing the distance makes no cost in terms of energy. This however should be confirmed by other simulations.

Figure 4 represents the coding gain, in dB, versus the overlapping, which is measured by the means of the probability  $P$ . The minimum distance is of 16 and we show different curves corresponding to different bit rates.

The higher the bit rate, the lower the coding gain. At the desired bit rate of 12 bits/symbol, the coding gain becomes null. Since the real coding gain is greater, we can conclude that *the desired bit rate is attained with some gain*.

For greater bit rates, we obtain negative coding gains. The number of symbols per set required is to high, one must use very high levels of energy.

This is not compensated by overlapping. Overlapping has a negligible influence on the coding gain. Curves are nearly horizontal, slowly decreasing

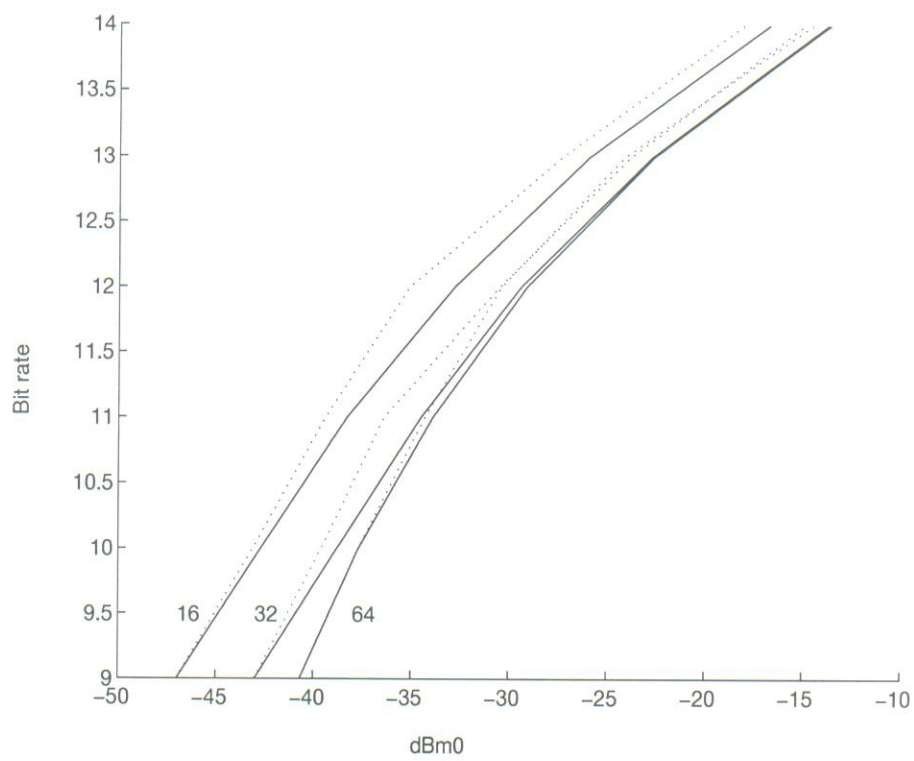


Figure 3: bit rate versus average energy

in the beginning and slowly increasing in the second part. This is confirmed by the simulations performed for  $d^2 = 32$  (see figure 5).

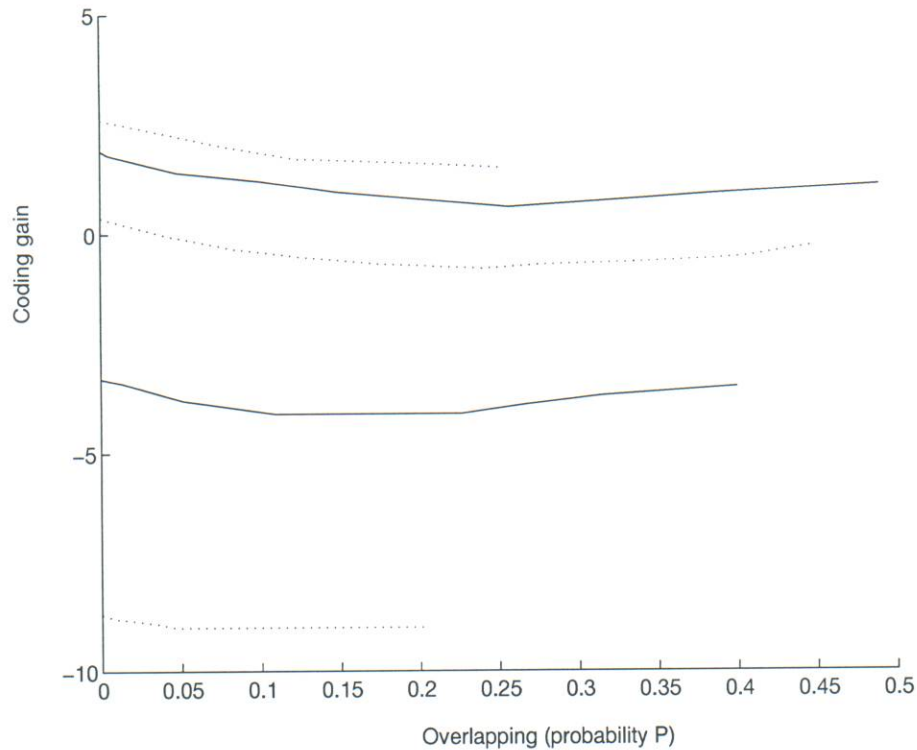


Figure 4: coding gain versus overlapping,  $d^2 = 16$

## 5 Conclusion

The effect of coding depends strongly on the bit rate : the higher the bit rate, the lower the coding gain. For high bit rates, the coding gain is null or even negative. However, the desired bit rate of 12 bits/symbol is attained with gain. If we want to get positive gains at higher rates, it is necessary to investigate codes with more numerous states.

Finally, arrange that the subsets overlapp does not bring much.

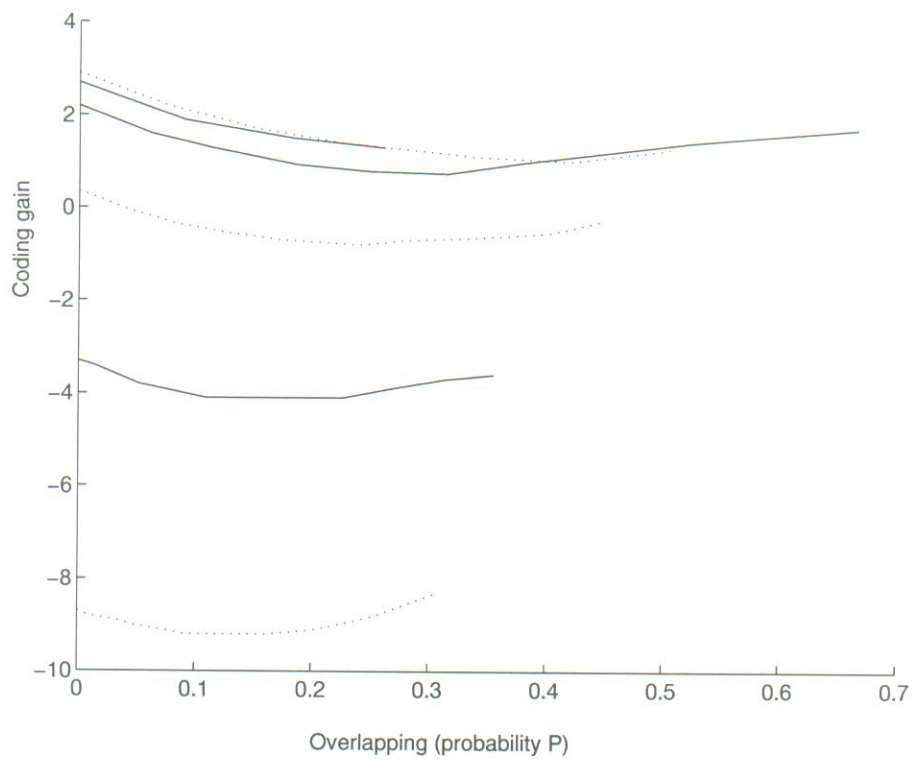


Figure 5: coding gain versus overlapping,  $d^2 = 32$