On the Rate Gap Between Multi- and Single-Cell Processing Under Opportunistic Scheduling

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Abstract—**Base station (BS) coordination is a key technique to handle inter-cell interference (ICI) in cellular networks. Nevertheless, recent work on scheduling indicates that the value of coordination is less prominent when the number of users grows large. More specifically, the loss in sum rate due to ICI in uncoordinated networks can be made arbitrarily small as the number of users goes to infinity. However, the gap in performance for a finite number of users has remained unknown so far. From this perspective we study the gains of multicell zero-forcing beamforming (ZFBF) on the downlink of a Wyner-type network. We first identify the beamforming weights and the optimal scheduling policy under a per-base power constraint. To compare ZFBF with single-cell processing (SCP) we focus on the extra number of users that is needed per cell to compensate for ICI. Specifically, we find the number of users n_1 with ZFBF and n_2 with SCP that gives the same mean post-scheduling SINR as an interference free network with n users. The results show that the ratio n_2/n_1 grows logarithmically with n. Finally, we demonstrate that the difference in sum-rate between SCP and multicell ZFBF goes to zero as $O\left(\frac{\ln \ln n}{\ln n}\right)$. As a consequence of the slow convergence there is a significant gain with multicell ZFBF for all practical numbers of users.**

Index Terms—**Base station coordination, zero-forcing beamforming, multiuser scheduling.**

I. Introduction

In conventional cellular systems signal transmission and reception are done independently on a per-cell basis. This may result in considerable inter-cell interference (ICI) which will ultimately limit the capacity. However, by interconnecting the BSs and coordinating their actions the ICI can be greatly reduced [1], [2]. A key driver for practical deployment of BS coordination is that the main complexity burden is on the network side and not the mobile users.

Recently there has been much work on the information theoretic nature of coordinated networks [3], [4]. In the ideal case the downlink can be viewed as vector broadcast channel in which dirty paper coding (DPC) is the capacity achieving strategy. Unfortunately, for many practical applications DPC is prohibitively complex. Sub-optimal techniques with lower complexities such as linear precoding are therefore of great interest.

In this paper we consider multicell zero-forcing beamforming (ZFBF) together with multiuser scheduling. We are particularly keen to compare the resulting sum-rate per cell, with that of single-cell processing (SCP) and optimal scheduling. The reason for this is twofold. First of all, there is an inevitable increase in complexity with any BS coordination scheme relative to conventional SCP. To justify the use of BS coordination there must therefore be an accompanied gain in performance. Second, recent work on scheduling shows that there can be substantial gains in SCP networks with multiple fading users. Specifically, in [5] and [6] it was shown that the loss in sum-rate incurred by ICI can be made arbitrarily small as the number of users go infinity. In [7] the focus was on inter-beam interference in single-cell beamforming. However, a reinterpretation of some quantities gives a similar conclusion. A corollary to these results is that the value of BS coordination will eventually diminish as the number of users increases. Nevertheless, the implications of this asymptotic behavior for a moderate to large number of users depend crucially on the rate of convergence.

The analytical study of BS coordination is notoriously difficult and previous authors have mainly resorted to network and interference model simplifications in order to obtain insights arising from closed-form expressions [3], [4], [8]–[10]. This will also be our approach here. Specifically, we assume a linear cell-array, where each user only receives a signal from the two closest BSs. This is a slight variation of Wyner's classical model introduced in [8]. Similar network and interference models were recently used in [3] and [4], with the exception that the cells were arranged on a circular array. However, this difference is insignificant as the number of cells goes to infinity.

In [3] the focus was on upper and lower bounds for the per-cell sum-rate under multicell DPC. In particular, the per-cell sum-rate was shown to scale as $\log \log n$ with the number of users n per cell. In [4] the performances of several suboptimal network coordination strategies were characterized. However, no explicit expressions for ZFBF together with Rayleigh fading were given. In [9] ZFBF and multiuser scheduling were studied using a model where each user could see the three closest BSs. A suboptimal scheduling strategy was proposed and shown to scale as $\log \log n$ which is the same as for optimal multicell DPC. However, the same optimal scaling can also be achieved with SCP and is therefore not sufficient to justify ZFBF in itself [11].

The goal of this work is to to evaluate the benefit of multicell ZFBF over SCP as the number of users grows large. To this end we derive explicit expressions for a set of beamforming weights satisfying the zero-forcing criterion and a per-base power constraint. Based on this preliminary result we identify

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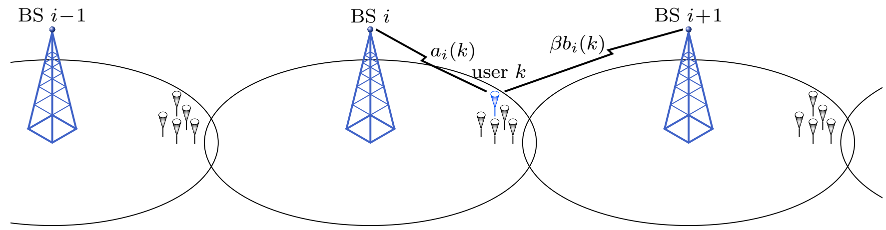

Fig. 1. Part of infinite linear cell-array. Each user receives a signal from the two closest BSs.

the optimal scheduling policy. To make a first comparison with SCP we note that the post-scheduling signal-to-interference-plus-noise ratio (SINR) can be viewed as the maximum of a random sample of size n. This observation allows us to draw on Extreme Value Theory (EVT) [12], [13] to characterize the asymptotic behavior of the mean SINR with the number of users. We scrutinize our findings further by giving some exact result as well as several upper and lower bounds.

Notably, we derive asymptotic expressions for the number of users n_1 and n_2 required to attain the same mean SINR with ZFBF and SCP respectively. Put differently, we find the extra number users needed per cell to compensate for the lack of coordination with SCP. Interestingly, the ratio n_2/n_1 is not bounded, but grows logarithmically with the number of users n. Finally, we demonstrate that the difference in sum-rate between ZFBF and SCP is significant for all practical values of number of users.

II. System Model

We consider a linear cell-array with n single-antenna users and one single antenna BS in each cell. For symmetry reasons we assume the cell-array to extend indefinitely in both directions. However, the choice is technical and a finite network would have no qualitative impact on the results. We further assume intra-cell time division multiplexing (TDM) with synchronous time slots (scheduling intervals) across the network. The time slots are assumed to be sufficiently short for the channel to be constant within a slot, yet contain enough symbols to employ capacity achieving codes. In the following we will focus on an arbitrary symbol transmission interval within an arbitrary time slot and omit explicit reference to time. The received signal for user k in cell i is given by

$$y_i(k) = a_i(k)x_i + \beta b_i(k)x_{i+1} + z_i(k), \tag{1}$$

where x_i,$x_{i+1} \in \mathbb{C}$ are the antennae outputs from BS i and BS $i+1$, $a_i(k)$,$b_i(k) \in \mathbb{C}N(0,1)$ are the corresponding fading coefficients and $z_i(k) \in \mathbb{C}N(0,1)$ is normalized Gaussian noise. The constant $\beta \in [0,1]$ reflects a difference in the path loss on the two signal paths.

In each time slot there is one user, denoted k_i^*, that is scheduled in each cell i. If we focus on the scheduled users we have the following input-output relationship

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{z}$$

where $\boldsymbol{y} = \{y_i(k_i^*)\}_{i\in\mathbb{Z}}$, $\boldsymbol{x} = \{x_i\}_{i\in\mathbb{Z}}$, $\boldsymbol{z} = \{z_i(k_i^*)\}_{i\in\mathbb{Z}}$ are infinite column vectors and H is a bidiagonal infinite matrix with

$$[H]_{i,j} = \begin{cases} a_i(k_i^*), & i = j \\ \beta b_i(k_i^*), & i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

In the case of multicell linear beamforming (preprocessing) one applies a matrix B such that $\boldsymbol{x} = B\boldsymbol{s}$ where $\boldsymbol{s} = \{s_i\}_{i\in\mathbb{Z}}$ is an infinite column vector. Here s_i is the information symbol intended for user k_i^*. In order to fulfill a per BS power constraint we require $\mathbb{E}|x_i|^2 \leq \rho$. With the assumption $\mathbb{E}|s_i|^2 = 1$ this is equivalent to the ℓ^2-norm of each row of B being no more than $\sqrt{\rho}$.

Finally, we assume that complete channel state information (CSI) is available to the BSs, while the users employ conventional single user receivers. The former assumption is clearly hard to fulfill in practical systems. Nevertheless, we will not focus on this aspect of BS coordination here.

III. Single-Cell network bound

As a reference we first consider the case with no inter-cell interference ($\beta = 0$). The channel model now reduces to

$$y_i(k) = a_i(k)x_i + z_i(k). \tag{2}$$

Conceptually this is equivalent to a network with one single isolated cell. The channel model in (2) is the prototype model for illustrating the potential gains of multiuser scheduling. The rate optimal scheduling policy is to select the user k with the largest gain $|a_i(k)|$ in cell i which yields the instantaneous SINR [14]

$$\Gamma^i_{\text{SCN}}(n) = \max_{1\leq k\leq n} \rho|a_i(k)|^2.$$

In the sequel we will drop the index i when denoting $\Gamma^i_{\text{SCN}}(n)$ since its distribution is independent of the particular cell. To find the distribution of $\Gamma_{\text{SCN}}(n)$ we first note that $\Gamma_{\text{SCN}} := \Gamma_{\text{SCN}}(1)$ is exponentially distributed with cdf

$$F_{\text{SCN}}(x) = 1 - e^{-x/\rho}, \quad x \geq 0.$$

Since $\Gamma_{\text{SCN}}(n)$ can be phrased as the largest order statistics of Γ_{SCN} the cdf F^n_{SCN} of $\Gamma_{\text{SCN}}(n)$ is [15]

$$F^n_{\text{SCN}}(x) = \big(1 - e^{-x/\rho}\big)^n, \quad x \geq 0.$$

It is well know that the corresponding mean is

$$\mathbb{E}\,\Gamma_{\text{SCN}}(n) = \int_0^\infty x\,\mathrm{d}F^n_{\text{SCN}} = \rho\,\mathrm{H}_n,$$

where $\mathrm{H}_n := \sum_{k=1}^{n} 1/k$ is the nth harmonic number [15]. For large n the asymptotic expression

$$\mathrm{H}_n = \ln n + \gamma + O\left(1/n\right)$$

is particularly useful. Here $\gamma = 0.577..$ is the Euler constant [16].

IV. Single-cell processing

In conventional SCP networks all signal transmissions are done independently on a per-cell basis. Specifically, the ith BS transmits $x_i = \sqrt{\rho}\, s_i$ directly without compensating for intercell interference. The instantaneous SINR with rate optimal scheduling is therefore

$$\Gamma_{\text{SCP}}(n) = \max_{1\le k\le n} \frac{|a_i(k)|^2}{1/\rho + \beta^2 |b_i(k)|^2}.$$

In [7] it is shown that the cdf F_{SCP} of $\Gamma_{\text{SCP}} := \Gamma_{\text{SCP}}(1)$ is

$$F_{\text{SCP}}(x) = 1 - \frac{e^{-x/\rho}}{1+\beta^2 x}, \qquad x \ge 0.$$

Hence, from the theory of order statistics we have that the cdf F^n_{SCP} of $\Gamma_{\text{SCP}}(n)$ is

$$F^n_{\text{SCP}}(x) = \left(1 - \frac{e^{-x/\rho}}{1+\beta^2 x}\right)^n, \qquad x \ge 0.$$

Unfortunately, exact analytical expressions based on the above distribution are hard to obtain and give little insight into the key quantities. Instead we will take an approach based on EVT in Section VI.

V. Multicell Zero-forcing beamforming

We now turn to BS coordination in the form of ZFBF. Interestingly, the considered interference model will allow us to shed some new light on the otherwise well known zero-forcing problem.

By definition of zero-forcing there should be no interference for scheduled users. Thus, we seek a beamforming matrix B such that $HB = D$ for some diagonal matrix $D = \operatorname{diag}(.., d_{-1}, d_0, d_1, ..)$. To ensure that an obtained solution is unique we require that the diagonal elements d_i are non-negative real numbers and that each BS transmits at full power, i.e. $\mathbb{E}|x_i|^2 = \rho$.

By combining $HB = D$ and $x = Bs$ we have

$$d_i s_i = a_i(k_i^*) x_i + \beta b_i(k_i^*) x_{i+1}, \qquad \forall\, i \in \mathbb{Z} \tag{3}$$

which is a difference equation in x_i. Without a constraint on the transmit power one can easily infer that

$$x_i = \sum_{j=i}^{\infty} \left(\prod_{l=i}^{j-1} -\beta \frac{b(k_l^*)}{a(k_l^*)} \right) \frac{d_j}{a(k_j^*)} s_j \tag{4}$$

is a valid solution. However, since (4) implies $\mathbb{E}\{s_i x_{i+1}\} = 0$ it follows from (3) that the power constraint is satisfied for

$$|d_i|^2 = \rho(|a_i(k_i^*)|^2 - \beta^2 |b_i(k_i^*)|^2) \tag{5}$$

given that

$$|a_i(k_i^*)| \ge \beta |b_i(k_i^*)|. \tag{6}$$

The effective channel after zero-forcing and scheduling is then

$$y(k_i^*) = \rho^{1/2}(|a_i(k_i^*)|^2 - |b_i(k_i^*)|^2)^{1/2} s_i + z(k_i^*). \tag{7}$$

Thus, the interference is eliminated at the expense of a power penalty. It also follows that the beamforming matrix is given as

$$[B]_{i,j} = \begin{cases} 0, & i > j \\ (1-|r_j|^2)^{1/2}, & i = j \\ (1-|r_j|^2)^{1/2} \prod_{l=i}^{j-1} r_l, & i < j, \end{cases}$$

where $r_i = -\beta \frac{b_i(k_i^*)}{a_i(k_i^*)}$.

The above solution builds on the assumption that $|a_i(k_i^*)| \ge \beta |b_i(k_i^*)|$. This says that channel gain to the host BS is stronger than the neighboring BS. This is a reasonable scheduling criteria in a multiuser setting. Nevertheless, to tackle the general case we redefine r_i to

$$r_i = -\beta \frac{b_i(k_i^*)}{a_i(k_i^*)} \Big/ \max\left\{1, \left|\beta \frac{b_i(k_i^*)}{a_i(k_i^*)}\right|\right\}.$$

The implications of this is that user k_i^* does not receive a useful signal whenever $|a_i(k_i^*)| < \beta |b_i(k_i^*)|$.

A. Scheduling

In order to characterize the performance of ZFBF we need to specify a particular scheduling policy. From (7) we can immediately conclude that rate optimal scheduling amounts to

$$k_i^* = \arg\max_{1\le k\le n} |a_i(k_i)|^2 - \beta^2 |b_i(k_i)|^2.$$

The instantaneous post-scheduling SINR is now

$$\Gamma_{\text{ZF}}(n) = \max_{1\le k\le n} \rho\big[|a_i(k)|^2 - \beta^2 |b_i(k)|^2\big]_+,$$

where $[\,\cdot\,]_+ := \max\{\,\cdot\,, 0\}$. Note that it is the received signal power after interference cancellation that determines the final performance. In the Appendix we find that the cdf of $\Gamma_{\text{ZF}} := \Gamma_{\text{ZF}}(1)$ is

$$F_{\text{ZF}}(x) = 1 - \frac{e^{-x/\rho}}{1+\beta^2}, \qquad x \ge 0. \tag{8}$$

Hence, the cdf F^n_{ZF} of $\Gamma_{\text{ZF}}(n)$ is

$$F^n_{\text{ZF}}(x) = \left(1 - \frac{e^{-x/\rho}}{1+\beta^2}\right)^n, \qquad x \ge 0.$$

For comparison, we also consider two suboptimal scheduling policies that have previously been proposed in the literature [4], [9], [11]. The first policy is to schedule the user with the largest gain to the host BS,

$$k_i^* = \arg\max_{1\le k\le n} |a_i(k_i^*)|^2.$$

To denote the resulting instantaneous SINR we use $\Gamma_{\text{ZF},2}(n)$. The second policy is to schedule the user with largest ratio between the gains to the host BS to the neighboring BS,

$$k_i^* = \arg\max_{1\le k\le n} \frac{|a_i(k_i^*)|^2}{|b_i(k_i^*)|^2}.$$

In line with the previous notation we use $\Gamma_{\text{ZF},3}(n)$ to denote the resulting instantaneous SINR.

VI. Asymptotic results for the mean SINR

In this section we obtain some asymptotic results on the performance of ZFBF and SCP. We first note that $\Gamma_{\text{SCN}}(n)$, $\Gamma_{\text{SCP}}(n)$ and $\Gamma_{\text{ZF}}(n)$ can all be viewed as the largest order statistics from a sample of size n. Based on this observation we make use of Extreme Value Theory (EVT) [12], [13], which is concerned with the asymptotic distribution of order statistics.

In the sequel, it will be convenient to extend the definitions of $\Gamma_{\text{SCN}}(y), \Gamma_{\text{SCP}}(y)$ and $\Gamma_{\text{ZF}}(y)$ to all $y \in \mathbb{R}_+$. To this end we take the distributions $F^y_{\text{SCN}}, F^y_{\text{SCP}}$ and F^y_{SCN} as definitions of $\Gamma_{\text{SCN}}(y), \Gamma_{\text{SCP}}(y)$ and $\Gamma_{\text{ZF}}(y)$ for non-integers y.

A. Some Extreme Value Theory

It is readily shown that Γ_χ, $\chi \in \{\text{SCN}, \text{SCP}, \text{ZF}\}$, are all in the domain of attraction of the Gumbel distribution (see the Appendix for technical conditions). Thus, according to EVT there exist normalizing functions $\mu_\chi(y)$ and $\nu_\chi(y)$ such that

$$\lim_{y\to\infty} F^y_\chi\big(\mu_\chi(y) + \nu_\chi(y)x\big) = G(x) \quad \text{for all } x, \tag{9}$$

where $G(x) := e^{-e^{-x}}$ is the Gumbel distribution. Furthermore, the normalizing functions can be selected to be

$$\mu_\chi(y) = g_\chi(y) \quad \text{and} \quad \nu_\chi(y) = g_\chi(ye) - g_\chi(y), \tag{10}$$

where $g_\chi(y) := F_\chi^{-1}(1 - 1/y)$.

The relationship in (9) corresponds to convergence in distribution. Additionally, one can also show that there is convergence in moments [17]. This means that once we obtain the normalizing functions we also have a characterization of the asymptotic behavior of the mean. In particular, by computing the first moment of the Gumbel distribution we get

$$\overline{\Gamma}_\chi(n) := \mathbb{E}\,\Gamma_\chi(n) \approx \mu_\chi(n) + \gamma\nu_\chi(n),$$

for large number of users n [12].

B. Explicit relationships for the normalizing functions

For Γ_{SCN} and Γ_{ZF} it is straightforward to find the normalizing functions from (10). In particular, we have

$$\mu_{\text{SCN}}(y) = \rho \ln y \tag{11}$$
$$\mu_{\text{ZF}}(y) = \rho \ln y - \rho\ln(1+\beta^2) \tag{12}$$
$$\nu_{\text{SCN}}(y) = \nu_{\text{ZF}}(y) = \rho.$$

Unfortunately, for Γ_{SCP} the normalizing functions can not be expressed in terms of elementary functions. To proceed we make use of the Lambert W function which is defined through the relation $W(x)e^{W(x)} = x$ [18]. We then obtain

$$\mu_{\text{SCP}}(y) = \rho W\left(\frac{y}{\beta^2\rho} e^{\frac{1}{\beta^2\rho}}\right) - \frac{1}{\beta^2},$$
$$\nu_{\text{SCP}}(y) = \rho W\left(\frac{ye}{\beta^2\rho} e^{\frac{1}{\beta^2\rho}}\right) - \rho W\left(\frac{y}{\beta^2\rho} e^{\frac{1}{\beta^2\rho}}\right) \underset{y\to\infty}{\longrightarrow} \rho,$$

where the limit can be inferred from $W(x) = \ln x - \ln\ln x + O(\frac{\ln\ln x}{\ln x})$ [18]. To gain more insight into the limiting behavior one can use more refined asymptotic expansions of $W(x)$. However, we will focus next on an alternative indirect characterization of $\mu_{\text{SCP}}(y)$.

C. Implicit relationships for the normalizing functions

Interestingly, we can express $\mu_{\text{SCP}}(y)$ and $\mu_{\text{ZF}}(y)$ implicitly in terms of $\mu_{\text{SCN}}(y)$. From (11) and (12) we see that

$$\mu_{\text{ZF}}\big(y(1+\beta^2)\big) = \mu_{\text{SCN}}(y).$$

Similarly, from the observation

$$1 - F_{\text{SCP}}(\mu_{\text{SCN}}(y)) = \frac{1}{y(1+\beta^2\rho\ln y)}$$

we obtain the following relationship

$$\mu_{\text{SCP}}\big(y(1+\beta^2\rho\ln y)\big) = \mu_{\text{SCN}}(y).$$

All in all we can infer from above that

$$\overline{\Gamma}_{\text{SCP}}\big(n(1+\beta^2\rho\ln n)\big) \approx \overline{\Gamma}_{\text{SCN}}(n) \approx \overline{\Gamma}_{\text{ZF}}\big(n(1+\beta^2)\big), \tag{13}$$

for large number of users n. Thus, to attain the same mean SINR as in a single-cell network with n users one needs asymptotically $n(1+\beta^2\rho\ln n)$ users per cell with SCP and $n(1+\beta^2)$ users per cell with ZFBF. It is interesting to note that ratio of required users with SCP to ZFBF is not bounded, but grows logarithmically with the number of users n. We also point out that the ratio is linear in ρ. Thus, ZFBF is increasingly beneficial with increasing SNRs which is consistent with previous results.

VII. Equalities and bounds for the mean SINR

Even though the above analysis reveals the asymptotic behavior of the mean SINRs it fails to say anything about the rates of convergence. Furthermore, EVT is not directly applicable to the study of $\Gamma_{\text{ZF},2}(n)$ and $\Gamma_{\text{ZF},3}(n)$ since they are not formulated as order statistics. Below we give some exact result together with several upper and lower bounds. The proofs can be found in the Appendix.

We first consider some results pertaining to ZFBF and suboptimal scheduling.

Proposition 1: *Let the user k with the largest ratio $|a_i(k)|^2/|b_i(k)|^2$ be scheduled in each cell i. The mean SINR with ZFBF has the following upper bound*

$$\overline{\Gamma}_{\text{ZF},3}(n) < 2\rho.$$

Proposition 1 is interesting because the upper bound is independent of the number of users per-cell. Clearly, the benefit of adding more users is severely limited. This is in contrast with the other suboptimal scheduling strategy which we consider below.

Proposition 2: *Let the user k with the largest gain $|a_i(k)|^2$ be scheduled in each cell i. The mean SINR with ZFBF is*

$$\begin{aligned}\overline{\Gamma}_{\text{ZF},2}(n) &= \rho\,\mathrm{H}_n - \rho\beta^2\Big(1 - nB\Big(\tfrac{1+\beta^2}{\beta^2}, n\Big)\Big) \\ &\le \rho\,\mathrm{H}_n - \rho\beta^2\frac{n}{n+1},\end{aligned} \tag{14}$$

where $B(x,y)$ denotes the beta function [16]. The inequality is strict for all $0 < \beta^2 < 1$.[1]

[1] Proposition 2 corrects a mistake in [19] where $\overline{\Gamma}_{\text{ZF},2}(n)$ was set equal to the second line of (14).

From (14) and the asymptotic expansion $\mathrm{H}_n \sim \ln n + \gamma$ it follows that

$$\overline{\Gamma}_{\mathrm{ZF},2}\big(ne^{\beta^2}\big) \approx \ln n + \gamma \approx \overline{\Gamma}_{\mathrm{SCN}}(n).$$

for n large. Thus, there is a performance degradation compared to optimal scheduling. To exemplify, for $\beta = 1$ one needs approximately 35% more users to attain the same mean SINR.

For the following results we will assume that $\beta \neq 0$ in order to obtain strict inequalities. For the special case $\beta = 0$ we have that Γ_{SCN}, Γ_{SCP} and Γ_{ZF} are identical and the results are trivial.

Proposition 3: *The mean SINR with ZFBF and optimal scheduling is*

$$\begin{aligned}\overline{\Gamma}_{\mathrm{ZF}}(n) &= \rho\,\mathrm{H}_n - \rho\sum_{k=1}^{n}\left(\frac{\beta^2}{1+\beta^2}\right)^k\frac{1}{k} \\ &> \rho\,\mathrm{H}_n - \rho\ln(1+\beta^2),\end{aligned} \tag{15}$$

where the last inequality is asymptotically tight. Additionally,

$$\overline{\Gamma}_{\mathrm{ZF}}\big(n(1+\beta^2)\big) < \overline{\Gamma}_{\mathrm{SCN}}(n) < \overline{\Gamma}_{\mathrm{ZF}}\big(n(1+\tfrac{n+1}{n}\beta^2)\big). \tag{16}$$

We next give an upper bound to the performance of SCP with optimal scheduling.

Proposition 4: *Assume SCP and optimal scheduling. The mean SINR satisfies the following upper bound*

$$\overline{\Gamma}_{\mathrm{SCP}}\big(n(1+\beta^2\rho\ln n)\big) < \overline{\Gamma}_{\mathrm{SCN}}(n). \tag{17}$$

Note that we already know from Section VI that the inequality is asymptotically tight.

VIII. Implications for the per-cell sum-rate

We now briefly consider the per-cell sum-rates. Define

$$C_\chi(n) := \mathbb{E}\log_2\big(1+\Gamma_\chi(n)\big)$$

for $\chi \in \{\mathrm{SCN},\mathrm{SCP},\mathrm{ZF}\}$. Unfortunately, the concavity of the $\log_2\big(1+(\cdot)\big)$ function prevents most of the results concerning the mean SINR to automatically carry over to the per-cell sum-rate. However, we still have the following results.

Proposition 5: *The per-cell sum-rate with SCP and optimal scheduling satisfies the following bounds*

$$\log_2(1+\rho\ln n) < C_{\mathrm{SCP}}\big(n(1+\beta^2\rho\ln n)\big) < \log_2(1+\rho\,\mathrm{H}_n).$$

The per-cell sum-rate with ZFBF and optimal scheduling satisfies

$$\log_2(1+\rho\ln n) < C_{\mathrm{ZF}}\big(n(1+\beta^2)\big) < \log_2(1+\rho\,\mathrm{H}_n),$$

for n sufficiently large.

The above results together with (13) suggest the approximation

$$C_{\mathrm{SCP}}\big(n(1+\beta^2\rho\ln n)\big) \approx C_{\mathrm{SCN}}(n) \approx C_{\mathrm{ZF}}\big(n(1+\beta^2)\big) \tag{18}$$

for n large. We will illustrate the accuracy of the above relations in the next section. Proposition 5 also implies that the difference in the per-cell sum-rate with SCP and ZFBF goes to zero as the number of users goes to infinity. Let $\Delta C(n) := C_{\mathrm{ZF}}(n) - C_{\mathrm{SCP}}(n)$ and consider the estimate

$$\begin{aligned}\Delta C(n) &\approx \log_2\big(1+\mu_{\mathrm{ZF}}(n)\big) - \log_2\big(1+\mu_{\mathrm{SCP}}(n)\big) \\ &= \log_2\left(1+\frac{\ln(1+\beta^2\rho\ln t)-\ln(1+\beta^2)}{1/\rho+\ln t}\right) \\ &\approx \log_2(e)\frac{\ln\big(\frac{\beta^2}{1+\beta^2}\rho\ln t\big)}{\ln t}\end{aligned} \tag{19}$$

where t is the unique solution to $n = t(1+\beta^2\rho\ln t)$. Hence $\Delta C(n)$ converges to zero, but only at the rate of $O\left(\frac{\ln\ln n}{\ln n}\right)$.

IX. Numerical results

In this section we illustrate some our results numerically. Since $\Gamma_\chi(n)$ is a direct function of exponential random variables we can easily evaluate $C_\chi(n)$ through Monte Carlo simulations ($\chi \in \{\mathrm{SCN},\mathrm{SCP},\mathrm{ZF}\}$). We first consider the approximate relationship in (18). Specifically, in Fig. 2 we plot the sum-rate per-cell corresponding to

(i) a SCN scenario with n users,
(ii) ZFBF with $n(1+\beta^2)$ users per-cell and
(iii) SCP with $n(1+\beta^2\rho\ln n)$ users per-cell

in the same plot. In all three cases the mean SNR is $\rho = 10$ dB and for (ii) and (iii) we have $\beta = 1$. Observe that there is a remarkably good fit between the three graphs even for small n. Thus, the approximations in (18) seems to be well justified. The magnified section of the plot also reveals that the ordering between (i) and (ii) is as expected from (16). However, we point out that part of the difference is likely to result from the concavity of the rate function. The ordering of (i) and (iii) is also as one would expect from (17). However, in this case the concavity of the rate function is likely to lead to a small decrease in the difference as one would otherwise expect.

The large difference in the number of users per cell between multicell ZFBF and SCP to attain the same performance is also interesting. To exemplify consider a SCN with $n = 10$ users. One then needs $n(1+\beta^2\ln n) \approx 240$ users with SCP as opposed to $n(1+\beta^2) \approx 20$ users with ZFBF to attain the same rate per-cell in a multicell network.

In Fig. 3 we plot the sum-rate per-cell corresponding to a SCN, multicell ZFBF and SCP for the same number of users. Note that there is a significant gain with ZFBF over SCP. In accordance with (19) there is little reduction in the gain even for very large number of users. The convergence of the two curves appears to have little impact in the pre-asymptotic user regime.

X. Conclusion

We have considered coordinated multicell ZBFB on the fading downlink of a linear cell-array. The beamforming coefficients and the optimal scheduling policy under a per-base power constraint were both identified. Furthermore, the resulting mean post-scheduling SINR was extensively studied. To put the performance in perspective SCP with optimal scheduling was used as a benchmark. Specifically, we gave asymptotic expressions for the additional number of users per cell to compensate for inter-cell interference with ZFBF and

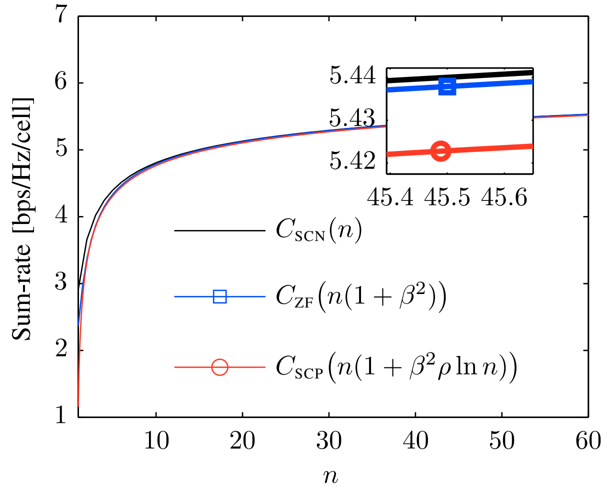

Fig. 2. The per-cell sum-rate for an interference free network with n users, ZFBF with $n(1+\beta^2)$ users, and SCP with $n(1+\beta^2\rho\ln n)$ users.

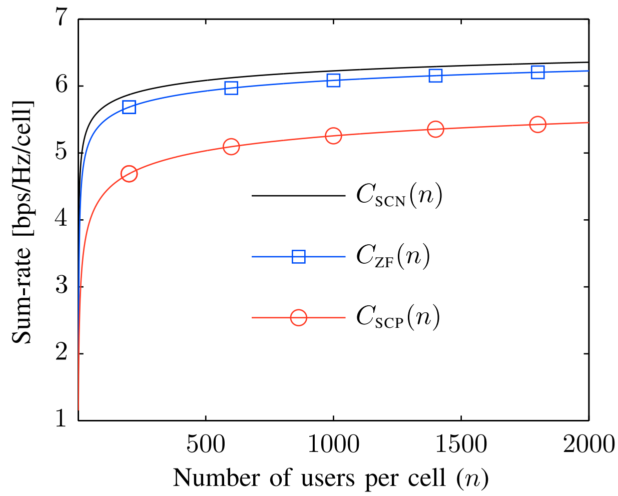

Fig. 3. The per-cell sum-rate for an interference free network, ZFBF and SCP as a function of the number of users per cell n.

SCP. The difference in per-cell sum-rate between SCP and multicell ZFBF goes to zero as the number of users goes to infinity. However, we demonstrated that the convergence is too slow to have any practical impact. Thus, for practical systems multicell ZFBF has a significant gain over SCP.

Appendix

A. $\Gamma_{\mathrm{SCN}}, \Gamma_{\mathrm{SCP}}$ *and* Γ_{ZF} *are in the domain of the Gumbel distribution*

The claim follows immediately from the following result due to Von Mises [20]:
Suppose X is random variable with cdf $F(x)$ and a pdf $f(x)$ which is positive and differentiable on a neighborhood of $x^* := \sup\{x|F(x) < 1\}$. If

$$\lim_{x\to x^*} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-F(x)}{f(x)}\right) = 0, \tag{20}$$

then X is in the domain of attraction of the Gumbel distribution.

B. *The distribution of* $\Gamma_{\mathrm{ZF}}(n)$ *is given according to* (8)

We have by definition $\Gamma_{\mathrm{ZF}} \overset{d}{=} \rho[|a_i(k)|^2 - \beta^2|b_i(k)|^2]_+$ for a fixed i and k. Since Γ_{ZF} cannot assume negative values we have $F_{\mathrm{ZF}}(x) = 0$ for $x < 0$. Let $F_{\mathrm{ZF}}(x|z)$ denote the cdf of Γ_{ZF} conditioned on $|b_i(k)|^2 = z$, let $F_{|a|^2}(x)$ denote the cdf of $|a_i(k)|^2$ and let $f_{|b|^2}(x)$ denote the pdf of $|b_i(k)|^2$. Note that $|a_i(k)|^2$ and $|b_i(k)|^2$ are exponential random variables with unit mean. By marginalizing over $|b_i(k)|^2$ the cdf of Γ_{ZF} can be expressed as

$$\begin{aligned} F_{\mathrm{ZF}}(x) &= \int_0^\infty F_{\mathrm{ZF}}(x|z) f_{|b|^2}(z)\,\mathrm{d}z \\ &= \int_0^\infty F_{|a|^2}\left(\tfrac{x}{\rho} + \beta^2 z\right) f_{|b|^2}(z)\,\mathrm{d}z \\ &= \int_0^\infty \left(1 - e^{-(\frac{x}{\rho}+\beta^2 z)}\right) e^{-z}\,\mathrm{d}z \\ &= 1 - \frac{e^{-x/\rho}}{1+\beta^2}, \end{aligned}$$

for $x > 0$.

C. *Proof of Proposition* 1

Let $A_k := |a_i(k)|^2, B_k := |b_i(k)|^2$ and $C_k := A_k/B_k$ for a fixed i. We seek $\mathbb{E}[A_{k^*} - \beta^2 B_{k^*}]_+$ where $k^* = \arg\max_{1\le k\le n} C_k$. The crucial point to observe is that knowing that C_{k^*} is the largest out of n variables do not give any extra information regarding A_{k^*} once the exact value of C_{k^*} is given. Thus,

$$f_{A_{k^*}}(x|C_{k^*} = z) = f_{A_k}(x|C_k = z)$$

for all k. Now since A_k and B_k have exponential distributions if follows that C_k has a F-distribution [21, p. 946] with pdf

$$f_{C_k}(z) = \frac{1}{(1+z)^2}, \quad z \ge 0.$$

Furthermore, C_k conditioned on A_k has an inverse exponential distribution with pdf

$$f_{C_k}(z|A_k = x) = \frac{x}{z^2} e^{-x/z}, \quad z \ge 0.$$

Based on Bayes' theorem we now obtain

$$\begin{aligned} f_{A_{k^*}}(x|C_{k^*} = z) &= \frac{f_{A_k}(x) f_{C_k}(z|A_k = x)}{f_{C_k}(z)} \\ &= \left(1 + \frac{1}{z}\right)^2 x\, e^{-\left(1+\frac{1}{z}\right)x}. \end{aligned}$$

This is a Gamma distribution [22, p. 103] with mean

$$\mathbb{E}\{A_{k^*}|C_{k^*} = z\} = \frac{2}{\left(1+\frac{1}{z}\right)^2} < 2.$$

Thus, regardless of the distribution of C_{k^*} we have $\mathbb{E}\{A_{k^*}\} < 2$. Finally,

$$\overline{\Gamma}_{\mathrm{ZF},3}(n) = \rho\mathbb{E}[A_{k^*} - \beta^2 B_{k^*}]_+ < 2\rho$$

which is the desired result.

D. *Proof of Proposition* 2

Throughout the proof of Proposition 2 we let $\rho = 1$ for simplicity. However, the general results follow by noting that the SINR is linear in ρ for ZFBF.

Let $A_k := |a_i(k)|^2$, $B_k := \beta^2 |b_i(k)|^2$ and $k^* := \arg\max_{1\le k\le n} A_k$. Since A_k and B_k are exponential random variables it follows that A_{k^*} has pdf

$$f_{A_{k^*}}(x) = ne^{-x}\big(1-e^{-x}\big)^{n-1}, \quad x \ge 0$$

and B_{k^*} has pdf

$$f_{B_{k^*}}(y) = \frac{1}{\beta^2} e^{-x/\beta^2}, \quad x \ge 0.$$

Now, define $B^{'}_{k^*}$ such that

$$[A_{k^*} - B_{k^*}]_+ = A_k - B^{'}_{k^*}.$$

The distribution of $B^{'}_{k^*}$ conditioned on A_{k^*} is then

$$F_{B^{'}_{k^*}}(y|A_{k^*}=x) = \begin{cases} 1-e^{-y/\beta^2}, & y \le x \\ 1, & y > x. \end{cases}$$

and the conditional mean is

$$\begin{aligned} \mathbb{E}\{B^{'}_{k^*}|A_{k^*}=x\} &= \int_0^{\infty} 1 - F_{B^{'}_{k^*}}(y|A_{k^*}=x)\mathrm{d}y \\ &= \beta^2\big(1-e^{-x}\big). \end{aligned}$$

Finally,

$$\begin{aligned} \overline{\Gamma}_{\text{ZF},2}(n) &= \mathbb{E}\,[A_{k^*} - B_{k^*}]_+ \\ &= \iint_{x,y\ge 0} (x-y) f_{A_{k^*}}(x) f_{B^{'}_{k^*}}(y|A_{k^*}=x)\,\mathrm{d}y\mathrm{d}x \\ &= \int_{x\ge 0} (x - \mathbb{E}\{B^{'}_{k^*}|A_{k^*}=x\}) f_{A_{k^*}}(x)\,\mathrm{d}x \\ &= \int_{x\ge 0} \big(x - \beta^2\big(1-e^{-x/\beta^2}\big)\big) f_{A_{k^*}}(x)\,\mathrm{d}x \\ &= \mathrm{H}_n - \int_{x\ge 0} \beta^2\big(1-e^{-x/\beta^2}\big) ne^{-x}\big(1-e^{-x}\big)^{n-1}\,\mathrm{d}x \\ &= \mathrm{H}_n - \beta^2 + \beta^2 n \int_0^1 t^{1/\beta^2}(1-t)^{n-1}\,\mathrm{d}x \\ &= \mathrm{H}_n - \beta^2 + \beta^2 n B(1+1/\beta^2, n) \\ &\le \mathrm{H}_n - \beta^2 + \beta^2 \frac{1}{n+1} \end{aligned}$$

where use the substitution $t = 1 - e^{-x}$. The inequality follows from observing that Beta-function is monotonically decreasing in both variables. Thus $B(1+1/\beta^2, n) \le B(2,n) = \frac{1}{n(n+1)}$ with equality only for $\beta^2 = 1$.

Before we prove Proposition 3 we state the following useful result on the harmonic numbers.

E. *Result on the harmonic numbers*

There exist monotonically decreasing functions $\epsilon(x)$ and $\eta(x)$ such that the harmonic numbers satisfy the following relations

$$\begin{aligned} \mathrm{H}_x &= \ln x + \gamma + \epsilon(x) &(21)\\ &= \ln x + \gamma + \frac{1}{2x} - \eta(x), &(22) \end{aligned}$$

for $x \ge 1$ [23].

F. *Proof of Proposition* 3

1) Proof of (15)*:* A direct calculation gives

$$\begin{aligned} \overline{\Gamma}_{\text{ZF}}(n) &= \int_0^{\infty} 1 - F^n_{\text{ZF}}(x)\,\mathrm{d}x \\ &= \int_0^{\infty} 1 - \left(1 - \frac{e^{-x/\rho}}{1+\beta^2}\right)^n \mathrm{d}x \\ &= \rho \int_{\frac{\beta^2}{1+\beta^2}}^1 \frac{1-z^n}{1-z}\,\mathrm{d}z \\ &= \rho \int_{\frac{\beta^2}{1+\beta^2}}^1 \sum_{k=1}^n z^{k-1}\,\mathrm{d}z \\ &= \rho \sum_{k=1}^n \frac{1}{k} - \rho \sum_{k=1}^n \left(\frac{\beta^2}{1+\beta^2}\right)^k \frac{1}{k} \\ &> \rho\,\mathrm{H}_n - \rho \ln(1+\beta^2) \end{aligned}$$

where we have used the substitution $z = 1 - \frac{e^{-x}}{1+\beta^2}$. The inequality follows from the identity [21, p. 68]

$$\ln(x) = \sum_{k=1}^{\infty} \left(\frac{x-1}{x}\right)^k \frac{1}{k}.$$

2) Proof of (16)*:* The left side follows from the following calculation

$$\begin{aligned} \overline{\Gamma}_{\text{ZF}}\big(n(1+\beta^2)\big) &= \int_0^{\infty} 1 - \left(1 - \frac{e^{-x/\rho}}{1+\beta^2}\right)^{n(1+\beta^2)} \mathrm{d}x \\ &< \int_0^{\infty} 1 - \big(1-e^{-x/\rho}\big)^n\,\mathrm{d}x \\ &= \overline{\Gamma}_{\text{SCN}}(n) \end{aligned}$$

where we use Bernoulli's inequality, $(1+x)^r > 1 + rx$ for $x > -1$ and $r > 1$ [24].

We now turn to the right hand side of the inequality. Let $y := n(1 + \frac{n+1}{n}\beta^2)$. From (15) and (22) we have

$$\begin{aligned} \overline{\Gamma}_{\text{ZF}}(y)/\rho &> \ln y + \gamma + \frac{1}{2y} - \eta(y) - \ln\big(1+\beta^2\big) \\ &= \ln n + \gamma + \frac{1}{2y} - \eta(y) + \ln\Big(1 + \frac{1}{n}\frac{\beta^2}{1+\beta^2}\Big) \end{aligned}$$

and

$$\overline{\Gamma}_{\text{SCN}}(n)/\rho = \ln n + \gamma + \frac{1}{2n} - \eta(n).$$

Thus, since $\eta(x)$ is monotonically decreasing it is sufficient to show

$$\ln\Big(1 + \frac{1}{n}\frac{\beta^2}{1+\beta^2}\Big) + \frac{1}{2n(1+\beta^2+\frac{1}{n}\beta^2)} \ge \frac{1}{2n}. \quad (23)$$

To proceed we use the following inequality [21, p. 68]

$$\ln\left(1+\frac{1}{x}\right) > \frac{1}{x+1}, \quad x>0.$$

Applied to the left side of (23) this gives

$$\frac{\beta^2}{n(1+\beta^2)+\beta^2} + \frac{1}{2n(1+\beta^2+\frac{1}{n}\beta^2)} = \frac{1+2\beta^2}{1+\beta^2+\frac{1}{n}\beta^2}\frac{1}{2n}.$$

Thus, $\overline{\Gamma}_{\text{ZF}}\left(n(1+\frac{n+1}{n}\beta^2)\right) > \overline{\Gamma}_{\text{SCN}}(n)$ for $n \geq 1$.

Before we prove Propostion 4 we will review the probability integral transform theorem [25].

G. *The probability integral transform theorem*

Suppose X is a random variable with continuous cdf F_X. By the integral transform theorem we have that $U := F_X(X)$ is a uniform random variable on $[0,1]$. Thus, $X = F_X^{-1}(U)$. The following extension is straight forward. Define $X_+ := [X]_+$ and let $F_{X_+}(x)$ denote its cdf. Then $F_{X_+}^{-1}(x) = [F_X^{-1}(x)]_+$. Therefore

$$X_+ = [F_X^{-1}(U)]_+ = F_{X_+}^{-1}(U).$$

H. *Proof of Proposition* 4

To prove (17) the following results will be convenient.

$$\Gamma_{\text{SCN}}(y) \overset{d}{=} \Gamma_{\text{SCP}}(y) + \rho\ln\left(1+\beta^2\Gamma_{\text{SCP}}(y)\right) \tag{24}$$

$$F_U\left(\mathbb{E}\,U^{1/y}\right) = 1-\frac{1}{y+1} > 1-\frac{1}{y} \tag{25}$$

$$\overline{\Gamma}_{\text{SCP}}(y) > \rho\ln n \tag{26}$$

$$\mathbb{E}\ln\bigl(1+\beta^2\Gamma_{\text{SCP}}(y)\bigr) > \ln\bigl(1+\beta^2\rho\ln n\bigr) \tag{27}$$

Here U is uniformly distributed on $[0,1]$ and n is the unique solution to $y = n(1+\beta^2\rho\ln n) \geq 1$. Assuming the above results to be true, we obtain

$$\begin{aligned}
\overline{\Gamma}_{\text{SCP}}(y) &= \overline{\Gamma}_{\text{SCN}}(y) - \rho\mathbb{E}\ln\left(1+\beta^2\Gamma_{\text{SCP}}(y)\right) \\
&< \rho\ln y + \rho\gamma + \rho\epsilon(y) - \rho\ln\left(1+\beta^2\rho\ln n\right) \\
&= \rho\ln n + \rho\gamma + \rho\epsilon(y) \\
&< \rho\ln n + \rho\gamma + \rho\epsilon(n) \\
&= \overline{\Gamma}_{\text{SCN}}(n).
\end{aligned}$$

which is the desired result. The last inequality follows follows from the fact that $\epsilon(x)$ is monotonically decreasing.

1) Proof of (24)*:* By the probability integral transform theorem we have

$$U \overset{d}{=} F_{\text{SCN}}^y\Bigl(\Gamma_{\text{SCN}}(y)\Bigr) \overset{d}{=} F_{\text{SCP}}^y\Bigl(\Gamma_{\text{SCP}}(y)\Bigr).$$

This in turn yields

$$\begin{aligned}
\Gamma_{\text{SCN}}(y) &\overset{d}{=} [F_{\text{SCN}}^y]^{-1} \circ F_{\text{SCP}}^y\Bigl(\Gamma_{\text{SCP}}(y)\Bigr) \\
&= -\rho\ln\left(1-\Bigl[F_{\text{SCP}}^y\bigl(\Gamma_{\text{SCP}}(y)\bigr)\Bigr]^{1/y}\right) \\
&= -\rho\ln\left(\frac{e^{-\Gamma_{\text{SCB}}(y)/\rho}}{1+\beta^2\Gamma_{\text{SCB}}(y)}\right) \\
&= \Gamma_{\text{SCB}}(y) + \rho\ln\bigl(1+\beta^2\Gamma_{\text{SCB}}(y)\bigr).
\end{aligned}$$

2) Proof of (25)*:* The pdf and cdf of U are $F_U(x) = x$, $f_U(x) = 1$, $0 \leq x \leq 1$. Thus,

$$F_U\left(\mathbb{E}\,U^{1/y}\right) = \mathbb{E}\,U^{1/y} = \int_0^1 f_U(x)x^{1/y}\mathrm{d}x = 1-\frac{1}{y+1}.$$

3) Proof of (26)*:* Applying the probability integral theorem we have $U \overset{d}{=} F_{\text{SCP}}^y\bigl(\Gamma_{\text{SCP}}(y)\bigr)$. Thus, $U^{1/y} \overset{d}{=} F_{\text{SCP}}\Bigl(\Gamma_{\text{SCP}}(y)\Bigr)$. Therefore, if F_{SCP} is concave we have

$$\mathbb{E}\,U^{1/n} \leq F_{\text{SCP}}\Bigl(\overline{\Gamma}_{\text{SCP}}(y)\Bigr)$$

by Jensen's inequality. This in turn gives

$$\overline{\Gamma}_{\text{SCP}}(y) \geq F_{\text{SCP}}^{-1}\left(\mathbb{E}\,U^{1/y}\right) > F_{\text{SCP}}^{-1}\Bigl(1-\frac{1}{y}\Bigr) = \rho\ln n \tag{28}$$

where the second inequality follows from (26) and the last equality from the relation

$$F_{\text{SCP}}(\rho\ln n) = 1-\frac{1}{n(1+\beta^2\ln n)}.$$

To prove the concavity of F_{SCN} we show that its second derivative is non-positive.

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{\text{SCN}}(x) &= \left(1-e^{-g(x)}\right)'' \\
&= \Bigl(e^{-g(x)}g'(x)\Bigr)' \\
&= -e^{-g(x)}\Bigl(\bigl(g'(x)\bigr)^2 - g''(x)\Bigr) \\
&\leq 0
\end{aligned}$$

where $g(x) := x/\rho + \ln(1+\beta^2 x)$.

4) Proof of (27)*:* Let $\Lambda(y) := \ln\Bigl(1+\beta^2\,\Gamma_{\text{SCP}}(y)\Bigr)$. The cdf F_Λ^y of $\Lambda(y)$ is then

$$\begin{aligned}
F_\Lambda^y(x) &= F_{\text{SCN}}^y\left(\frac{e^x-1}{\beta^2}\right) \\
&= \left(1-e^{-x+\frac{e^x-1}{\rho\beta^2}}\right)^y.
\end{aligned}$$

If $F_\Lambda := F_\Lambda^1$ is concave we now have

$$\begin{aligned}
\mathbb{E}\ln\bigl(1+\beta^2\Gamma_{\text{SCP}}(y)\bigr) &= \mathbb{E}F_\Lambda^{-1}\left(U^{1/y}\right) \\
&\geq F_\Lambda^{-1}\left(\mathbb{E}\,U^{1/y}\right) \\
&= \ln\Bigl(1+\beta^2 F_{\text{SCP}}^{-1}\left(\mathbb{E}\,U^{1/y}\right)\Bigr) \\
&> \ln\bigl(1+\beta^2\rho\ln n\bigr),
\end{aligned}$$

where we use the probability integral transform theorem, Jensen's inequality and finally (26). To prove the concavity of F_Λ we demonstrate that its second derivative is non-positive.

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_\Lambda(x) &= \left(1-e^{-g(x)}\right)'' \\
&= -e^{-g(x)}\Bigl(\bigl(g'(x)\bigr)^2 - g''(x)\Bigr) \\
&= -e^{-g(x)}\left(\Bigl(1+\frac{e^x}{\rho\beta^2}\Bigr)^2 - \frac{e^x}{\rho\beta^2}\right) \\
&< 0,
\end{aligned}$$

where $g(x) := x + \frac{e^x-1}{\rho\beta^2}$.

I. *Proof of Proposition* 5

From Jensen's inequality and Proposition 4 we have

$$\begin{aligned}C_{\text{SCP}}\big(n(1+\beta^2\rho\ln n)\big) &= \mathbb{E}\log_2\Big(1+\Gamma_{\text{SCP}}\big(n(1+\beta^2\rho\ln n)\big)\Big)\\ &< \log_2\Big(1+\mathbb{E}\,\Gamma_{\text{SCP}}\big(n(1+\beta^2\rho\ln n)\big)\Big)\\ &< \log_2\Big(1+\mathbb{E}\,\Gamma_{\text{SCN}}(n)\Big)\\ &= \log_2\big(1+\rho\,\mathrm{H}_n\big)\end{aligned}$$

Likewise, from Jensen's inequality and Proposition 3 we have

$$C_{\text{ZF}}\big(n(1+\beta^2)\big) < \log_2\big(1+\rho\,\mathrm{H}_n\big).$$

From (27) it immediately follows that

$$C_{\text{SCP}}\big(n(1+\beta^2\rho\ln n)\big) > \log_2\big(1+\rho\ln n\big).$$

Finally we turn to the claim,

$$C_{\text{ZF}}\big(n(1+\beta^2)\big) > \log_2(1+\rho\ln n)$$

for n sufficiently large. We first introduce the notation

$$R(y) := \log_2\big(1+\Gamma_{\text{ZF}}(y)\big)$$

and $R := R(1)$. The cdf of R is then $F_R(x) = F_{\text{ZF}}\big(2^x-1\big)$. To prove the desired result we postulate the existence of a random variable Z with cdf F_Z such that $u(x) := F_Z^{-1}\circ F_R(x)$ is concave and

$$F_Z\big(\mathbb{E}\,Z(y)\big) > 1-\frac{1}{y} \tag{29}$$

for y sufficiently large. Here $Z(y)$ is defined through its cdf $F_{Z(y)}(x) = \big(F_Z(x)\big)^y$. By the integral transform theorem we then have

$$R(y) \overset{d}{=} F_R^{-1}\circ F_Z\big(Z(y)\big) = u^{-1}\left(Z(y)\right)$$

where $u^{-1}(x)$ is convex since $u(x)$ is concave. The desired result then follows from Jensen's inequality since

$$\begin{aligned}C_{\text{ZF}}\big(n(1+\beta^2)\big) &= \mathbb{E}R\big(n(1+\beta^2)\big)\\ &\geq F_R^{-1}\circ F_Z\Big(\mathbb{E}\,Z\big(n(1+\beta^2)\big)\Big)\\ &> F_R^{-1}\left(1-\frac{1}{n(1+\beta^2)}\right)\\ &= \log_2\Big(1+F_{\text{ZF}}^{-1}\Big(1-\frac{1}{n(1+\beta^2)}\Big)\Big)\\ &= \log_2(1+\rho\ln n).\end{aligned}$$

To prove the existence of Z we introduce the following quantities

$$\begin{aligned}h_1(x) &:= \tfrac{\beta^2}{1+\beta^2}+\tfrac{1}{1+\beta^2}\tfrac{2^x-1}{\rho}\\ x_m &:= h_1^{-1}\Big(1-\tfrac{e^{-1}}{1+\beta^2}\Big)\\ c_2 &:= h_1^{'}(x_m)\\ h_2(x) &:= 1-\tfrac{e^{-1}}{1+\beta^2}+c_2(x-x_m)\\ x_e &:= h_2^{-1}(1).\end{aligned}$$

We now define Z to have support $[0,x_e]$ and cdf

$$F_Z(x) := \begin{cases} h_1(x), & 0\leq x\leq x_m\\ h_2(x), & x_m < x\leq x_e.\end{cases}$$

Note that F_Z has a continuous derivative on its support. To prove the concavity of $u(x)$ we fist show that the second derivative of $u(x)$ is negative on $\Big[0,F_R^{-1}\Big(1-\frac{e^{-1}}{1+\beta^2}\Big)\Big)$ and then on $\Big(F_R^{-1}\Big(1-\frac{e^{-1}}{1+\beta^2}\Big),\infty\big)$. Since $u(x)$ has a continuous derivative it follows that $u(x)$ is concave on the whole of $[0,\infty)$.

For $x\in\Big[0,F_R^{-1}\Big(1-\frac{e^{-1}}{1+\beta^2}\Big)\Big)$ we have

$$\begin{aligned}u(x) &= \log_2\big(1+\rho\big((1+\beta^2)F_R(x)-\beta^2\big)\big)\\ &= \log_2\Big(1+\rho\Big(1-e^{-\frac{2^x-1}{\rho}}\Big)\Big).\end{aligned}$$

Now let $v(x)$ denote the argument of $\log_2(\cdot)$ above. By taking the second derivative of $u(x)$ we obtain

$$\begin{aligned}u''(x) &= \left(\frac{1}{\ln 2}\frac{v'(x)}{v(x)}\right)'\\ &= \frac{1}{\ln 2}\frac{v''(x)}{v(x)}-\frac{1}{\ln 2}\frac{\big(v'(x)\big)^2}{v(x)^2}\\ &= \frac{\ln 2\, 2^x e^{-\frac{2^x-1}{\rho}}}{v(x)}\cdot\left\{1-\frac{2^x e^{-\frac{2^x-1}{\rho}}}{1+\rho\Big(1-e^{-\frac{2^x-1}{\rho}}\Big)}-\frac{2^x}{\rho}\right\}.\end{aligned}$$

By applying the inequality $e^{-x}\leq 1-x$ twice inside the curly brackets we get

$$u''(x) \leq -\frac{\ln 2\, 2^x e^{-\frac{2^x-1}{\rho}}}{v(x)}\frac{1}{\rho} < 0.$$

For $x\in\Big(F_R^{-1}\Big(1-\frac{e^{-1}}{1+\beta^2}\Big),\infty\Big)$ we have

$$u(x) = x_m+\frac{1}{c_2}\Big(F_R(x)+\frac{e^{-1}}{1+\beta^2}-1\Big).$$

By taking the second derivative we obtain

$$\begin{aligned}u''(x) &= \frac{1}{c_2}\left(1-\frac{e^{-\frac{2^x-1}{\rho}}}{1+\beta^2}\right)''\\ &= \frac{1}{c_2}\left(\frac{e^{-\frac{2^x-1}{\rho}}}{1+\beta^2}\frac{2^x}{\rho}\ln 2\right)'\\ &= \frac{1}{c_2}\left(\frac{e^{-\frac{2^x-1}{\rho}}}{1+\beta^2}\frac{2^x}{\rho}(\ln 2)^2\right)\cdot\Big\{1-\frac{2^x}{\rho}\Big\},\end{aligned}$$

which is negative for $x>\log_2(\rho)$. Hence $u''(x)$ is negative for $x>F_R^{-1}\Big(1-\frac{e^{-1}}{1+\beta^2}\Big) = \log_2(1+\rho)$.

To prove (29) we introduce the function

$$h_3(x) := \frac{\beta^2}{1+\beta^2}+c_3 x,$$

with $c_3 := \frac{1-e^{-1}}{(1+\beta^2)x_m}$. Note that $h_3(x)$ satisfies $h_3(x) > h_1(x)$

for $x \in (0, x_m)$. Hence,

$$\begin{aligned}
\mathbb{E}\, Z(y) &= \int_0^{x_e} 1 - \big(F_Z(x)\big)^y \,\mathrm{d}x \\
&= \int_0^{x_m} 1 - \big(h_1(x)\big)^y \,\mathrm{d}x + \int_{x_m}^{x_e} 1 - \big(h_2(x)\big)^y \,\mathrm{d}x \\
&> \int_0^{x_m} 1 - \big(h_3(x)\big)^y \,\mathrm{d}x + \int_{x_m}^{x_e} 1 - \big(h_2(x)\big)^y \,\mathrm{d}x \\
&= x_e - \frac{1/c_3}{y+1}\Big[\big(1 - \tfrac{e^{-1}}{1+\beta^2}\big)^{y+1} - \big(\tfrac{\beta^2}{1+\beta^2}\big)^{y+1}\Big] \\
&\qquad\qquad - \frac{1/c_2}{y+1}\Big[1 - \big(1 - \tfrac{e^{-1}}{1+\beta^2}\big)^{y+1}\Big].
\end{aligned}$$

Since $\mathbb{E}\, Z(y)$ goes to x_e with increasing y we have for y sufficiently large

$$F_Z\big(\mathbb{E}\, Z(y)\big) = 1 - \tfrac{e^{-1}}{1+\beta^2} + c_2\big(\mathbb{E}\, Z(y) - x_m\big).$$

Substituting with the lower bound for $\mathbb{E}\, Z(y)$ we obtain

$$F_Z\big(\mathbb{E}\, Z(y)\big) > 1 - \frac{\big(\frac{c_2}{c_3} - 1\big)\big(1 - e^{-1}\big)^{y+1} + 1}{y+1}.$$

This completes the proof since

$$\frac{\big(\frac{c_2}{c_3} - 1\big)\big(1 - e^{-1}\big)^{y+1} + 1}{y+1} < \frac{1}{y}$$

for y sufficiently large.

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