Bayesian and Deterministic CRBs for Semi-Blind Channel Estimation in SIMO Single Carrier Cyclic Prefix Systems

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***Abstract*—Traditionally, the performance of different semi-blind channel estimation algorithms has been assessed and compared to a certain lower bound. One of these famous lower bounds that has been extensively used in the literature is the Cramer Rao Bound (CRB). Depending on how we treat the symbols and the channel, different versions of CRB have been derived. There are two possible cases on how to treat the symbols and/or the channel namely, deterministic unknowns or random. Moreover, the symbols are either jointly estimated with the channel or eliminated. In other words, we have six different cases to be handled. In this paper we present the CRBs that exist in the literature and fit to some of these cases and derive the others in the context of SIMO single carrier cyclic prefix systems (SC-CP). On the top of that we present a unified framework that permits to derive all versions of CRBs in a concrete manner. All the derived CRBs are validated numerically by conducting limited Monte-Carlo simulations.**

I. Introduction

Traditionally, the transmitter sends some known information to the receiver to aid the latter in estimating the channel. However, in wireless communication the channel varies rapidly with time and as a consequence more training sequence/pilots are required. This process wastes a lot of bandwidth as a result of augmenting the transmission rate to maintain the throughput. In the last two decades a new branch of channel estimation has emerged focusing on accomplishing this task blindly i.e. without the need for a training sequence. Nevertheless, most wireless standards that have evolved during this period are still relying on the training sequence/pilots to estimate the channel. This is due probably to the unsatisfactory results of the blind channel estimation algorithms. On the other hand, some powerful channel estimation algorithms that take advantage of both aforementioned techniques have been also developed during the same era. These are known as semi-blind where a superior performance is achieved although few training sequence/pilots are transmitted. As usual the performance of these algorithms are lower bounded. We will focus in this paper on the most famous lower bound used by the statisticians namely, the CRB. Different versions of semi-blind CRBs are shown and derived in the sequel. Basically, there are two approaches on how to tackle the problem of semi-blind channel estimation depending on how we treat the transmitted symbols. The first approach is based on jointly estimating the symbols with the channel while the second approach is based on estimating the channel and marginalizing the symbols. Moreover, in the first approach we have the choice to consider the channel and/or the symbols as either deterministic unknowns or random with known probability density function (pdf). Hence, there are four methodologies to jointly estimate the channel and the symbols. However, in the second approach we can only marginalize the symbols if we consider them as random with known pdf regardless of how we treat the channel. Therefore, there are only two methodologies to estimate the channel while marginalizing the symbols. Overall we have six cases to be handled. It should be noted that treating the channel as random rather than deterministic in the context of blind and semi-blind channel estimation has been introduced in [1] and developed recently in [2]. Once the channel is treated as random, we are within the framework of Bayesian semi-blind channel estimation. In [2] the Bayesian and the deterministic algorithms are evaluated by running Monte-Carlo simulations. In this paper we will derive the lower bounds that correspond to the different algorithms elaborated in [2]. This paper is organized as follows: In section II we develop the SIMO SC-CP transmission system model, while in section III we show a general framework that permits the derivation of the different CRBs that belong to the two approaches stated above. In section IV we make use of the framework developed in section III to derive the different CRBs. In section V we show a summary of the CRBs and in section VI we conduct some Monte-Carlo simulations to pictorially compare different CRBs with their corresponding algorithms. Finally, in section VII we draw some conclusions and in section VIII we show the acknowledgments.

II. SIMO FIR SC-CP Tx System Model

In (semi-)blind channel identification, a multichannel framework can be obtained from oversampling a received signal and leads to a Single Input Multiple Output (SIMO) vector channel representation. The multiple FIR channels we obtain in this representation can also be obtained from multiple signals received from an array of antennas (in the context of mobile digital communications [3]) or from a combination of both. To further develop the case of oversampling, consider a linear digital modulation over a linear channel with additive noise so that the received signal $y(t)$ has the following form:

$$y(t) \ = \ \sum_k h(t-kT)a(k) \ + \ v(t). \tag{1}$$

In (1) $a(k)$ are the transmitted symbols, T is the symbol period, $h(t)$ is the channel impulse response and v(t) designates noise. The channel is assumed to be FIR with length NT. If the received signal is oversampled at the rate $\frac{m}{T}$ (or if m different samples of the received signal are captured by m sensors every T seconds, or a combination of both), the discrete input-output relationship can be written as:

$$\boldsymbol{y}(k)=\sum_{i=0}^{N-1}\boldsymbol{h}(i)a(k-i)+\boldsymbol{v}(k)=\boldsymbol{H}A_N(k)+\boldsymbol{v}(k) \quad (2)$$

where $\boldsymbol{y}(k)=[y_1^H(k)\cdots y_m^H(k)]^H$, $\boldsymbol{h}(i)=\left[h_1^H(i)\cdots h_m^H(i)\right]^H$, $\boldsymbol{v}(k)=[v_1^H(k)\cdots v_m^H(k)]^H$, $\boldsymbol{H}=[\boldsymbol{h}(N-1)\cdots\boldsymbol{h}(0)]$, $A_N(k)=\left[a(k-N+1)^H\cdots a(k)^H\right]^H$ and superscript ^{H} denotes Hermitian transpose. Let $\mathbf{H}(z)=\sum_{i=0}^{N-1}\boldsymbol{h}(i)z^{-i}=[\mathrm{H}_1^H(z)\cdots\mathrm{H}_m^H(z)]^H$ be the SIMO channel transfer function, and $\boldsymbol{h}=\left[\boldsymbol{h}^H(N-1)\cdots\boldsymbol{h}^H(0)\right]^H$. As for the noise, we consider an additive independent white Gaussian circular noise $\boldsymbol{v}(k)$ with $r_{\boldsymbol{vv}}(k-i)=\mathrm{E}\,\boldsymbol{v}(k)\boldsymbol{v}(i)^H=\sigma_v^2 I_m\,\delta_{ki}$. Assume we receive a (OFDM or single-carrier) CP block transmission system with M samples per block. The introduction of a cyclic prefix of L samples means that the last L samples of the current block (corresponding to M samples) are repeated before the actual block. If we assume w.l.o.g. that the current block starts at time 0, then samples $a[M-L]\cdots a[M-1]$ are repeated at time instants $-L,\ldots,-1$. This means that the output at sample periods $0,\ldots,M-1$ can be written in matrix form as

$$\begin{bmatrix}\boldsymbol{y}[0]\\ \vdots\\ \boldsymbol{y}[M-1]\end{bmatrix}=\boldsymbol{Y}[0]\;=\;\mathcal{T}(\boldsymbol{h})\;A[0]+\boldsymbol{V}[0] \quad (3)$$

where the matrix $\mathcal{T}(\boldsymbol{h})$ is not only (block) Toeplitz but even (block) circulant: each row is obtained by a cyclic shift to the right of the previous row and $[\boldsymbol{h}(0)\;0_{m\times(M-N)}\;\boldsymbol{h}(N-1)\cdots\boldsymbol{h}(1)]$ is the first block row. We shall simplify the notation in (3) to

$$\begin{aligned}\boldsymbol{Y}\;=\;\mathcal{T}(\boldsymbol{h})\,A+\boldsymbol{V}\;&=\;\mathcal{T}_K(\boldsymbol{h})\,A_K+\mathcal{T}_U(\boldsymbol{h})\,A_U+\boldsymbol{V}\\ &=\;\mathcal{A}_K\boldsymbol{h}+\mathcal{A}_U\boldsymbol{h}+\boldsymbol{V}\;.\end{aligned} \quad (4)$$

Where $\mathcal{T}_K(\boldsymbol{h})$ and $\mathcal{T}_U(\boldsymbol{h})$ represent respectively the portions of $\mathcal{T}(\boldsymbol{h})$ that correspond to A_k (M_K known symbols) and A_U (M_U unknown symbols), see (5). Here we assume for simplicity that the known symbols are gathered at the beginning of the block. On the other hand, $\mathcal{A}$ is a block circulant matrix filled with the elements of A whereas $\mathcal{A}_K$ and $\mathcal{A}_U$ are also block circulant matrices filled with the elements of A_K and A_U respectively.

$$\mathcal{T}(\boldsymbol{h})=\left[\begin{array}{c|c}\mathcal{T}_K(\boldsymbol{h}) & \mathcal{T}_U(\boldsymbol{h})\end{array}\right] \quad (5)$$

III. A Unified Framework for different CRBs

As we have stated before, there are six possible cases that can be classified into two categories. In the first category the channel and the unknown symbols are estimated jointly by making some assumptions on the channel and the unknowns symbols. It is worthy to note that in this category the estimation of the channel and symbols from one side and the noise variance estimation from the other side are decoupled. Hence, the estimation of the noise variance is excluded in this category. If we denote by θ the unknown parameters to be estimated then it is given by:

$$\theta=[A_U^H,\boldsymbol{h}^H]^H \quad (6)$$

The joint probability density function is given by:

$$f(Y,\theta)=f(Y/\theta)f(\theta) \quad (7)$$

Where $f(\theta)$ stands for the probability density function (pdf) of θ, $f(Y,\theta)$ stands for the joint probability density function of Y and θ and $f(Y/\theta)$ stands for the pdf of Y conditioned on θ is given or known. Once we substitute (6) in (7) we get:

$$f(Y,A_U,\boldsymbol{h})=f(Y/A_U,\boldsymbol{h})f(A_U)f(\boldsymbol{h}) \quad (8)$$

Since the symbols and the channel are independent of each other we can write $f(\theta)=f(A_U)f(\boldsymbol{h})$. Of course on the basis of how we treat the symbols and the channel both $f(A_U)$ and $f(\boldsymbol{h})$ differ from one estimator to another as we shall see in the sequel. Knowing that the CRB and consequently the Fisher Information Matrix (FIM) requires the application of the log function to the joint pdf in (8), we get:

$$\ln[f(Y,A_U,\boldsymbol{h})]=\ln[f(Y/A_U,\boldsymbol{h})]+\ln[f(A_U)]+\ln[f(\boldsymbol{h})] \quad (9)$$

Now, let J represents the Fisher Information matrix (FIM), it is given by [4]:

$$\begin{aligned}J_{\theta\theta}\;&=\;\mathrm{E}\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta^*}\right)\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta^*}\right)^H\\ &=\;-\,\mathrm{E}\,\frac{\partial}{\partial\theta^*}\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta^*}\right)^H\end{aligned} \quad (10)$$

As we shall observe later, since we are treating complex parameters we also need, besides $J_{\theta\theta}$, $J_{\theta\theta^*}$ which is defined by:

$$\begin{aligned}J_{\theta\theta^*}\;&=\;\mathrm{E}\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta^*}\right)\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta}\right)^H\\ &=\;-\,\mathrm{E}\,\frac{\partial}{\partial\theta}\left(\frac{\partial\ln[f(Y,A_U,\boldsymbol{h})]}{\partial\theta^*}\right)^H\end{aligned} \quad (11)$$

When $J_{\theta\theta^*}\neq 0$ we shall resort to θ_R defined below:

$$\theta_R=\begin{bmatrix}Re(\theta)\\ Im(\theta)\end{bmatrix}=\mathcal{M}\begin{bmatrix}\theta\\ \theta^*\end{bmatrix},\mathcal{M}=\frac{1}{2}\begin{bmatrix}I & I\\ -jI & jI\end{bmatrix} \quad (12)$$

Knowing that $J_{\theta\theta}=J^*_{\theta^*\theta^*}$ and $J_{\theta\theta^*}=J^*_{\theta^*\theta}$ then (12) yields:

$$J_{\theta_R\theta_R}=\mathcal{M}\begin{bmatrix}J_{\theta\theta} & J_{\theta\theta^*}\\ J^*_{\theta\theta^*} & J^*_{\theta\theta}\end{bmatrix}\mathcal{M}^H \quad (13)$$

On the other side, when $J_{\theta\theta^*=0}$ then $J_{\theta_R\theta_R}$ is defined totally by $J_{\theta\theta}$. This holds true for all the cases where we jointly estimate the channel and the symbols as we shall notice later. Under some assumptions and regularity conditions [5], the error covariance matrix of an unbiased channel estimator $\widehat{\boldsymbol{h}}(Y)$, which is defined as:

$$C(\widehat{h})=\mathrm{E}\left\{\left[\widehat{\boldsymbol{h}}(Y)-\boldsymbol{h}\right]\left[\widehat{\boldsymbol{h}}(Y)-\boldsymbol{h}\right]^H\right\} \quad (14)$$

satisfies the following inequality:

$$C(\widehat{\boldsymbol{h}}) \geq \{J_{\theta_R\theta_R}\}^{-1} \triangleq CRB \tag{15}$$

We usually focus on comparing the Mean Square Error, MSE = tr $\left\{C(\widehat{\boldsymbol{h}})\right\}$ to the minimum error variance which is defined by tr $\{CRB\}$ where tr stands for the trace of a matrix.

However, in the second category the channel and the noise variance are the only parameters to be estimated while the symbols are supposed to be marginalized during the estimation process. Here we can't exclude the estimation of the noise variance because it is coupled to the estimation of the channel. Thus,

$$\theta = [\boldsymbol{h}^H, \sigma_v^2]^H \tag{16}$$

Once we substitute θ in (7) we get:

$$f(Y, \boldsymbol{h}, \sigma_v^2) = f(Y/\boldsymbol{h}, \sigma_v^2) f(\boldsymbol{h}) f(\sigma_v^2) \tag{17}$$

Again, we apply the log function on both sides of (17) to get:

$$\ln[f(Y, \boldsymbol{h}, \sigma_v^2)] = \ln[f(Y/\boldsymbol{h}, \sigma_v^2)] + \ln[f(\boldsymbol{h})] + \ln[f(\sigma_v^2)] \tag{18}$$

As for FIM, both (10) and (11) are still applicable where only θ is redefined as in (16).

IV. Derivations of Different CRBs

We shall develop in this section the CRBs of all the cases that belong to both categories and provide a closed-form formula where it is possible. This will be done by exploiting the framework introduced in the previous section. To commence with this task, we shall explain the way by which we call the different CRBs. First of all, to differentiate between the CRBs that correspond to the deterministic and Bayesian channels we call them respectively DCRB and BCRB. However, to differentiate between CRBs where we treat the symbols as deterministic and random we use respectively CRB$_{det}$ and CRB$_{sto}$. On the other hand, to differentiate between joint estimation and marginalization we use respectively CRB$_{joint}$ and CRB$_{marg}$.

A. DCRB$_{det,joint}$

In this lower bound [4] both the unknown symbols and the channel are considered as deterministic unknowns to be estimated. Hence it belongs to the first category and consequently the joint pdf is given by (9). Moreover, since both are deterministic we have $f(\boldsymbol{h}) = \boldsymbol{h}^o \delta(\boldsymbol{h} - \boldsymbol{h}^o)$ and $f(A_U) = A_U^o \delta(A_U - A_U^o)$ where $\boldsymbol{h}^o$ and A^o represent respectively the true values of the channel and the symbols. It can be easily noticed that the pdfs of both the unknown symbols and the channel have no effect on the computation of the FIM. Hence, $\ln[f(Y, A_U, \boldsymbol{h})]$ is replaced by $\ln[f(Y/A_U, \boldsymbol{h})]$ in (10) where $f(Y/A_U, \boldsymbol{h}) = \frac{1}{(\pi\sigma_v^2)^{Mm}} exp[-\frac{1}{\sigma_v^2}(Y - \mathcal{T}(h)A)^H(Y - \mathcal{T}(h)A)]$. After a little treatment (10) yields:

$$J_{\theta\theta} = \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) & \mathcal{T}_U^H(\boldsymbol{h})\mathcal{A} \\ \mathcal{A}^H \mathcal{T}_U(\boldsymbol{h}) & \mathcal{A}^H\mathcal{A} \end{bmatrix} \tag{19}$$

With a little manipulation we can easily show that $J_{\theta\theta^*} = 0$. Hence, by applying the Schur's complement on (19) we get:

$$DCRB_{det,joint} = J_{\boldsymbol{hh}}^{-1} = \sigma_v^2 \left(\mathcal{A}^H P_{\mathcal{T}_U(\boldsymbol{h})}^{\perp} \mathcal{A}\right)^{-1} \tag{20}$$

Where $P_{\mathcal{T}_U(\boldsymbol{h})}^{\perp} = I - P_{\mathcal{T}_U(\boldsymbol{h})}$ and $P_{\mathcal{T}_U(\boldsymbol{h})} = \mathcal{T}_U(\boldsymbol{h})(\mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}))^{-1}\mathcal{T}_U^H(\boldsymbol{h})$ is the projection matrix on $\mathcal{T}_U(\boldsymbol{h})$.

B. DCRB$_{sto,joint}$

The corresponding blind CRB appeared first in [6]. In this novel lower bound (see [7] for a profound analysis of its blind counterpart) we consider the unknown symbols as random with Gaussian distribution while the channel is considered deterministic to be jointly estimated with the unknown symbols. This estimator also belongs to the first category, thus the joint pdf is given by (9). Moreover, $f(A_U) = \frac{1}{(\pi\sigma_a^2)^{M+N-1-M_K}} exp[-\frac{A_U^H A_U}{\sigma_a^2}]$ and $f(\boldsymbol{h}) = \boldsymbol{h}^o \delta(\boldsymbol{h} - \boldsymbol{h}^o)$. It is obvious here that $\ln[f(\boldsymbol{h})]$ can be omitted without affecting the computation of FIM. Hence, (10) yields:

$$J_{\theta\theta} = \mathrm{E}_{Y,A_U/\boldsymbol{h}} \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) + \frac{\sigma_v^2}{\sigma_a^2} I_{M_U} & \mathcal{T}_U^H(\boldsymbol{h})\mathcal{A} \\ \mathcal{A}^H \mathcal{T}_U(\boldsymbol{h}) & \mathcal{A}^H\mathcal{A} \end{bmatrix} \tag{21}$$

Denoting $\mathrm{E}_A\{\mathcal{A}\} = \mathcal{A}_K^{'}$ and $\mathrm{E}_A\left\{\mathcal{A}^H\mathcal{A}\right\} = C_K$ where $C_k = \mathcal{A}_K^{'H}\mathcal{A}_K^{'} + M_U \sigma_a^2 I_{mN}$ and noting that $J_{\theta\theta^*=0}$, then by applying the Schur's complement on (21) we get:

$$DCRB_{sto,joint} = J_{\boldsymbol{hh}}^{-1} = \sigma_v^2 \left(C_K - \mathcal{A}_K^{'H}\mathcal{T}_U(\boldsymbol{h})[\mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) + \frac{\sigma_v^2}{\sigma_a^2} I]^{-1}\mathcal{T}_U^H(\boldsymbol{h})\mathcal{A}_K^{'}\right)^{-1} \tag{22}$$

C. DCRB$_{sto,marg}$

This lower bound [4], [8] belongs to the second category, hence we are interested in estimating the channel and the variance of the noise only while the unknown symbols are supposed to be eliminated during the estimation process. Furthermore, the joint pdf is given by (18) where we consider the channel and the noise variance to be deterministic while the unknown symbols have a Gaussian distribution. Here again, $\ln[f(\boldsymbol{h})]$ and $\ln[f(\sigma_v^2)]$ have no influence on computing the FIM. Substituting $f(Y/\boldsymbol{h}, \sigma_v^2) = \frac{1}{(\pi)^{(Mm}|C_{YY}|} exp[-(Y - m_Y)^H C_{YY}^{-1}(Y - m_Y))]$ Where $m_Y = \mathcal{T}_K(\boldsymbol{h})A_K$ and $C_{YY} = E\ (\boldsymbol{Y} - m_Y)(\boldsymbol{Y} - m_Y)^H = \sigma_a^2 \mathcal{T}_U(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h})^H + \sigma_v^2 I_{M_U m}$ in (18) after omitting $\ln[f(\boldsymbol{h})]$ and $\ln[f(\sigma_v^2)]$ then with a little manipulation (10) and (11) yield:

$$J_{\theta\theta}^{sto}(i,j) = \mathrm{tr}\left\{C_{YY}^{-1}\frac{\partial C_{YY}}{\partial \theta_i^*} C_{YY}^{-1}\left(\frac{\partial C_{YY}}{\partial \theta_j^*}\right)^H\right\} + \left[\mathcal{A}_K^H C_{YY}^{-1}\mathcal{A}_K\right]_{i,j}$$
$$J_{\theta\theta^*}^{sto}(i,j) = \mathrm{tr}\left\{C_{YY}^{-1}\frac{\partial C_{YY}}{\partial \theta_i^*} C_{YY}^{-1}\left(\frac{\partial C_{YY}}{\partial \theta_j^*}\right)\right\} \tag{23}$$

Where $[B]_{i,j}$ denotes the element that lies in the ith row and jth column of matrix B. We have used in the derivation of (23) the following facts: $\frac{\partial C_{YY}}{\partial \boldsymbol{h}_i^*} = \sigma_a^2 \mathcal{T}_U(\boldsymbol{h})\mathcal{T}_U(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{h}_i^*})^H$, $\frac{\partial C_{YY}}{\partial \sigma_v^2} = \frac{1}{2}$, $\frac{\partial ln|C_{YY}|}{\partial \theta_i^*} = \text{tr}\left\{C_{YY}^{-1}\frac{\partial C_{YY}}{\partial \theta_i^*}\right\}$ and $\frac{\partial}{\partial \theta_i^*}\text{tr}\left\{C_{YY}^{-1}\right\} = -\text{tr}\left\{C_{YY}^{-1}\frac{\partial C_{YY}}{\partial \theta_i^*}C_{YY}^{-1}\right\}$. Once we compute both $J_{\theta\theta}$ and $J_{\theta\theta^*}$ from (23), we substitute them in (13) to compute $J_{\theta_R\theta_R}$. Consequently, by using Schur's complement we can extract easily $J_{\boldsymbol{hh}}$ from $J_{\theta_R\theta_R}$ then $DCRB_{sto,marg} = J_{\boldsymbol{hh}}^{-1}$ follows directly.

D. $BCRB_{sto,joint}$

In this lower bound [9] (see also [10] for its application in cooperative-OFDM system) both the channels and the unknown symbols are assumed random with Gaussian distribution and are supposed to be estimated jointly. Hence, this lower bound in its turn belongs to the first category and its joint pdf is given by (9). By substituting the terms in (9) by their corresponding functions we deduce the corresponding FIM as follows:

$$J_{\theta\theta} = \frac{1}{\sigma_v^2}\ \text{E}_{Y,\boldsymbol{h},A} \begin{bmatrix} \mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) + \frac{\sigma_v^2}{\sigma_a^2}I_{M_U} & \mathcal{T}_U^H(\boldsymbol{h})\mathcal{A} \\ \mathcal{A}^H\mathcal{T}_U(\boldsymbol{h}) & \mathcal{A}^H\mathcal{A} + \sigma_v^2 C_h^{0-1} \end{bmatrix} \tag{24}$$

Assuming that both the channel and the symbol distributions have a zero mean as stated above we get:

$$J_{\theta\theta} = \frac{1}{\sigma_v^2} \begin{bmatrix} \text{E}_{Y,\boldsymbol{h},A}\ \left\{\mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) + \frac{\sigma_v^2}{\sigma_a^2}I_{M_U}\right\} & 0 \\ 0 & C_K + \sigma_v^2 C_h^{0-1} \end{bmatrix} \tag{25}$$

The corresponding CRB for the channel can be readily extracted from (25) as follows:

$$BCRB_{sto,joint} = \sigma_v^2\left(C_K + \sigma_v^2 C_h^{0-1}\right)^{-1} \tag{26}$$

It is obvious that this CRB is independent of the number of training symbols used. Moreover, (28) is a block diagonal matrix which means that the estimation of the channel and the symbols are decoupled. Of course, this is not true in case of semi-blind channel estimation except if all the transmitted symbols are known but in that case we are no more estimating the symbols. As a consequence, this CRB is considered to be too optimistic.

E. $BCRB_{det,joint}$

This lower bound called Bayesian CRB for deterministic symbols $\text{BCRB}_{det,joint}$ is novel. However, it is considered as a variation of $\text{BCRB}_{sto,joint}$ that has been derived in the previous section. The main difference with $\text{BCRB}_{sto,joint}$ is that we consider the symbols here to be deterministic unknowns while there we consider them to be random with Gaussian distribution. Hence, with this lower bound we introduce the concept of semi-blind Bayesian CRB for channel estimation by treating the channel as random with Gaussian distribution $f(\boldsymbol{h}) = \frac{1}{(\pi)^{mN}|C_h^o|}exp[-\boldsymbol{h}^H C_h^{o-1}\boldsymbol{h}]$. However, the unknown symbols are considered as deterministic to be jointly estimated with the channel hence, this estimator belongs to the first category where the joint pdf is given by (9). Moreover, here again $\ln[f(A_U)]$ has no effect on computing FIM so it can be omitted. Therefore, (10) yields:

$$J_{\theta\theta} = \text{E}_{Y,\boldsymbol{h}/A}\ \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h}) & \mathcal{T}_U^H(\boldsymbol{h})\mathcal{A} \\ \mathcal{A}^H\mathcal{T}_U(\boldsymbol{h}) & \mathcal{A}^H\mathcal{A} + \sigma_v^2 C_h^{0-1} \end{bmatrix} \tag{27}$$

Assuming that the channel distribution has a zero mean as stated above we get:

$$J_{\theta\theta} = \frac{1}{\sigma_v^2} \begin{bmatrix} \text{E}_{Y,\boldsymbol{h}/A}\ \left\{\mathcal{T}_U^H(\boldsymbol{h})\mathcal{T}_U(\boldsymbol{h})\right\} & 0 \\ 0 & \mathcal{A}^H\mathcal{A} + \sigma_v^2 C_h^{0-1} \end{bmatrix} \tag{28}$$

the corresponding CRB for the channel can be readily extracted from (28) as follows:

$$BCRB_{det,joint} = \sigma_v^2\left(\mathcal{A}^H\mathcal{A} + \sigma_v^2 C_h^{0-1}\right)^{-1} \tag{29}$$

This CRB is also too optimistic for the same reasons discussed in the case of $\text{BCRB}_{sto,joint}$.

F. $BCRB_{sto,marg}$

This lower bound called Bayesian CRB for stochastic symbols ($\text{BCRB}_{sto,marg}$) is by its turn novel. It belongs to the second category since the symbols are supposed to be eliminated. It can be considered as an extension to $DCRB_{sto,marg}$ by exploiting the prior information that exists about the channel. The joint pdf is given by (18) but this time $\ln[f(\boldsymbol{h})]$ can't be omitted. Substituting the terms in (18) by their corresponding functions and following the same steps mentioned in $\text{DCRB}_{sto,marg}$ section we get:

$$\begin{cases} J_{\theta\theta} &= \text{E}_h\ \{J_{\theta\theta}^{sto}\} + \begin{bmatrix} C_h^{0-1} & 0 \\ 0 & 0 \end{bmatrix} \\ J_{\theta\theta^*} &= \text{E}_h\ \{J_{\theta\theta^*}^{sto}\} \end{cases} \tag{30}$$

Now we can resort to (13) to compute $J_{\theta_R\theta_R}$. Consequently, by using Schur's complement we can extract easily $J_{\boldsymbol{hh}}$ from $J_{\theta_R\theta_R}$ then $BCRB_{sto,marg} = J_{\boldsymbol{hh}}^{-1}$ follows directly.

V. SUMMARY

Therefore, with the extension of some existing blind CRBs and deriving novel ones, the picture is broadened considerably and to sum up we depict the current picture in Table 1. On the other hand, since some CRBs correspond to one realization of the channel and/or the symbols while others correspond to random channel and/or symbols then these CRBs, in their current form, are not suitable to be compared together. This problem can be overcome readily by computing the expectation of the CRBs that correspond to one realization of the channel and/or the symbols. Hence, in the simulation section we are going to compare the following: $\text{E}_h\,\text{E}_{A_U}\left\{DCRB_{det,joint}\right\}$, $\text{E}_h\left\{DCRB_{sto,joint}\right\}$, $\text{E}_h\left\{DCRB_{sto,marg}\right\}$, $\text{E}_{A_U}\left\{BCRB_{det,joint}\right\}$, $BCRB_{sto,joint}$ $BCRB_{sto,marg}$. However, due to the difficulties that we face when we try to carry on the expectation operator in some situations, we are going to run Monte-Carlo simulations to perform the averaging over the ensemble of the symbols and/or the channel realizations.

CRB Type	Unknown Sym	Channel	Elm of Sym	Novel
$DCRB_{det,joint}$	Det	Det	No	No
$DCRB_{sto,joint}$	Gauss	Det	No	Yes
$DCRB_{sto,marg}$	Gauss	Det	Yes	No
$BCRB_{sto,joint}$	Gauss	Gauss	No	No
$BCRB_{det,joint}$	Det	Gauss	No	Yes
$BCRB_{sto,marg}$	Gauss	Gauss	Yes	Yes

TABLE I
SUMMARY OF CRBS

VI. SIMULATIONS

In this section we plot the different CRBs to verify some of their aspects that we mentioned in the paper. In each Monte-Carlo simulation we generate different realizations of the channel, the symbols and the noise. As for the channel, we generate a Rayleigh fading channel with exponentially decaying power delay profile (PDP) as follows: e^{-wn} where $n = 0 : N-1$ and $w = 2$. Hence, $C^o_{\mathbf{h}}$ is the diagonal matrix $C^o_{\mathbf{h}} = I_m \otimes C$ where $C = \text{diag}\{e^{-wn}, n = 0 : N-1\}$. As for the symbols, we generate random QPSK symbols to reflect the real world case. The performance of the different CRBs is evaluated by means of the Normalized MSE (NMSE) vs. SNR. The SNR is defined as: SNR $= \frac{||\mathcal{T}(h)A||^2}{mM\,\sigma_v^2}$. The NMSE is defined as $\frac{avg\ \text{tr}\,(CRB)}{avg\,||\boldsymbol{h}||^2}$ where avg stands for average. In Figure 1 we plot the NMSE of all the CRBs that have been derived in this paper versus the number of iterations at SNR = 10 dB. To validate our comments about the looseness of the Bayesian CRBs elaborated in this paper, we plot in the same figure the results of the algorithms derived in [2] which correspond to these CRBs. To be more specific the algorithms SB-ML-ML, SB-ML-GMAP, SB-GMAP-ML, SB-GMAP-GMAP, SB-GMAP-Elm-ML and SB-GMAP-Elm-GMAP are lower bounded respectively by DCRB$_{det,joint}$, BCRB$_{det,joint}$, DCRB$_{sto,joint}$, BCRB$_{sto,joint}$, DCRB$_{sto,marg}$ and BCRB$_{sto,marg}$. Well, at this moderate SNR it is clear that none of the algorithms attain its corresponding CRB. This holds true also at high SNR except for DCRB$_{det,joint}$ and DCRB$_{sto,marg}$ which can be attained asymptotically in SNR by SB-ML-ML and SB-GMAP-Elm-ML respectively. Apart from the fact that they are loose, we can observe that all the Bayesian CRBs and DCRB$_{sto,joint}$ are so close to each other.

VII. CONCLUSION

We introduced previously the concept of blind Bayesian channel estimation and extended it to the semi-blind case by proposing a bunch of useful algorithms. In this paper, we have presented a framework that permits to derive a complete set of CRBs that correspond to the various Deterministic and Bayesian cases. Some of these algorithms already exist in the literature and the others are novel. The main conclusion that can be drawn is that the Bayesian Cramer Rao Bound is loose and there is a need for another lower bound which is tighter. This result is valid regardless of how we treat the symbols namely, deterministic or random and it is even valid when we marginalize the symbols. Furthermore, this result extends

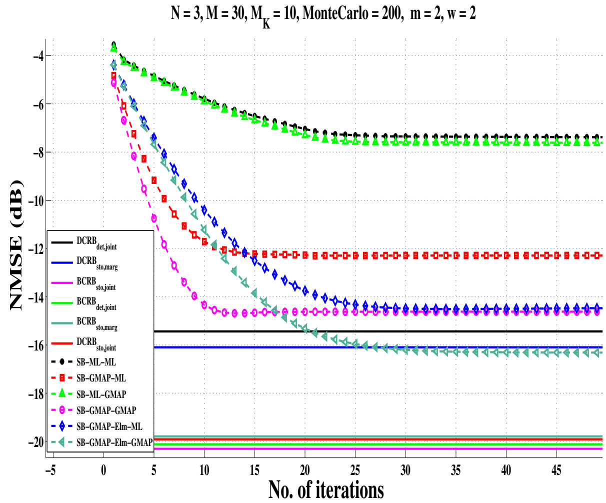

Fig. 1. NMSE vs. No.of iterations of all CRBs at SNR = 10 dB

also to DCRB$_{sto,joint}$ which corresponds to joint estimation of deterministic channel and random symbols. Hence, not only Bayesian CRBs but also some deterministic CRBs requires tighter alternatives. This point is under investigation and is subject for further research.

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