The Impact of Routing on Multicast Error Recovery

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Abstract

We investigate the relationship between multicast routing algorithms and reliable multicast communication. To capture the impact of the multicast tree topology onto reliable multicast, we consider two performance measures, namely the probability mass function of successful receptions and the expected number of retransmissions needed to successfully deliver a packet from the source to all receivers. Since the expected number of retransmissions is computationally expensive we also give a tight approximation. We finally evaluate the impact of routing algorithms on the performance of reliable multicast transmission and propose a realistic generic model for a multicast tree.

Key words: Reliable Multicast, MBONE, Multicast Routing, Performance Evaluation, ARQ

1 Introduction

The MBONE [1] has given raise to a number of conferencing applications such as vat, ivs, or vic where *timely* delivery is most important and packet loss can be tolerated. However, there is another class of dissemination-oriented applications where *reliable* multicast delivery from one source to many receivers is required such as

- Information delivery e. g. software distribution, newspaper excerpts, and software updates.
- Distributed Simulation where state information must be exchanged.
- Web caching and replication.

When designing or evaluating reliable multicast transport protocols one needs to be able to compute performance measures such as delay or the number of retransmissions. We will derive the formulas for computing

- the probability mass function (*pmf*) for the number of receivers that successfully receive a packet that is emitted once.
- the mean number of retransmissions until all receivers have successfully received a packet.

Since the exact expression for the mean number of retransmissions is difficult to compute we also give a simple approximation.

Our aim is to investigate reliable transmission for multicast communication and explore its re-

lationship to multicast routing. Very little work [2–4] was done in this area and the effect of the topology on reliable multicast is not well understood.

Recent multicast routing algorithms have been evaluated in terms of cost and delay [5–7], blocking probability [8,9] and overhead [10]. The impact of the routing algorithm on reliable multicast transmission has not yet been studied. We will demonstrate the impact of multicast routing algorithms on reliable transmission for two multicast routing algorithms that are known to perform best in terms of cost and delay.

Nearly all performance studies [11–15] of reliable multicast communication assume multicast trees where the loss on any link affects only a single receiver.

We will consider this special case of a multicast tree, referred to as $MFAN$ (see figure 1), consider and compare it both, with trees that are the outcome of multicast routing algorithms and with two other generic multicast trees. We will show that the full binary tree (see figure 3) is a more realistic model for a multicast tree than $MFAN$.

2 Multicast Trees

The formulas we derive are valid for all types of multicast trees, i.e. they are independent of the topology of the multicast trees. In order to evaluate the formulas we define three generic multicast trees and additionally use two of the most popular multicast routing algorithms to compute multicast trees for artificially generated networks. A 1:R – multicast connection forms a tree rooted at the source. The loss in a multicast tree is dependent on the topology. A tree topology has several parameters, each of them having a different influence on loss: (i) tree height, (ii) number R of receivers (members in the multicast group), (iii) number of nodes in
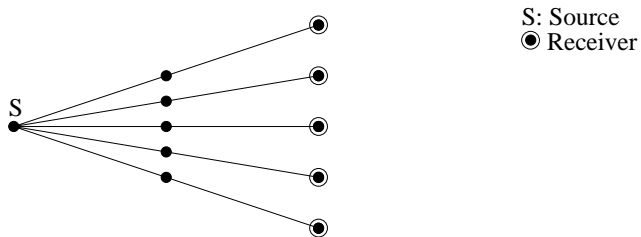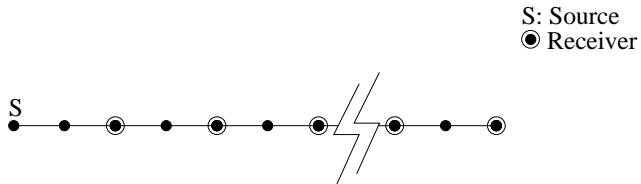

Fig. 1. Multi-hop-Fanout ($MFAN$).
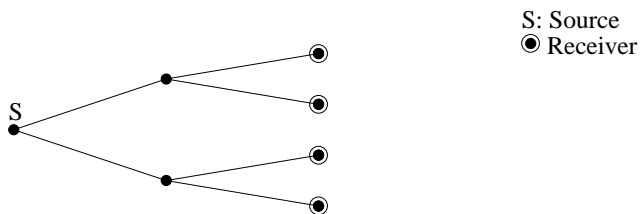

Fig. 2. Linear Chain (LC).

Fig. 3. Full Binary Tree (FBT).

the tree [2], and (iv) the number of receivers affected by a loss over a single link.

We have chosen the following three generic multicast trees because they behave very differently with respect to the impact of packet loss on a single link:

- For $MFAN$ (figure 1), always only a single receiver is affected.
- For the linear chain LC (figure 2), depending on what link the loss occurs, the number of affected receivers can range from one to all receivers.
- For the full binary tree FBT (figure 3), the impact of loss lies between the one for $MFAN$ and LC, affecting either a single receiver or a subgroup of all receivers.

By keeping the ratio of the number of receivers and the number of tree nodes approximately at 0.5 for all three trees (see Table 1) we collapse the two parameters (ii) and (iii) that influence

	MFAN	FBT	LC
$\dfrac{receivers}{nodes}$	$\dfrac{1}{2+\frac{1}{R}}$	$\dfrac{1}{2-\frac{1}{R}}$	$\dfrac{1}{2+\frac{1}{R}}$
tree height	2	$\log_2(R)$	$2R$

Table 1
The characteristic of the three generic multicast trees with respect to the number R of receivers.

loss into a single one. However, as the tree grows, the tree height will vary if we keep the ratio of receivers and nodes in the tree fixed (see Table 1). To generate "real" multicast trees we use two different multicast routing algorithms that optimize either cost or delay:

Cost optimization tries to minimize the sum of the edge costs in the multicast tree. The Kou Markovsky Berman algorithm [16], referred to as *KMB*, is a well known heuristic to approach the optimal cost solution for a multicast tree. It constructs a **Heuristic Steiner Tree** (*HST*) [17] based on the minimum spanning tree algorithm.

Delay optimization minimizes the delay from the source to every receiver. The Shortest Path Algorithm analyzed by Doar [6] optimizes delay and constructs a **shortest path tree** (*SPT*) that connects every receiver to the source via the shortest path.

10 random networks with 200 nodes and an average outdegree of 3.0 were constructed following a method proposed by Waxman [18] with the modification of Doar [6] that avoids the influence of the number of nodes on the average outdegree. The method of Waxman is commonly used by the Multicast Routing community [5,6,18,19] to compare the performance of

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² The number of edges in a tree is not stated, since for a tree: $edges = nodes - 1$.

different Multicast Routing Algorithms on random networks.

On each of the 10 random nets, 100 multicast groups with varying group sizes (5...140) and receivers at random locations had been routed by the two algorithms for Cost (*HST*) and Delay (*SPT*) optimization. Two sample multicast trees generated by the *SPT* algorithm and the *HST* algorithm for the same network and the same group of 5 receivers are shown in figure 4.
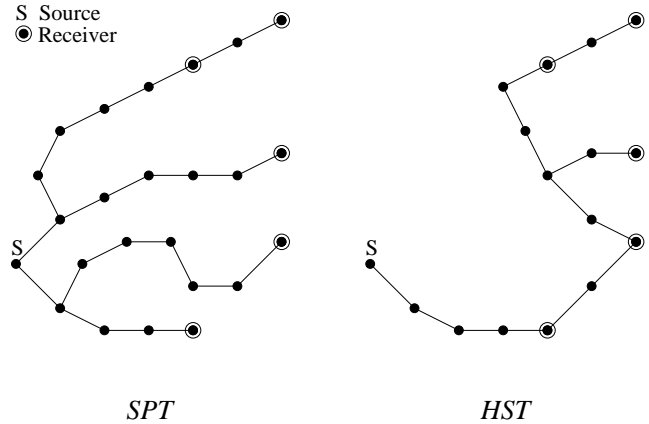

Fig. 4. Multicast trees *SPT* and *HST* for the same group of $R = 5$ receivers in the same random network.

Figure 5 and figure 6 show the characteristics of an average *SPT* and *HST* dependent on the number R of receivers. Comparing the characteristics of *SPT*s and *HST*s to the characteristics of the generic trees given in table 1, it can be stated that the ratio $receivers/nodes$ of the generic trees is about 0.5, comparable to *HST*s and *SPT*s for the case of $R = 40$ receivers (see figure 5). The tree height of *SPT*s and *HST*s is the maximum number of links between the source and any receiver. Figure 6 shows the tree height of *SPT*s and *HST*s as a function of the number R of receivers. The tree height is modeled best by the *FBT* with its logarithmically increasing height (table 1) for a small number of receivers $R < 50$. For a larger number of receivers $R \geq 50$, the tree height of *SPT* and *HST* is constant, since the growth of the tree is limited by the network diameter.
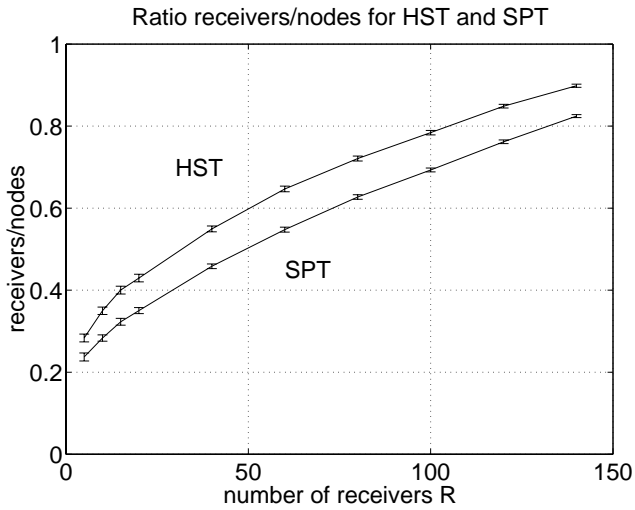
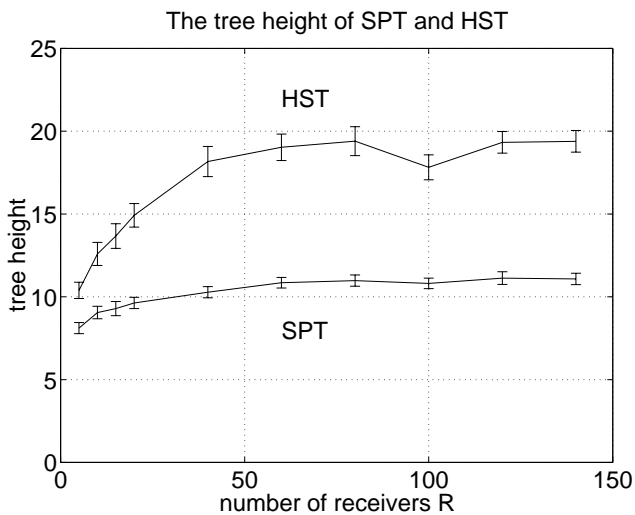Fig. 5. The ratio $\frac{receivers}{nodes}$ for HST and SPT.

Fig. 6. Characteristic of multicast trees SPT and HST.

3 Loss characteristics of a multicast tree

Loss in a multicast tree affects several receivers if it happens on a link that leads to several receivers. We will call such a link a **shared link**.

Reliable multicast transmission must deal with two major problems:

- **Feedback implosion**: Receivers in a reliable multicast communication must provide the source with the status of the reception. Loss on shared links causes loss at several receivers and increases the amount of feedback.

- **High number of retransmissions**: The higher the number of receivers, the higher the number of links in the multicast tree and the average number of retransmissions.

We derive a formula to analytically evaluate the **feedback implosion** at the source, by calculating the probability mass function (*pmf*) of successful and unsuccessful receptions at R receivers for *a single packet emission*. We also show that shared links have no influence on the expected number of successful receptions.

We give the expected number of **retransmissions** needed to deliver one packet to all receivers and propose a tight approximation that enables loss prediction for adaptive error control mechanisms.

3.1 The number of successful receptions in a multicast tree

Supposed that a packet is sent **once**, we are interested in the *pmf* of the number of receivers that successfully receive this packet.

Given is a multicast tree *mct* with:

- source S as the root
- R receivers placed at arbitrary nodes and at all leaves. We allow at most one receiver at any node in the tree and we assume not to have a receiver at the source.
- homogeneous link loss probability q of a packet.

Let X_S be the number of receivers out of the R receivers in the multicast tree rooted at S that receive the packet successfully when transmitted once from S. We will give a method to calculate the corresponding probability mass function for $P(X_S = k)$ that enables us to capture the loss characteristic of different multicast trees. For the definition of the variables used in

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the following see table 2.

The *pmf* can now be calculated in a recursive way, starting at the leaves of the multicast tree. We need to distinguish two cases:

Node n is a leaf. Then there are no receivers located behind node n and the probability that no receiver is receiving a packet is 1 and the *pmf*

evaluates trivially to:

$$P(X_n = 0) = 1 \qquad (1)$$

Node n is not a leaf. Then $P(X_n = k)$ is given by the sum of the probabilities of all different combinations of k successful receptions in the tree rooted at n. The recursive way of calculating the *pmf* allows the use of already known probabilities $P(X_i = a_n(i))$ at the children $i \in child(n)$ of n. For every node n we have therefore just to look at the adjacent links leading to the children.

We must sum over all the combinations of link success that allow in total k successful receiving receivers located at the children i of n and in the subtrees rooted at each of the children.

For one combination s_n of link success the number of successful receptions at the direct children, being also receivers, is given by the inner product $s_n^T x_n$. The number of receptions in the subtrees rooted at the children is given by $s_n^T a_n$.

To obtain the number k of successful receptions for a given s_n the following condition must hold:

$$k = s_n^T(a_n + x_n) \qquad (2)$$

Since x_n is constant and s_n is given, equation (2) selects a subset A_n of combinations of receptions in the subtrees rooted at the children of n:

$$A_n(s_n) = \{a_n \mid k = s_n^T(a_n + x_n)\}$$

Different number of receptions in subtrees behind a failing link does not change the probability $P(X_n = k)$. $A_n(s_n)$ can therefore be reduced by masking the number of receptions in subtrees behind failing links.

$$A_n(s_n) = \{ a_n \mid k = s_n^T(a_n + x_n) \qquad (3)$$
$$\wedge \ \forall i : \ s_n(i)a_n(i) = a_n(i)\}$$

The probability for one combination s_n of link success and one $a_n \in A_n(s_n)$ is then given by the product over the children:

$$P(a_n, s_n) = \prod_{i \in child(n)} (1-q)s_n(i)P(X_i = a_n(i)) \\ + \quad q(1 - s_n(i)) \qquad (4)$$

Since the link to child i is successful $(s_n(i) = 1)$ with probability $(1-q)$ and the probability of $a_n(i)$ successful receptions in the subtree rooted at child i is $P(X_i = a_n(i))$. The packet gets lost $((1 - s_n(i)) = 1)$ on the link to child i with probability q and $a_n(i)$ has no contribution.

The probability $P(X_n = k)$ is then given by summing over all link success combinations s_n and all $a_n \in A_n(s_n)$:

$$P(X_n = k) = \sum_{s_n} \sum_{a_n \in A_n(s_n)} P(a_n, s_n) \qquad (5)$$

We depict $P(X_S = k)$ for the generic multicast trees with a link loss probability of $q = 0.03$ in figure 7 for $R = 64$ receivers and for $R = 128$ receivers in figure 8.

We can see that the *pmf*s vary significantly for the three generic multicast trees. This is due to the fact that the number of receivers affected by a loss on a single link also differs widely for the three generic multicast trees.

The *pmf* of the *MFAN* is the binomial *pmf*, the *pmf* of the *LC* approximates the geometric *pmf* for a large number of receivers. The curve of the *FBT* is multi-modal with peaks at $k = 2^{h-1}, 2^{h-1} + 2^{h-2}, \ldots$. These peaks are due to a high number of full binary subtrees with $2^{h-2}, 2^{h-3}, \ldots$ receivers and therefore a high number of possible combinations that amount to a sum of $k = 2^{h-1}, 2^{h-1} + 2^{h-2}, \ldots$ successful receptions, whereas for $k + 1$ successful receptions the number of possible combinations of full binary subtrees is much lower.

Fig. 7. The probability mass function $P(X_S = k)$ for *FBT*, *MFAN* and *LC* for 64 receivers and $q = 0.03$.
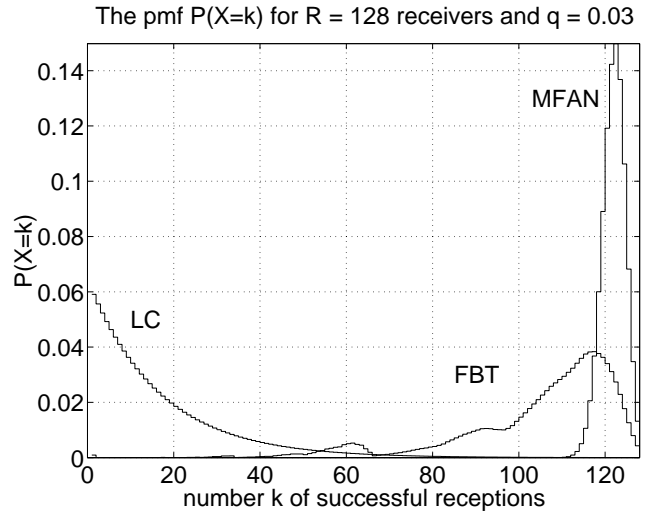
Fig. 8. The probability mass function $P(X_S = k)$ for *FBT*, *MFAN* and *LC* for 128 receivers and $q = 0.03$.

The *pmfs* for the *SPT* and the *HST* for the same multicast group on the same network (figures 9 and 10) indicate that the variance of the number of successful receptions for the *HST* is higher than for the *SPT*. The high probabilities for low numbers of successful receivers are due to loss on shared links near the source. We observe that the *pmfs* for the *SPT* and the *HST* resemble most closely the *pmf* for the *FBT*.
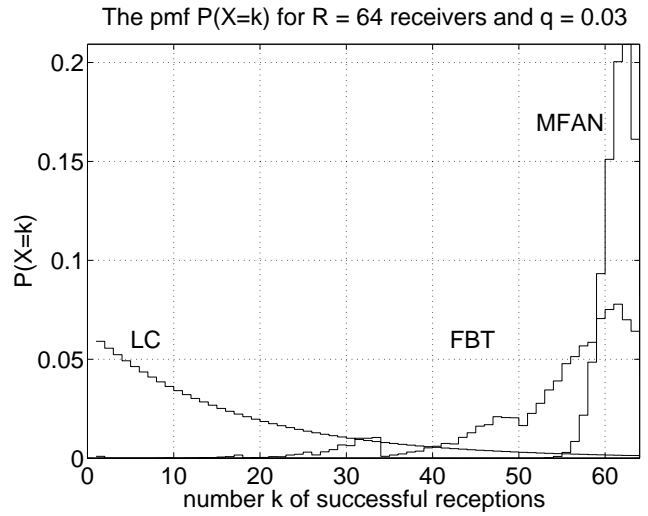
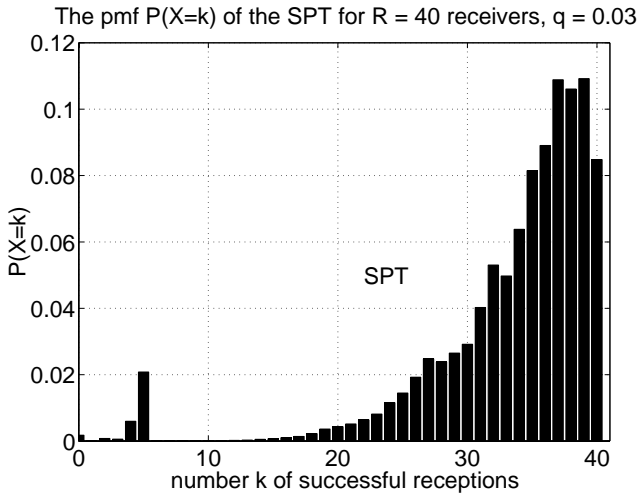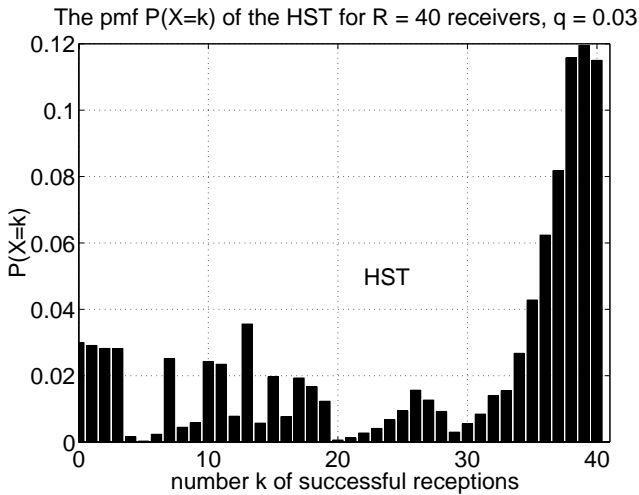Fig. 9. The probability mass function $P(X_S = k)$ for SPT with $R = 40$ receivers and $q = 0.03$.

Fig. 10. The probability mass function $P(X_S = k)$ for HST with $R = 40$ receivers and $q = 0.03$.

3.2 The number of responses

Given a packet is emitted once by the source, we are interested in the number of responses, which can be either positive or negative ACKs, that can be expected from the R receivers in the multicast tree. We make the assumption that the feedback channel from the receivers to the source is loss–free, in which case the number of ACKs/NAKs is identical to the number of receivers that have received or have not received a packet.

The number X_S of successful receptions in the whole multicast tree is the sum of receptions $X_{S,r} \in \{0, 1\}$ of all single receivers r: $X_S = \sum_{r=1}^{R} X_{S,r}$. Since we assume uniform link loss q on all links, the probability of a successful reception for receiver r, which lies h_r hops away from the source, is $(1 - q)^{h_r}$. The expected number of ACKs for every single receiver is therefore $E[X_{S,r}] = (1 - q)^{h_r}$. The expected number of successful receptions $E[X_S]$ in a tree with R receivers is then:

$$E[X_S] = E[\sum_{r=1}^{R} X_{S,r}] = \sum_{r=1}^{R} (1 - q)^{h_r} \qquad (6)$$

Please note that $E[X_S]$ is *not* dependent on the number of shared links, since in (6) the path from the source to every receiver accounts by its full length. We can also express $E[X_S]$ dependent on the receiver distribution over the tree levels h, by accumulating receivers that have the same distance from the source. Let n_h be the number of receivers that lie in tree level h, e.g. h hops from the source, then the expected number of ACKs is given as:

$$E[X_S] = \sum_{h=1}^{h_{max}} n_h (1 - q)^h \qquad (7)$$

The expected number $E[X_S]$ of ACK-packets at the source is shown in figure 11 as a function of the number of receivers in the multicast group for a link loss probability $q = 0.03$. For HST, the number of ACKs is slightly lower than for SPT, accounting for the fact that the number of links traversed between the source and a receiver is higher for HST than for SPT.

The error control feedback scheme may use positive ACKs or negative ACKs (NAKs). Let $Y_S = R - X_S$ be the random variable that describes the number of unsuccessful receptions, then the *pmf* of Y_S is:
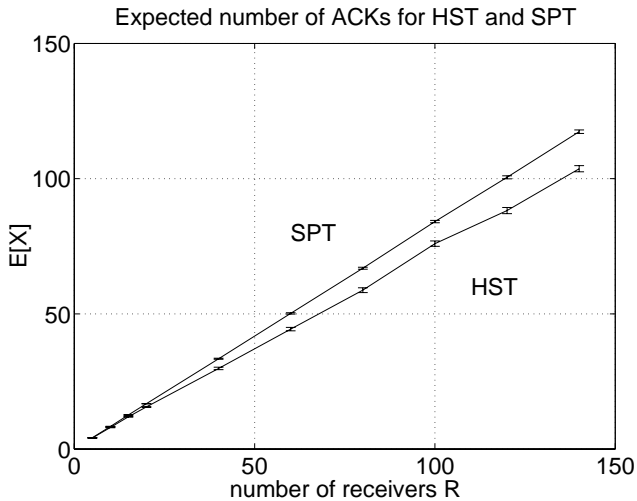
$$P(Y_S = k) = P(X_S = R - k)$$

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Fig. 11. Expected number $E[X_S]$ of ACK-packets at the source for a link loss probability of $q = 0.03$.

and the expected number of NAKs for R receivers is given as:

$$E[Y_S] = R - E[X_S].$$

3.3 The expected number of transmissions for reliable delivery

The expected number of multicast transmissions to deliver a packet to **all** receivers is an important measure in reliable multicast communication. The expected number of transmissions captures the global packet loss behavior in the tree and the cost and the time of a reliable multicast delivery. The expected number of multicast transmissions depends on the link loss probability q and the topology of the multicast tree.

In [3], the expected number of multicast retransmissions is given for the case of loss at **nodes** in the multicast tree. It is more appropriate to consider loss on a link due to two reasons: loss at the source node is unlikely and link loss can be associated with loss in output buffers in routers. In [20], the expected number of multicast transmissions for homogeneous link loss is given by a slight modification of the formula given in [3].

Let M_n be the random variable describing the number of transmissions of a packet until it is received by node n and all receivers in the subtree rooted at n, given that the packet is always successfully received by the predecessor (parent) of node n. The Cumulative Distribution Function (CDF) of M_n, $F_n(i) = P(M_n \leq i)$, can be calculated in a recursive fashion, starting at the leaves of the multicast tree. It must be distinguished if node n is a leaf, an internal node, or the source S:

Node n is a leaf. Then, the probability that fewer than $i + 1$ transmissions are needed to deliver the packet over one link from the parent to the leaf is:

$$F_n(i) = P(M_n \leq i) = 1 - q^i \qquad (8)$$

Node n is an internal node. Then there exists one link leading to n and at least one child c. If there are i attempts to deliver the packet over the link leading to node n and it is lost exactly u times with the probability $q^u(1-q)^{(i-u)}$, then a copy of the packet is forwarded $i - u$ times on every outgoing link to every child. The conditional probability that all children of n and the nodes in the subtrees rooted at the children are receiving the packet during these $i - u$ times is $\prod_{c \in child(n)} F_c(i - u)$. So we obtain $F_n(i)$ by summing over all possible u:

$$F_n(i) = \sum_{u=0}^{i} \binom{i}{u} q^u (1-q)^{(i-u)} \prod_{c \in child(n)} F_c(i - u)$$
$$(9)$$

Node n is the source S. Then there is no link leading to S and consequently only the loss experienced by its children c has to be considered:

$$F_S(i) = \prod_{c \in child(S)} F_c(i) \qquad (10)$$

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Using $F_S(i)$, the expected number $E[M_S]$ of multicast transmissions from the source S is:

$$E[M_S] = \sum_{i=0}^{\infty} (1 - F_S(i)) \qquad (11)$$

The expected number of retransmissions is:

$$E[M_S - 1] \stackrel{(11)}{=} \sum_{i=1}^{\infty} (1 - F_S(i)) \qquad (12)$$

3.4 An Approximation for the Number of Retransmissions

Reliable multicast protocols need to know the expected number of retransmissions. However, the exact calculation of $E[M_S]$ as derived above is not practical:

- The expected number of retransmissions is hard to calculate, since the calculation of the recursive *CDF* in Eq. (9) is computationally intensive for arbitrary topologies.
- Adaptive transport protocols need simple but effective mechanisms to decide.

We give a tight and very simple approximation. The expected number of retransmissions is approximately the product of the link loss probability q and the number of links L in the multicast tree:

$$E[M_S - 1] \approx qL \qquad (13)$$

Consequently, the number $E[M_S]$ of transmissions can be approximated by $1 + qL$.

Proof For the sake of clarity and to shorten the proof of (13) Lemma 1 is used. The exact Lemma 1 and its proof is given in appendix A. Lemma 1 states that $F_S(i)$ can be expressed in the form

$$F_S(i) = 1 - \sum_{j_S^-} (Q_{j_S^-})^i + \sum_{j_S^+} (Q_{j_S^+})^i,$$

where the $Q_{j_S^-}$ and $Q_{j_S^+}$ are polynoms in q: $Q = \sum_k \zeta_k q^k$, with a minimal exponent $k_{min} \geq 1$. The difference between the sum $\Sigma_S^- = \sum_{j_S^-} \zeta_{1_{j_S^-}}$ of the ζ_1 of all the polynoms $Q_{j_S^-}$ with $k_{min} = 1$ and the sum $\Sigma_S^+ = \sum_{j_S^+} \zeta_{1_{j_S^+}}$ of the ζ_1 of all the polynoms $Q_{j_S^+}$ with $k_{min} = 1$ equals the number L_S of links in the tree rooted at the source S:

$$L_S = \Sigma_S^- - \Sigma_S^+$$

Using Lemma 1 the proof of Eq. (13) proceeds as follows. The expected number of retransmissions is:

$$E[M_S - 1] = \sum_{i=1}^{\infty} (1 - F_S(i))$$

$$= \sum_{j_S^-} \frac{Q_{j_S^-}}{1 - Q_{j_S^-}} - \sum_{j_S^+} \frac{Q_{j_S^+}}{1 - Q_{j_S^+}}$$

Then, the ratios $\dfrac{Q}{1 - Q}$ are approximated by Q, yielding

$$E[M_S - 1] \approx \sum_{j_S^-} Q_{j_S^-} - \sum_{j_n^+} Q_{j_n^+}$$

Finally are we interested in the term q of the polynom Q, due to its relevance compared with the terms $q^2, q^3, \ldots$. Every polynom $Q = \sum_k \zeta_k q^k$ is approximated by $\zeta_1 q$, resulting in an approximation for the expectation of retransmissions as:

$$E[M_S - 1] \approx q(\sum_{j_S^-} \zeta_{1_{j_S^-}} - \sum_{j_n^+} \zeta_{1_{j_S^+}})$$

$$= q(\Sigma_S^- - \Sigma_S^+) = qL \qquad \square$$

The last approximation, where higher order terms are suppressed, also gives the condition for which the whole approximation of the expected number of retransmissions (Eq. 13) is valid:

$$qL < 1$$

For $qL \geq 1$, the polynoms $Q = \sum_k \zeta_k q^k$ can not be approximated by $\zeta_1 q$, since higher order terms become more important. For example, a second order term q^2 accounts at least as one additional link in the approximation of the expectation: $q^2 \cdot L = q(qL) \geq q \cdot 1$.

We compare the quality of the approximation the two most extreme cases of multicast topologies. The first one is called linear chain (LC) and is just a chain of L links, the other one is the $MFAN$. The $MFAN$[3] has one separate link from the source to each of the L receivers. In both cases, we have L links. LC is the deepest $MFAN$ the broadest multicast tree that can be built with L links. Figure 12 shows that the approximation lies between the number of retransmissions of LC and $MFAN$ for a wide range of link loss probabilities q and number L of links. The high number of retransmissions of LC compared to $MFAN$ further shows that the tree height has a major impact on the number of retransmissions.

The approximation qL is also compared to the number of retransmissions of various HSTs and SPTs via simulation of reliable multicast transmission. The link loss probability is $q = 0.01$. Only SPTs and HSTs with up to 100 links were considered in order to meet the condition $qL < 1$ for the approximation. Figure 13 shows that qL approximates very well the number of retransmissions for SPTs and HSTs with $L < 40$

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[3] To get the most extreme multicast tree, we reduce the tree height of the $MFAN$ to 1, compared to the previous definition of $MFAN$ with a tree height of 2 (figure 1).
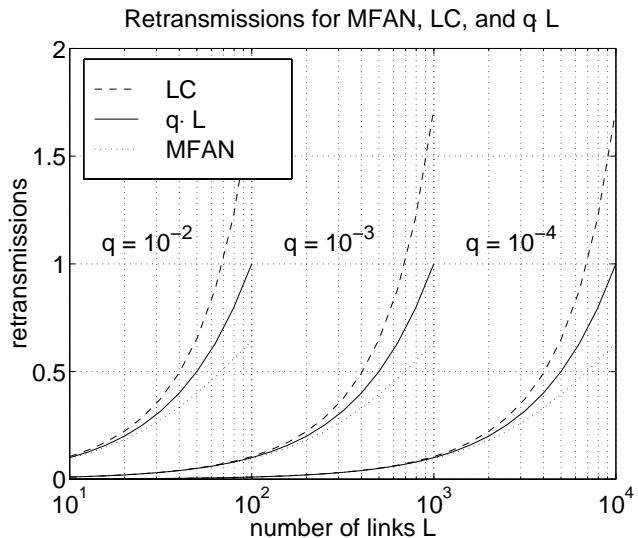
Fig. 12. Expected number of retransmissions for LC and $MFAN$ and the approximation qL.
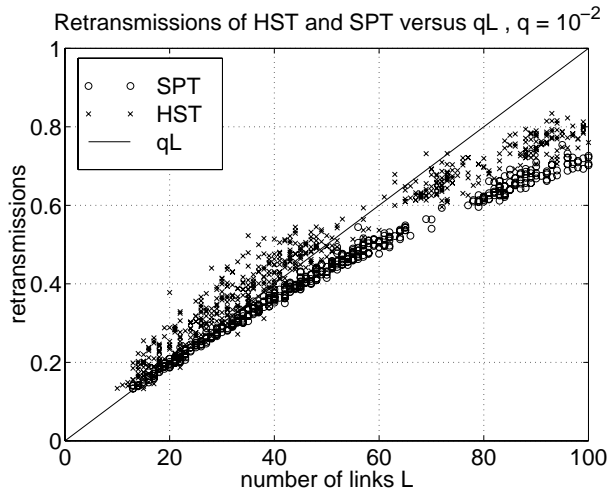
Fig. 13. The Approximation qL versus the real number of retransmissions for SPT and HST and $q = 0.01$.

links. Note that the SPTs and HSTs represent a wide range of various tree topologies.

4 Implications of our work

We demonstrate the impact of our results in the following two domains:

- We show that multicast routing algorithms that optimize delay achieve better delay and

throughput performance for reliable multicast communication than algorithms that optimize cost.

- We show that the *FBT* is a good generic model of a multicast connection and that more realistic results are obtained than with the usual used *MFAN*.

### 4.1 Impact of Routing on Error Recovery

Multicast routing algorithms have been designed that take mainly into account cost and delay. However, the impact of the routing on the performance for reliable transmission is left aside.

For a given loss rate, the performance of error recovery schemes for point-to-point connections is determined by the Round Trip Time (*RTT*) between the source and the receiver. We define the Round Trip Time as two times the sum of the propagation and transmission delays of the links on the path from the source to the receiver. In the following, the impact of *SPT* routing and *HST* routing on the performance of reliable delivery is evaluated by comparing the *RTT* and the number of retransmissions.

For a multicast connection, the receiver connected to the source via the longest path (in terms of delay) is the bottleneck for the error recovery scheme that uses positive ACKs. The *RTT* of a multicast connection is therefore defined as two times the sum of the propagation and transmission time on the links on this *longest* path and depends on the routing algorithm.

The number of retransmissions for *SPT* and *HST* is obtained via simulation, since the computation of $E[M_S - 1]$ (12) via (10), (9) and (8) is very expensive. The link loss probability is $q = 0.01$. Every point in figure 15 is obtained as an average for 100 trees, each tree being constructed for a different set of R receivers,
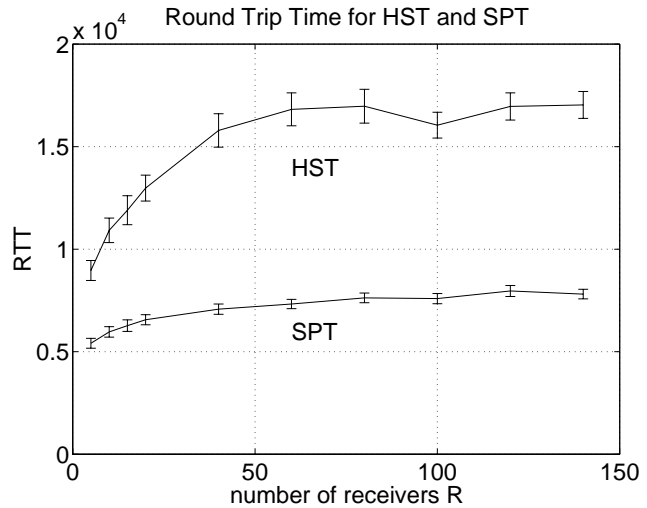
Fig. 14. The Round Trip Time *RTT* of *SPT* and *HST*.
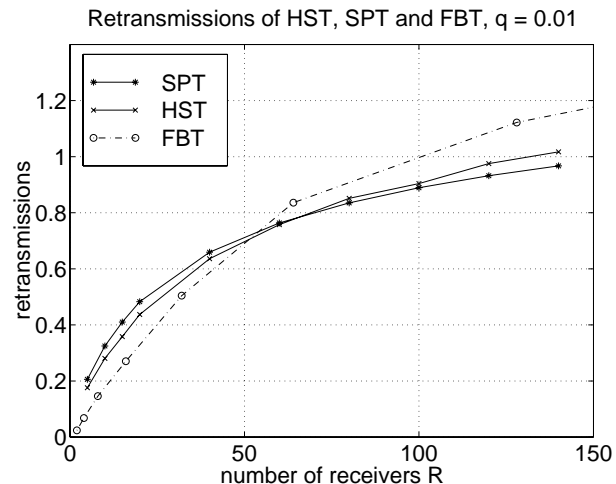
Fig. 15. The expected number of retransmissions of *SPT* and *HST* for $q = 0.01$.

where a different random network is used every 10 trees. Figure 15 shows that the difference between *HST* and *SPT* in terms of the number of retransmissions is minor. However, on the other hand is the *RTT* for a *HST* about two times higher than than the *RTT* for the *SPT* (see figure 14).

From the two observations, we conclude that delay optimization (*SPT*) in multicast routing algorithms yields better delay and throughput performance for reliable transmission than does cost optimization (*HST*).

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In addition, applications with a stringent time–constraint profit also from routing algorithms that optimize delay (SPT). In recent years, routing algorithms have been designed that optimize cost and try to meet a delay–constraint. However, most of the algorithms optimizing cost do not support dynamic multicast group membership changes – the SPT does.

We believe that SPT routing is the best solution for multicast routing. Due to its simplicity, it can use the routing of the underlying unicast algorithm, it supports dynamic membership changes, and assures good performance for reliable transmission as well as for time–constraint delivery.

4.2 A good multicast tree model: Full Binary Tree

We saw in previous sections that the loss characteristics of the FBT are very close to the loss characteristics of HST and SPT.

To confirm that the FBT is a good generic model for a multicast tree, we compare the **link share** in different trees, i.e. to what degree do receivers in a tree share common paths.

Let L be the number of links and R be the number of receivers in the multicast tree, then the link share of one link l_i, $i = 1, \ldots, L$ can be defined as the number of receivers $rd(l_i)$ that share the cost on link l_i divided by the total number of receivers: $ls(l_i) = \frac{rd(l_i)}{R}$. The link share ls for the entire tree mct is defined as the average link share of all links:

$$ls(mct) = \frac{1}{L} \sum_{i=1}^{L} \frac{rd(l_i)}{R} \qquad (14)$$

For a tree, there are several methods to define a measure of link share. We compared measures of link share and found that the definition given
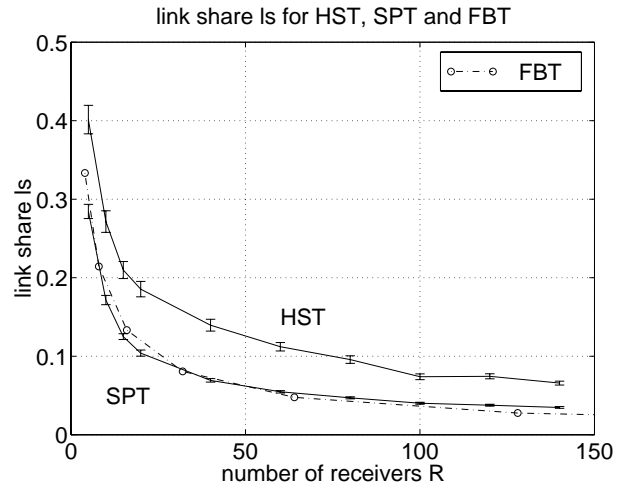

Fig. 16. The link share ls for SPT, HST and FBT.

in (14) reflects well the degree to which receivers share links in a tree. For a further discussion on definitions of link share see [21].

The link share of the FBT is nearly identical with the link share of the SPT (see figure 4.2). The HST has a higher link share than the SPT since the routing algorithm tries to connect the receiver set with a minimal cost, resulting in a high number of receivers that share an average single link in the multicast tree.

The choice of the FBT as a good multicast tree model due to the degree to which receivers share the links is also based on the pmf of the number of successful receptions. The number of successful receptions is highly dependent on the tree topology, since loss may affect several receivers - due to shared links. The similarity in shape of the pmf for the FBT (figure 7) and the pmfs of SPT (figure 9) and HST (figure 10) suggest the FBT as a good tree model. The FBT tree model is further confirmed by the number of retransmissions needed for reliable delivery. Figure 15 shows that the performance of the FBT topology is close to the performance of HST and SPT.

Our results so far are based on a homogeneous link loss probability. For heterogeneous link loss the pmf of the number of successful reception mainly depends on the location of bottleneck

links with high loss probability and on the number of receivers in the subtree rooted at such a bottleneck link. The *FBT* provides a rich model that allows for a variety of heterogeneous link loss settings, including: several bottlenecks and bottlenecks in sequence.

Another model for a multicast tree is proposed in [2]. The tree model is the outcome of loss measurements on the MBONE. The authors report high loss at the source and high loss at the receivers, while backbone loss is minor. This spatial loss correlation among receivers is reflected by the proposed tree model, referred to as **modified star**: a source is connected via one link to a *MFAN* topology [1].

For homogeneous link loss, the modified star models exactly a GEO (geosynchronous earth orbit) satellite with one uplink and multiple downlinks. For more complex tree topologies the modified star does not reflect well shared loss, since only one shared link exists. Therefore the *pmf* of the number of successful receptions for the modified star is comparable to the one of the *MFAN* with an additional peak at $X_S = 0$ (compare figure 7).

For heterogeneous link loss, the authors [2] derive link loss probabilities from the loss measurements, such that the modified star can serve as a tree model for a small number of receivers in a real world scenario.

5   Conclusion

We evaluated the impact of multicast routing on reliable multicast and obtained two main results. First, multicast routing that optimizes delay achieves better throughput and delay performance for reliable multicast than cost optimal routing. Second, the full binary tree (*FBT*)

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[1]   A *MFAN* that connects every receiver with one link to the source

is a good generic model for the loss characteristics of real multicast trees and provides more realistic results than the *MFAN* for which a loss affects always only one receiver. We derived two characterizations that enable the comparison of routing algorithms and error recovery mechanisms with respect to the multicast tree topology, namely a *pmf* for the number of successful receptions when a packet is emitted once from the source and the expected number of retransmissions needed to deliver a packet from the source to all receivers. We also show that the product qL of the link loss probability q and the number of links L in an arbitrary multicast tree tightly approximates the expected number of retransmissions under the condition that $qL < 1$.

Acknowledgment

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A   Lemma 1

We proof a fundamental relation between routing and error recovery for 1:R − multicast communication. Given a homogeneous link loss probability

q in a multicast connection (tree) with L links, where retransmissions are multicast we show that the expected number of retransmissions $E[M_S - 1]$ needed to deliver a packet from the source to *all* receivers can be approximated for *any* topology by qL, as long as $qL < 1$:

$$E[M_S - 1] \approx qL$$

This approximation is based on the following Lemma 1.

Lemma 1

The *CDF* $F_n(i)$ can always be expressed in the following form:

$$F_n(i) = 1 - \sum_{j_n^-} (Q_{j_n^-})^i + \sum_{j_n^+} (Q_{j_n^+})^i \qquad \text{(A.1)}$$

Where the $Q_{j_n^-}$ and $Q_{j_n^+}$ are polynoms in q: $Q = \sum_k \zeta_k q^k$, with the following properties:

- the smallest exponent in Q is $k_{min} \geq 1$ and the coefficient of $q^{k_{min}}$ is a natural number $\zeta_{k_{min}} \geq 1$.

- Let
  - N_n^- be the number of polynoms $Q_{j_n^-}$ in $-\sum_{j_n^-} (Q_{j_n^-})^i$, e.g. the number of polynoms $Q_{j_n^-}$ indexed by j_n^-.
  - N_n^+ be the number of polynoms $Q_{j_n^+}$ indexed by j_n^+.

  Then it is always

$$N_n^- = N_n^+ + 1$$

- Let
  - Σ_n^- be the sum of the coefficients ζ_1 of all polynoms $Q_{j_n^-}$ that have a minimal exponent $k_{min} = 1$. It is possible to sum over all $Q_{j_n^-}$: $\Sigma_n^- = \sum_{j_n^-} \zeta_{1_{j_n^-}}$, since for polynoms with a minimal exponent $k_{min} > 1$ is $\zeta_1 = 0$.
  - Σ_n^+ be the sum of the coefficients ζ_1 of all polynoms $Q_{j_n^+}$ that have a minimal exponent $k_{min} = 1$. As for Σ_n^- we can sum as $\Sigma_n^+ = \sum_{j_n^+} \zeta_{1_{j_n^+}}$.

- L_n be the number of links in the subtree rooted at n.

Then, when there is a link leading to n it is:

$$L_n + 1 = \Sigma_n^- - \Sigma_n^+$$

In the case where n is the source, then there is no link leading to n and it is:

$$L_n = \Sigma_n^- - \Sigma_n^+$$

A.1   Proof of Lemma 1

The proof proceeds by induction over the children c in $child(n)$ using Eqs. (8), (9), (10) and the binomial theorem.

$$(a + b)^i = \sum_{u=0}^{i} \binom{i}{u} a^u b^{(i-u)} \qquad \text{(A.2)}$$

When proving Lemma 1 for node n the induction assumption is that Lemma 1 holds for the children of node n. The induction over the children must distinguish the cases of node n as a leaf, as the source, and as an intermediate node – just as in the definition of the *CDF* $F_n(i)$ given in (8), (9) and (10).

Case o)

The case where n is a **leaf** gives the induction basis.

$$F_n(i) =_{(8)} 1 - q^i = 1 - \sum_{j_n^-} (Q_{j_n^-})^i + \sum_{j_n^+} (Q_{j_n^+})^i$$

Where j_n^- indexes just one polynom $Q = q$ with the smallest exponent $k = 1 \geq 1$ and the coefficient $\zeta_1 = 1 \geq 1$. j_n^+ does not index any polynom. Trivially, there is one more polynom in $-\sum_{j_n^-} (Q_{j_n^-})^i$, than in $+\sum_{j_n^+} (Q_{j_n^+})^i$ and

$$N_n^- = N_n^+ + 1$$

holds. The one polynom $Q = (\zeta_1 q + \ldots) = q$ with exponent $k = 1$ and $\zeta_1 = 1$, yields

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$\Sigma_n^- = \sum_{j_n^-} \zeta_{1_{j_n^-}} = 1$ and since there's no polynom $Q_{j_n^+}$ it is $\Sigma_n^+ = 0$. The number of links in the subtree rooted at l is $L_n = 0$ and there is a link leading to l and the equation

$$L_n + 1 = 1 = \Sigma_n^- - \Sigma_n^+$$

is true.

Case i)
For a node n, not a leaf we must distinguish two cases: the node n is the source S, or n is neither source, nor leaf. In both cases the induction assumption (I.A.) is that Lemma 1 is true for every child $c \in child(n)$ of n.

Case i.1.) n is the source S
Then due to the definition of $F_S(i)$ in (10):

$$
\begin{aligned}
F_S(i) =_{(10)} & \prod_{c \in child(S)} F_c(i) \\
=_{I.A.} & \prod_{c \in child(S)} \left(1 - \sum_{j_c^-}(Q_{j_c^-})^i + \sum_{j_c^+}(Q_{j_c^+})^i\right)
\end{aligned}
$$

The proof that $F_S(i)$ can be expressed as:

$$F_S(i) = 1 - \sum_{j_S^-}(Q_{j_S^-})^i + \sum_{j_S^+}(Q_{j_S^+})^i$$

with the given properties in Lemma 1 Eq. (A.1) is relatively straightforward and proceeds via another inner induction over the number $w = 1, \ldots, z$ of children $c \in child(S)$. For the ease of indexing we assume the children of S named $1, \ldots, z$.

Case i.1.o) The induction base is one child of source S ($w = 1$):

$$
\begin{aligned}
F_S(i) =_{(10)} & \prod_{c=1}^{w} F_c(i) = F_1(i) \\
=_{I.A.} & 1 - \sum_{j_1^-}(Q_{j_1^-})^i + \sum_{j_1^+}(Q_{j_1^+})^i \\
= & 1 - \sum_{j_S^-}(Q_{j_S^-})^i + \sum_{j_S^+}(Q_{j_S^+})^i
\end{aligned}
$$

and Lemma 1 Eq. (A.1) is true due to the outer induction assumption, since $c = 1$ indexes one child of n and for a child the induction assumption holds. Therefore:

$$N_S^- = N_S^+ + 1$$

is true for $F_S(i)$, since the number of polynoms indexed by j_1^- and j_1^+ does not change and $N_1^- = N_1^+ + 1$ was true for the child by induction assumption.

For child 1 it is $L_1 + 1 = \Sigma_1^- - \Sigma_1^+$, since there is a link leading to child 1.

There is only one child ($w = 1$) of the source S and the number of links in the tree rooted at S is just the number of links in the tree rooted at the child (L_1) plus one for the link from S to the child: $L_S = L_1 + 1$. It is $\Sigma_S^- = \Sigma_1^-$ and $\Sigma_S^+ = \Sigma_1^+$ for the source S as for the child 1, since $-\sum_{j_S^-}(Q_{j_S^-})^i = -\sum_{j_1^-}(Q_{j_1^-})^i$ and $+\sum_{j_S^+}(Q_{j_S^+})^i = +\sum_{j_1^+}(Q_{j_1^+})^i$. It is therefore

$$L_S = L_1 + 1 =_{I.A.} \Sigma_1^- - \Sigma_1^+ = \Sigma_S^- - \Sigma_S^+$$

Since there is no link leading to S, Lemma 1 is proven for the case of one child of the source.

Case i.1.i) The induction step from w to $w + 1$ children of the source S.
This case uses the inner induction assumption, referred to as ($i.I.A.$) that for w children at the source S Lemma 1 is true for S. The induction step is adding one child, called $w + 1$, for which Lemma 1 is also true by the outer induction assumption ($I.A.$). An extended notation $S(w)$ will be used to describe the source node S with w children. $F_{S(w+1)}(i)$ can be expressed as:

$$F_{S(w+1)}(i) =_{(10)} F_{w+1}(i) F_{S(w)}$$

Where:

$$F_{w+1}(i) =_{I.A.} 1 - \sum_{j_{w+1}^-}(Q_{j_{w+1}^-})^i + \sum_{j_{w+1}^+}(Q_{j_{w+1}^+})^i$$

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$$F_{S(w)} =_{i.I.A.} 1 - \sum_{j^-_{S(w)}} (Q_{j^-_{S(w)}})^i + \sum_{j^+_{S(w)}} (Q_{j^+_{S(w)}})^i$$

The product $F_{w+1}(i) F_{S(w)}$ results again in the following form for $F_{S(w+1)}(i)$:

$$F_{S(w+1)}(i) = 1 - \sum_{j^-_{S(w+1)}} (Q_{j^-_{S(w+1)}})^i$$
$$+ \sum_{j^+_{S(w+1)}} (Q_{j^+_{S(w+1)}})^i$$

The polynoms $Q_{j^-_{S(w+1)}}$ and $Q_{j^+_{S(w+1)}}$ are products of polynoms $Q_{j^+_{w+1}}$, $Q_{j^+_{S(w)}}$, $Q_{j^-_{w+1}}$ and $Q_{j^-_{S(w)}}$ as can be seen below.

$$- \sum_{j^-_{S(w+1)}} (Q_{j^-_{S(w+1)}})^i = - \sum_{j^-_{w+1}} (Q_{j^-_{w+1}})^i \qquad (A.3)$$
$$- \sum_{j^+_{w+1}, j^-_{S(w)}} (Q_{j^+_{w+1}} Q_{j^-_{S(w)}})^i$$
$$- \sum_{j^-_{S(w)}} (Q_{j^-_{S(w)}})^i$$
$$- \sum_{j^-_{w+1}, j^+_{S(w)}} (Q_{j^-_{w+1}} Q_{j^+_{S(w)}})^i$$

$$+ \sum_{j^+_{S(w+1)}} (Q_{j^+_{S(w+1)}})^i = + \sum_{j^+_{w+1}} (Q_{j^+_{w+1}})^i \qquad (A.4)$$
$$+ \sum_{j^+_{w+1}, j^+_{S(w)}} (Q_{j^+_{w+1}} Q_{j^+_{S(w)}})^i$$
$$+ \sum_{j^+_{S(w)}} (Q_{j^+_{S(w)}})^i$$
$$+ \sum_{j^-_{w+1}, j^-_{S(w)}} (Q_{j^-_{w+1}} Q_{j^-_{S(w)}})^i$$

Due to the induction assumption the $Q_{j^+_{w+1}}$, $Q_{j^-_{w+1}}$, $Q_{j^+_{S(w)}}$ and $Q_{j^-_{S(w)}}$ are all polynoms $Q = \sum_k \zeta_k q^k$ in q with a minimal exponent $k_{min} \geq 1$ and the coefficient of $q^{k_{min}}$ is a natural number $\zeta_{k_{min}} \geq 1$. A product $Q_1 Q_2$ of two polynoms Q_1 and Q_2 with this property results in

a polynom in q with a minimal exponent $k_{min} \geq 2 \geq 1$ and a coefficient of $q^{k_{min}}$ that is again a natural number $\zeta_{k_{min}} \geq 1$. The polynoms $Q_{j^+_{S(w+1)}}$ and $Q_{j^-_{S(w+1)}}$ have therefore also a minimal exponent $k_{min} \geq 1$ and $\zeta_{k_{min}} \geq 1$.

The total number of polynoms $N^-_{S(w+1)}$ in the sum $- \sum_{j^-_{S(w+1)}} (Q_{j^-_{S(w+1)}})^i$ and the total number of polynoms $N^+_{S(w+1)}$ in the sum $+ \sum_{j^+_{S(w+1)}} (Q_{j^+_{S(w+1)}})^i$ has again the property:

$$N^-_{S(w+1)} - N^+_{S(w+1)} = 1$$

To prove this, the number of polynoms in the above expressions (A.3) and (A.4) will be evaluated:

$$N^-_{S(w+1)} = N^-_{(w+1)} + (N^+_{(w+1)})(N^-_{S(w)})$$
$$+ N^-_{S(w)} + (N^-_{(w+1)})(N^+_{S(w)})$$
$$N^+_{S(w+1)} = N^+_{(w+1)} + (N^+_{(w+1)})(N^+_{S(w)})$$
$$+ N^+_{S(w)} + (N^-_{(w+1)})(N^-_{S(w)})$$

The induction assumption gives

$$N^-_{(w+1)} - N^+_{(w+1)} = 1$$
$$N^-_{S(w)} - N^+_{S(w)} = 1$$

and can be applied to $N^-_{S(w+1)} - N^+_{S(w+1)}$:

$$N^-_{S(w+1)} - N^+_{S(w+1)} =_{i.I.A.} (N^-_{(w+1)} - N^+_{(w+1)})$$
$$+ N^+_{(w+1)} - N^-_{(w+1)} + 1$$
$$=_{I.A.} 1$$

In order to complete the proof of Lemma 1, we need to show that the number $L_{S(w+1)}$ of links in the tree rooted at the source S with $w + 1$ children equals the difference of the sums of the coefficients ζ_1 of the polynoms in (A.3) and (A.4):

$$L_{S(w+1)} = \Sigma^-_{S(w+1)} - \Sigma^+_{S(w+1)}$$

The number $L_{S(w+1)}$ of links in the tree rooted at S in the case of $w + 1$ children is just the number

$L_{S(w)}$ of links in the case of w children at the source plus the number L_{w+1} of links in the tree rooted at child $w+1$ plus 1 for the link leading from the source S to this child:

$$L_{S(w+1)} = L_{S(w)} + L_{w+1} + 1$$

As stated before the products Q_1Q_2 of polynoms Q_1 and Q_2 in Eq. (A.3) and Eq. (A.4) have a minimal exponent $k_{min} \geq 2$, which means that the coefficients $\zeta_{k_{min}}$ of these products do not influence $\Sigma^-_{S(w+1)}$ and $\Sigma^+_{S(w+1)}$. Therefore only the polynoms in the expressions $\sum_{j^-_{w+1}}(Q_{j^-_{w+1}})^i$ and $\sum_{j^-_{S(w)}}(Q_{j^-_{S(w)}})^i$ in (A.3) and the expressions $\sum_{j^+_{w+1}}(Q_{j^+_{w+1}})^i$ and $\sum_{j^+_{S(w)}}(Q_{j^+_{S(w)}})^i$ in (A.4) have to be considered. The sum of the coefficients ζ_1 in (A.3) and (A.4) is:

$$\Sigma^-_{S(w+1)} = \sum_{j^-_{S(w+1)}} \zeta_{1_{j^-_{S(w+1)}}}$$
$$= \sum_{j^-_{w+1}} \zeta_{1_{j^-_{w+1}}} + \sum_{j^-_{S(w)}} \zeta_{1_{j^-_{S(w)}}}$$
$$= \Sigma^-_{w+1} + \Sigma^-_{S(w)}$$
$$\Sigma^+_{S(w+1)} = \sum_{j^+_{S(w+1)}} \zeta_{1_{j^+_{S(w+1)}}}$$
$$= \sum_{j^+_{w+1}} \zeta_{1_{j^+_{w+1}}} + \sum_{j^+_{S(w)}} \zeta_{1_{j^+_{S(w)}}}$$
$$= \Sigma^+_{w+1} + \Sigma^+_{S(w)}$$

This yields:

$$\begin{aligned}
\Sigma^-_{S(w+1)} - \Sigma^+_{S(w+1)} &= (\Sigma^-_{w+1} + \Sigma^-_{S(w)}) \\
&\quad - (\Sigma^+_{w+1} + \Sigma^+_{S(w)}) \\
&=_{i.I.A.} \Sigma^-_{w+1} - \Sigma^+_{w+1} + L_{S(w)} \\
&=_{I.A.} L_{w+1} + 1 + L_{S(w)} \\
&= L_{S(w+1)}
\end{aligned}$$

Case i.2.) n is neither the source S, nor a leaf

We need to consider definition (9) of $F_n(i)$ for this case:

$$F_n(i) = \sum_{u=0}^{i-1} \binom{i}{u} q^u (1-q)^{i-u} \prod_{c \in child(n)} F_c(i-u)$$

The result from the preceding proof for the source S can be reused, since the expression $\prod_{c \in child(n)} F_c(i-u)$ has the same form as $F_S(i) = \prod_{c \in child(S)} F_c(i)$. Lemma 1 has been proven for the source and $\prod_{c \in child(n)} F_c(i-u)$ can therefore be expressed in the following form:

$$\prod_{c \in child(n)} F_c(i-u) = 1 - \sum_{j^-_m}(Q_{j^-_m})^{i-u} + \sum_{j^+_m}(Q_{j^+_m})^{i-u} \tag{A.5}$$

Here the node variable m instead of n is used, since n is not the source and there is a link leading to n. m describes the node n without the link leading from the parent to n. Because the properties in Lemma 1 have been proven for the source S, m has the same properties:
The $Q_{j^-_m}$ and $Q_{j^+_m}$ are polynoms in q: $Q = \sum_k \zeta_k q^k$.

- The smallest exponent in Q is $k_{min} \geq 1$ and the coefficient of $q^{k_{min}}$ is a natural number $\zeta_{k_{min}} \geq 1$.

- $N^-_m = N^+_m + 1$

- $L_m = \Sigma^-_m - \Sigma^+_m$

Substituting $\prod_{c \in child(S)} F_c(i)$ by (A.5) in $F_n(i)$ and applying the binomial theorem (A.2) yields:

$$\begin{aligned}
F_n(i) &=_{(9)} \sum_{u=0}^i \binom{i}{u} q^u (1-q)^{i-u} \prod_{c \in child(n)} F_c(i-u) \\
&=_{(A.5)} \sum_{u=0}^i \binom{i}{u} q^u (1-q)^{i-u} \\
&\quad - \sum_{u=0}^i \binom{i}{u} q^u \sum_{j^-_m}((1-q)Q_{j^-_m})^{i-u} \\
&\quad + \sum_{u=0}^i \binom{i}{u} q^u \sum_{j^+_m}((1-q)Q_{j^+_m})^{i-u}
\end{aligned}$$

$$=_{(A.2)} 1$$
$$- \sum_{j_m^-} (q + (1-q)Q_{j_m^-})^i$$
$$+ \sum_{j_m^+} (q + (1-q)Q_{j_m^+})^i$$
$$= 1 - \sum_{j_n^-} (Q_{j_n^-})^i + \sum_{j_n^+} (Q_{j_n^+})^i)$$

Where the indexes are just renamed $j_n^- = j_m^-$ and $j_n^+ = j_m^+$ and cover the same range. The $Q_{j_n^-}$ and $Q_{j_n^+}$ are again polynoms with the following relation to the $Q_{j_m^-}$ and $Q_{j_m^+}$:

$$Q_{j_n^-} = q + (1-q)Q_{j_m^-} \qquad (A.6)$$
$$Q_{j_n^+} = q + (1-q)Q_{j_m^+} \qquad (A.7)$$

The $Q_{j_m^-}$ and $Q_{j_m^+}$ are polynoms in q: $Q = \sum_k \zeta_k q^k$. The $Q_{j_n^-}$ and $Q_{j_n^+}$ have then the following form:

$$q + (1-q)Q = (\zeta_1 + 1)q + (\zeta_2 - \zeta_1)q^2 + \ldots \ (A.8)$$

These polynoms are again polynoms in q and have again a minimal exponent $k_{min} = 1 \geq 1$ and the coefficient $\zeta_{k_{min}} = \zeta_1 + 1 \geq 1$ is again a natural number, since ζ_1 has this property. The bijective mapping of polynoms Q_{j_m} to Q_{j_n} in (A.6) and (A.7) means further that the number of polynoms in $-\sum_{j_n^-} (Q_{j_n^-})^i$ is the same as in $-\sum_{j_m^-} (Q_{j_m^-})^{i-u}$ and the number of polynoms in $+\sum_{j_n^+} (Q_{j_n^+})^i$ is the same as in $+\sum_{j_m^+} (Q_{j_m^+})^{i-u}$:

$$N_n^- = N_m^- \qquad N_n^+ = N_m^+$$

Since $N_m^- = N_m^+ + 1$ holds it is also $N_n^- = N_n^+ + 1$.

$$L_m = \Sigma_m^- - \Sigma_m^+$$

is true for m, since m describes node n as the source, without the link leading to n. The link to n exists and the property

$$L_n + 1 = \Sigma_n^- - \Sigma_n^+$$

in Lemma 1 has to be proven. The ζ_1s in the coefficients $(\zeta_1 + 1)$ of the first order terms q in (A.8) add up to $\Sigma_m = \sum_{j_m} \zeta_{1_{j_m}}$. The 1s in the coefficients $(\zeta_1 + 1)$ of the first order term q in (A.8) add N_m^- (N_m^+) to Σ_n^- (Σ_n^+) − one time for every polynom $Q_{j_m^-}$ $(Q_{j_m^+})$ of the N_m^- (N_m^+) polynoms. Such that:

$$\Sigma_n^- = \Sigma_m^- + N_m^-$$
$$\Sigma_n^+ = \Sigma_m^+ + N_m^+$$

This yields:

$$\Sigma_n^- - \Sigma_n^+ = \Sigma_m^- - \Sigma_m^+ + N_m^- - N_m^+$$
$$= L_m + 1$$
$$= L_n + 1$$

□