

# FAST ACCESS TO DISTRIBUTED ATOMIC MEMORY \*

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**Abstract.** We study efficient and robust implementations of an atomic read-write data structure over an asynchronous distributed message-passing system made of reader and writer processes, as well as a number of servers implementing the data structure. We determine the exact conditions under which every read and write involves *one round* of communication with the servers. These conditions relate the number of readers to the tolerated number of faulty servers and the nature of these failures.

**Key words.** Atomic registers, Byzantine failures, Distributed algorithms, Fault-tolerance, Time-complexity, Shared-memory emulations.

## 1. Introduction.

**1.1. Background.** Assigning a value to a variable or fetching a value from a variable are probably the most common instructions of any program. When several programs cooperate to achieve a common task, it is natural to provide them with means to perform those instructions through *shared* variables.

The *atomic* read-write data structure allows concurrent processes, each possibly running a different program, to share information through a common variable, as if they were accessing this variable in a sequential manner. This abstraction, usually called an *atomic register* or simply a *register*, is fundamental in distributed computing and is at the heart of a large number of distributed algorithms [16, 5].

We study distributed implementations of this abstraction in an asynchronous message-passing system with no actual physical shared memory: instead, a set of server processes provide the illusion to a set of reader and writer processes (clients) that the abstraction is a physical memory accessible to the client processes. We consider *robust* [4] implementations where any read or write invocation by some client process eventually returns, independently of (a) the operational status of other clients: all of them might have stopped their computation (this aspect of robustness is also called *wait-freedom* [16]); and (b) the failure of some of the servers. Such implementations have recently attracted a lot of interest as they underly reliable distributed storage systems [25, 2, 17], which constitute appealing alternatives to traditional centralized storage infrastructures.

Ensuring both atomicity and robustness is not trivial. Informally, atomicity requires that, even though each read or write operation may overlap and take an arbitrary period of time to complete, they appear to execute at some indivisible instant during their respective period of execution [18]. This requires ordering operations in a way that respects their real-time order as well as the expected sequential specification of a read-write data structure: namely, a read should return the last written value.

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To illustrate how an implementation can be robust yet achieve atomicity and motivate our quest for fast implementations, consider the classical implementation from [4] in a single-writer multi-reader case, also called a SWMR register implementation [18]. In [4], readers and servers are the same set, the writer is one of the servers, a minority of processes may fail by crashing, i.e., halting all their activities without warning, whereas the rest of the processes execute the algorithm assigned to them.

This implementation ensures atomicity by associating timestamps with every written value. To write some value  $v$ , the writer increments its local timestamp, and sends  $v$  with the new timestamp  $ts$  to all servers. Every server, on receiving such a message, stores  $v$  and  $ts$  and then sends an acknowledgment (an ack) to the writer. On receiving acks from a majority, the writer terminates the write. In a read operation, the reader first gathers value and timestamp pairs from a majority of servers, and selects the value  $v$  with the largest timestamp  $ts$ . Then the reader sends  $v$  and  $ts$  to all servers, and returns  $v$  on receiving acks from a majority of processes. Given the single-writer setting, and since only the writer introduces new timestamps in the system, the writer always knows the latest timestamp. Unlike the writer, a reader does not know the latest timestamp in the system, and hence, needs to spend one communication round-trip to discover the latest value, and then another round-trip to propagate the value to a majority of servers. The second round-trip is “required” because the latest value learned in the first round-trip might be present at only a minority of servers: a subsequent read might thus miss this value. In a sense, every read includes, in its second communication round-trip, a “write phase”, with the input parameter being the value selected in the first round-trip.

It is easy to see how to reduce the time-complexity of a read by using a simple decentralization scheme combined with a *max-min* technique. First, the reader sends messages to all servers. Every server, on receiving such a message, broadcasts its timestamp to all servers. On receiving timestamps from a majority of servers, every server selects the *maximum* timestamp, adopts the timestamp and its associated value, and sends the pair to the reader. On receiving such messages from a majority of servers, the reader returns the value with the *minimum* timestamp. To see why this ensures atomicity, observe that, when a write completes, its timestamp, say  $ts$ , is stored at a majority of servers. In any subsequent read, every server sees a timestamp that is at least  $ts$ , before the server sends the message to the reader. Hence, the read returns a value that is not older than the written value. On the other hand, if a read returns a value with timestamp  $ts$ , then a majority of servers have a timestamp at least  $ts$ , and no subsequent read returns an older value.

But can we do better? Is there a *fast* implementation where none of the operations (read or write) require more than one round-trip of communication between a client and servers?

This would clearly be optimal in terms of time-complexity. Besides theoretical interest, such implementation might for instance be of practical relevance in the context of distributed storage systems where fast access to shared information might be of primary importance [1, 7, 29].

Clearly, the difficulty is related to the multiplicity of readers. With a single reader, it is easy to modify the algorithm of [4] such that the read takes only one round-trip [18]: the read can return the latest value learned from the servers in the first round trip, provided it is not older than the value returned in the previous read. Otherwise,

the reader returns the same value as in the previous read. Since there is only one reader, this clearly orders the reads and ensures atomicity. To illustrate this case, suppose the writer writes  $v$  with timestamp  $ts$ , and the write message is received only by one server  $s$  (the write is incomplete). The first reader gets information from a majority of servers that includes  $s$ . The read must return  $v$  because the reader does not know whether the write of  $v$  is complete or not, and this reader has to return the value of the last preceding write. Consider now the situation with two readers. The second reader invokes a read, queries a majority of servers, and misses  $s$ . Clearly, the second read returns a value with a timestamp lower than  $ts$ , violating atomicity: the second read returns an older value than the preceding read.

At first glance, it seems impossible to have a fast implementation with two readers when any minority of servers can be faulty. But what if we further restrict the number of tolerated server failures?

**1.2. Contributions.** We show in this paper that, interestingly, the existence of a fast SWMR implementation depends on the maximum number  $R$  of readers. Notice here the focus on SWMR implementations; in this paper we, therefore, assume  $R \geq 2$ . We consider a general model of computation where  $t$  among the set  $S$  of server processes on which the data structure is implemented can fail; in this paper, we assume  $t \geq 1$ . A server can fail by crashing, or even by deviating arbitrarily from its algorithm and be malicious (also called Byzantine [26]). We denote by  $b$  the number of arbitrary server failures, where  $0 \leq b \leq t$ . In this paper, we consider the authenticated arbitrary failure model in which processes can rely on unforgeable digital signatures [28].

The main contribution of this paper is a theorem stating the following:

**THEOREM 1.1.** *There is a fast implementation of a SWMR register if and only if the number of readers  $R$  is less than  $\frac{S+b}{t+b} - 2$ .*

The paper proves this theorem by giving a fast implementation and then proving it optimal (in terms of number of readers). Both the algorithm and the lower bound proof are, we believe, interesting in their own rights. The algorithm uses a new *trace-based* memory access technique whereas the lower bound uses a *sieve-based* run construction scheme.

- **Algorithm.** Our fast implementation relies on the idea of *traces* left by readers in the servers they access, even if they expedite their operation in one round-trip. These traces are then used to determine which value to return while preserving atomicity. It is important to notice at this point that this idea has not been used in the register transformations literature e.g., [4] because, in a read/write shared memory, a process cannot read a value and at the same time leave a trace. We exploit the idea to obtain a crash-stop fast implementation (i.e., assuming  $b = 0$  and  $R < \frac{S}{t} - 2$ ) and then a Byzantine-resilient one (i.e., assuming  $b \neq 0$ ).

To get an intuition of the idea in the crash-stop case, consider the classical algorithm of [4], sketched above, and the following observation: if a reader sees the latest timestamp  $ts$  at  $x$  servers, then any subsequent reader sees  $ts$  or a higher timestamp at  $x - t$  servers; this is because, in a fast implementation, the first reader does not propagate  $ts$ , and the second reader might miss  $t$  servers seen by the first reader. A generalization of this observation

helps determine when some reader can safely return the value associated with the latest timestamp. This is not entirely trivial because the atomicity of a value cannot be simply deduced from the number of servers that has seen the value. To determine whether a value is safe to return, we make every server maintain, besides the latest value, the set of readers to which the server has sent that value. This is the actual *trace* left by the readers.

- **Lower bound.** Consider  $S$  server processes,  $t$  among which can be faulty, and none is Byzantine. We prove by contradiction that there is no fast implementation with  $R \geq \frac{S}{t} - 2$  (recall here that we assume  $R \geq 2$  and  $t \geq 1$ ). We illustrate the proof in Figure 1.1 for  $S = 4$ ,  $t = 1$  and  $R = 2$ .

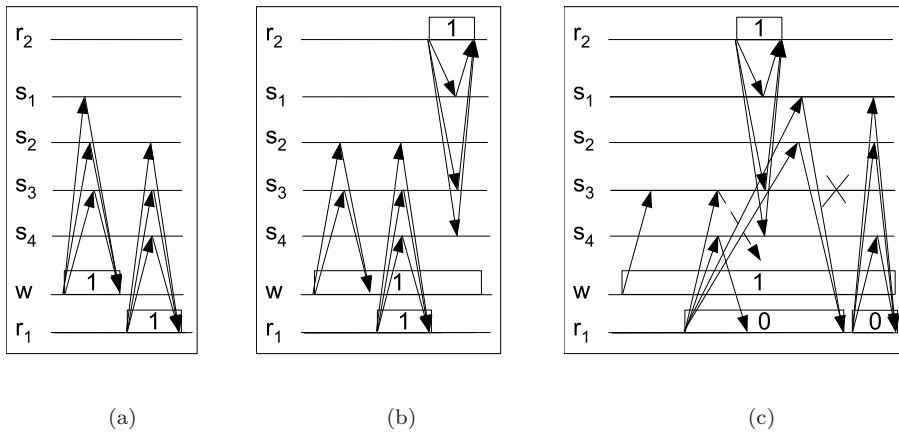


FIG. 1.1. Sketch of the lower bound proof for  $S = 4$ ,  $t = 1$  and  $R = 2$

Given a fast implementation with  $R \geq \frac{S}{t} - 2$ , we consider a partial run which contains a *write(1)* that misses  $t$  servers, to which we append a read that misses  $t$  other servers (see Fig. 1.1(a)). Then we delete all the steps in the partial run that are not “visible” to the reader (basically, the steps of the  $t$  servers that the read missed). By atomicity, the read returns 1 in the resulting partial run. Now we iteratively append reads, each acting like a *sieve*, by distinct readers, and delete the steps in the partial run that are not visible to the last reader, until we exhaust all the readers (Fig. 1.1(b)). To ensure atomicity, the last read of each partial run returns 1. In the final partial run (obtained after exhausting all readers) the steps of *write(1)* are almost deleted. We modify this partial run to construct several additional partial runs, one of which violates atomicity. In the special case shown in Figure 1.1(c) we obtain atomicity violation by reusing the first reader  $r_1$ : a) the first read invocation by  $r_1$ , concurrent with that of reader  $r_2$ , cannot return 1 because it cannot read value 1 from any server (the response from  $s_3$  is never received since  $s_3$  crashes) and b) for similar reasons, the following read by  $r_1$  cannot return 1, however a preceding read by  $r_2$  already returned 1 — an atomicity violation.

The lower bound proof is then extended to the case where  $b \leq t$  servers can fail in an arbitrary manner: we show that a fast implementation is possible

only if the number of readers is less than  $\frac{S+b}{t+b} - 2$ .

Finally, to complete the picture, we prove that it is impossible to have a *one-round* read algorithm with multiple writers [20] (MWMMR atomic register) even if only one server can fail and it can only do so by crashing.

**1.3. Roadmap.** The paper is organized as follows. Section 2 gives the system model. Section 3 defines atomic (and fast) register implementations. We present a fast implementation assuming  $R < \frac{S}{t} - 2$  in Section 4. We prove a tight bound for  $R$  in Section 5. Section 6 extends the previous results to the arbitrary failure model. Section 7 considers the multi-writer case. We discuss related work in Section 8. Section 9 summarizes the main results of the paper.

## 2. System Model.

**2.1. Basics.** The distributed system we consider consists of three *disjoint* sets of processes: a set *servers* (also denoted by  $\Sigma$ ) of cardinality  $S$  containing processes  $\{s_1, \dots, s_S\}$ , a set *writer* containing a single process  $\{w\}$  (we discuss the multi-writer case in Section 7), and a set *readers* of cardinality  $R$  containing processes  $\{r_1, \dots, r_R\}$ . We refer to the elements from  $writer \cup readers$  as *clients*. We also denote a set  $\Sigma \setminus \Sigma'$ , where  $\Sigma' \subseteq \Sigma$  by  $\Sigma'$ . Every pair of processes communicate by message-passing using a bi-directional reliable communication channel. Notice that the reliable channels assumption is needed to ensure *wait-freedom*, but not *atomicity* (we define these notions later). We also assume the existence of a global clock; however, processes cannot access the global clock.

A distributed algorithm  $A$  is a collection of automata, where  $A_p$  is the automata assigned to process  $p$  [22]. Computation proceeds in *steps* of  $A$ . A *run* is an infinite sequence of steps of  $A$ . A *partial run* is a finite prefix of some run. A (partial) run  $r$  *extends* some partial run  $pr$  if  $pr$  is a prefix of  $r$ . At the end of a partial run, all messages that are sent but not yet received are said to be *in transit*. In any given run, any number of readers, the writer, and  $t$  out of  $S$  servers may crash. We extend our failure model to allow for arbitrary failures in Section 6.

**2.2. Details of the System Model.** The state of communication channels is viewed as a set of messages *mset* containing messages that are sent but not yet received. We assume that every message has two tags which identify the sender and the receiver of the message. A distributed algorithm  $A$  is a collection of automata, where  $A_p$  is the automata assigned to process  $p$ . Computation proceeds in *steps* of  $A$ . A step of  $A$  is denoted by a pair of process id and message set  $\langle p, M \rangle$  ( $M$  might be  $\emptyset$ ). In step  $\langle p, M \rangle$ , process  $p$  atomically does the following: (1) remove the messages in  $M$  from *mset*, (2) apply  $M$  and its current state  $st_p$  to  $A_p$ , which outputs a new state  $st'_p$  and a set of messages to be sent, and then (3)  $p$  adopts  $st'_p$  as its new state and puts the output messages in *mset*.

Given any algorithm  $A$ , a *run* of  $A$  is an infinite sequence of steps of  $A$  such that the following properties hold for each process  $p$ : (1) initially,  $mset = \emptyset$ , (2) the current state in the first step of  $p$  is a special state *Init*, (3) for each step  $\langle p, M \rangle$ , and for every message  $m \in M$ ,  $p$  is the receiver of  $m$  and *mset* contains  $m$  immediately before the step  $\langle p, M \rangle$  is taken, and (4) (*reliable channels*) if there is a step that puts a message  $m$  in *mset* such that  $p$  is the receiver of  $m$  and both  $p$  and the sender of  $m$  take an infinite number of steps, then there is a step  $\langle p, M \rangle$  such that  $m \in M$ .

A *partial run* is a finite prefix of some run. We say that a process is *correct* in a run if it takes an infinite number of steps in that run. Otherwise the process is *faulty*. In a run of our model, any number of readers or the writer may be faulty, and at most

$t \leq S$  servers might be faulty. We say that a (faulty) process  $p$  *crashes* at step  $sp$  in a run, if  $sp$  is the last step of  $p$  in that run. Notice that the assumption of reliable channels does not guarantee that correct processes always receive messages sent by faulty processes.

For presentation simplicity, we do not explicitly model the initial state of a process, nor the invocations and responses of operations. We assume that the algorithm  $A$  initializes the processes, and schedules invocation/response of operations (i.e.,  $A$  modifies the states of the processes accordingly). However, we say that  $p$  invokes an operation  $op$  at step  $sp$ , if  $A$  modifies the state of a process  $p$  in step  $sp$  so as to invoke an operation (and similarly for response).

In any run, we say that an operation  $op1$  *precedes* operation  $op2$  (or  $op2$  *follows*  $op1$ ) if the response step of  $op1$  precedes the invocation step of  $op2$  in that run. If neither  $op1$  nor  $op2$  precedes the other, the operations are said to be *concurrent*. We say that an operation is *complete* in a (partial) run if the run contains a response step for that operation. We assume that all (partial) runs are *well-formed*, i.e., no process  $p$  invokes a new operation before all operations previously invoked by  $p$  have completed.

A *history* of a partial run is a sequence of invocation and response steps of operations in the same order as they appear in the partial run. An *incomplete* operation in a history  $H$  is an operation whose invocation step is in  $H$ , but the matching response step is not in  $H$ . We say that a history  $H1$  *completes* history  $H2$  if  $H1$  can be obtained through the following modification of  $H2$ : for each incomplete operation  $op$  in  $H2$ , either invocation step of  $op$  is removed from  $H2$ , or any valid matching response for that invocation is appended to the end of  $H2$ .

**3. Atomic Register.** A sequential register is a data structure accessed by multiple processes in a non-concurrent manner (i.e., no two operations on a sequential register are concurrent). It provides two operations:  $\text{write}(v)$ , which stores  $v$  in the register, and  $\text{read}()$ , which returns the last value stored. Only readers invoke *reads* on the register and only the writer invokes *writes* on the register. We further assume that there is a special write  $wr_0$  operation that initializes the register by writing a special value  $\perp$  (which is not a valid input value for other writes) such that  $wr_0$  precedes all other operations.

An atomic register is a distributed data structure that may be concurrently accessed by multiple processes and yet provides an “illusion” of a sequential register to the accessing processes. An algorithm *implements* an atomic register if every run of the algorithm satisfies *termination* and *atomicity* properties, defined in the following.

**3.1. Definition.** Termination states that if a correct process invokes an operation, then eventually the operation completes (even if all other client processes have crashed).

A run satisfies atomicity, if for every history  $H'$  of any of its partial runs, there is a history  $H$  that completes  $H'$  and  $H$  satisfies the properties A1-A3 below (Lemma 13.16 of [21]). Let  $\Pi$  be the set of all operations in  $H$ . There is an irreflexive partial ordering  $\prec$  of all the operations in  $H$  such that: (A1) if  $op1$  precedes  $op2$  in  $H$  then it is not the case that  $op2 \prec op1$ , (A2) if  $op1$  is a write operation in  $\Pi$  and  $op2$  is any other operation in  $\Pi$ , then either  $op1 \prec op2$  or  $op2 \prec op1$  in  $\Pi$ , and (A3) the value returned by each read operation is the value written by the last preceding write operation according to  $\prec$ .

In our single writer setting assuming well-formed runs, atomicity properties A1-A3 can be simplified. Namely, in the single-writer case, the relation *precedes* totally

orders write operations (and, likewise, the values stored by these). Before we give the equivalent definition of atomicity in the single writer setting, we introduce some additional notation.

In a given run, we denote by  $wr_k$  the write that is preceded by exactly  $k$  writes (including the initial write  $wr_0$ ). In other words, for  $k \geq 1$ ,  $wr_k$  denotes the  $k^{th}$  write by the writer in a given run (note that  $wr_0$  is not invoked by the writer). Then, we say that an operation  $op$  returns a *timestamp*  $k$ , if: a)  $op$  is  $wr_k$ , or b)  $op$  is a read that returns the value stored by  $wr_k$ .

Consider a relation  $\prec$  such that  $op_1 \prec op_2$  if and only if the timestamp returned by  $op_1$  is *smaller* than the timestamp returned by  $op_2$ . Then, it is straightforward to show that the  $\prec$  is a partial ordering that satisfies atomicity properties A1-A3, if the following properties are satisfied:

- (SWA1) If a read returns, it returns a non-negative timestamp.
- (SWA2) If a read  $rd$  returns timestamp  $l$  and  $rd$  follows write  $wr_k$ , then  $l \geq k$ .
- (SWA3) If a read  $rd$  returns timestamp  $k$ , then  $rd$  does not precede  $wr_k$ .
- (SWA4) If reads  $rd_1$  and  $rd_2$  return timestamps  $l_1$  and  $l_2$ , respectively, and if  $rd_2$  follows  $rd_1$ , then  $l_2 \geq l_1$ .

Indeed, it is straightforward to see that Property A1 of  $\prec$  is implied by Properties SWA2 and SWA4, whereas property A3 is implied by properties SWA1, SWA2 and SWA3. Finally, Property A2 follows immediately from our definition of  $\prec$ , the ordering of write operations (well-formedness) and SWA1. Hence, to show that a single writer register implementation satisfies atomicity, it is sufficient to show that it satisfies the properties SWA1-SWA4.

**3.2. Time-Complexity of Implementations.** We define the time-complexity of atomic register implementation in terms of communication *round-trips*. An operation  $op$  invoked by client  $c$  consists of a sequence of round-trips, where a round-trip contains the following three phases.

1. Client  $c$  sends messages to a subset of processes in a step. (In the first round-trip of the operation, the invocation step of operation precedes this send step.)
2. Upon receiving a message  $m$  in step  $sp1 = \langle p, M \rangle$  ( $m \in M$ ), where  $m$  is sent by the client  $c$  in phase 1 of the round-trip, a process  $p$  replies to  $c$  either in step  $sp1$  itself, or in a subsequent step  $sp2$ , such that  $p$  does not receive any message in any step between  $sp1$  and  $sp2$ , including  $sp2$ . (In other words, upon receiving  $m$ ,  $p$  replies to the client before receiving any other messages. Intuitively, this requirement forbids the processes to wait for some other message before replying to  $m$ .)
3. Upon receiving a sufficient number of above replies, client  $c$  either returns from  $op$  or moves to the next round-trip.

We would like to note the following points to emphasize the generality of the above definition: (1) steps from phase 2 and phase 3 of a round-trip may interleave, (2) round-trips of different operations (by different clients) may overlap, and (3) there is no requirement on the communications between a pair of processes both of which are distinct from the client (e.g., the servers may communicate among themselves using any message exchange pattern). Also, note that the above definition is close to the time-complexity definition in [13], where a *round-trip* in the above definition corresponds to a *round* in [13].

**3.3. Fast Implementations.** A read or a write operation is *fast* if it completes in one communication round-trip. We say that an atomic register implementation is *fast* if both its read and write operations are fast.

Recall that implementations need to tolerate the crash of any client and up to  $t$  servers. Hence, in order to ensure termination in a fast implementation, a reader (or a writer) cannot wait for replies from any other client, or more than  $S - t$  servers, in the first round-trip of the operation. For an implementation that has fast reads, we can say without ambiguity that the messages sent by a reader, on invoking a read, are of type READ, and the reply sent by a process to the reader, on receiving a READ message, are of type READACK. Similarly, we define WRITE and WRITEACK messages for fast writes.

We would like to note that no register implementation can have all its write invocations return in the first round-trip before receiving any WRITEACK message. Suppose by contradiction that a  $\text{write}(v)$  by writer  $w$  returns in the first round-trip before receiving any WRITEACK message. Then,  $w$  cannot distinguish this operation from another incomplete operation where all WRITE messages of the operation are in transit. In the latter operation, suppose the operation returns before any WRITE message is received,  $w$  crashes, and no WRITE messages are ever received. Then, no subsequent read can recover and return  $v$ . Thus, in terms of worst-case time-complexity, it is not possible to improve over fast implementations.

**4. A Fast Atomic Register Implementation.** In the following, we first describe our fast implementation assuming  $R < \frac{S}{t} - 2$  and then prove its correctness. For simplicity of presentation, we first present our algorithm assuming that the writer writes timestamps, and the readers read back timestamps. Later we explain how to simply generalize our algorithm such that the writer and the readers associate some value with a timestamp.

**4.1. Algorithm.** The pseudocode of our fast implementation is given in Figure 4.1. The write procedure is similar to that of [4]. On invoking a write, the writer increments its timestamp (initialized to 0) and sends a WRITE message with the timestamp to all servers (lines 4-5). Upon receiving the message, servers update the timestamp and send WRITEACK messages back to the writer (lines 22-29). The writer returns OK once it has received WRITEACK messages from  $S - t$  servers (lines 6-7).

Implementing a fast read is more involved. Like several previous atomic register implementations, our read procedure collects timestamps from  $S - t$  servers (by sending READ messages and receiving READACK messages from the servers), and selects the highest timestamp, denoted by  $\text{maxTS}$  in line 15, Figure 4.1. However, notice that in our fast implementation server  $s_i$ , besides storing the highest received timestamp in its local variable  $ts_i$ , maintains also the set  $\text{updated}_i$  that contains all clients to which  $s_i$  has sent the current value of  $ts_i$ . Basically,  $\text{updated}_i$  contains all clients to which  $s_i$  has sent an update (in the form of a READACK or a WRITEACK message) about its highest timestamp. This information is read in our read procedure along with the timestamps. Hence, a reader collects timestamps and sets  $\text{updated}_*$  from  $S - t$  servers using READ and READACK messages (lines 12-13 and 32).

Upon receiving  $S - t$  READACK messages (collected in the set  $\text{rcvMsg}$ , line 14), the read is precluded from waiting for more messages (the remaining  $t$  servers may be faulty). Moreover, in order to have a fast implementation, the read may only perform some local computations and then it must return the value. In our fast implementation, the essence of this local computation is captured by predicate *admissible* (line 8)



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0: **at the writer  $w$ :**

1: **procedure** initialization:

2:    $ts \leftarrow 0$

3: **procedure** write():

4:    $ts \leftarrow ts + 1$

5:   send(WRITE,  $ts$ ) to all servers

6:   **wait until** receive(WRITEACK,  $ts$ ) from  $S - t$  servers

7:   return(OK)

---

**at each reader  $r_i$ :**

8:  $admissible(TS, Msg, a) \equiv \exists \mu \subseteq Msg, \forall m \in \mu :$   
 $(m.ts = TS) \wedge (|\mu| \geq S - at) \wedge (|\bigcap_{m' \in \mu} m'.updated| \geq a)$

9: **procedure** initialization:

10:    $maxTS \leftarrow 0$

11: **procedure** read():

12:   send(READ,  $maxTS$ ) to all servers

13:   **wait until** receive(READACK, \*, \*) from  $S - t$  servers

14:    $rcvMsg \leftarrow \{m | r_i \text{ received (READACK, *, *) in line 13}\}$

15:    $maxTS \leftarrow \mathbf{Max}\{ts \mid (READACK, ts, *) \in rcvMsg\}$

16:   **if**  $\exists a \in [1, R + 1] : admissible(maxTS, rcvMsg, a)$  **then**

17:     return( $maxTS$ )

18:   **else**

19:     return( $maxTS - 1$ )

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**at each server  $s_i$ :**

20: **procedure** initialization:

21:    $ts_i \leftarrow 0; updated_i \leftarrow \emptyset$

22: **procedure** update( $ts, c$ ):

23:   **if**  $ts > ts_i$  **then**

24:      $ts_i \leftarrow ts; updated_i \leftarrow \{c\}$

25:   **else**

26:      $updated_i \leftarrow updated_i \cup \{c\}$

27: **upon** receive(WRITE,  $ts$ ) from writer  $w$  **do**

28:   **update**( $ts, w$ )

29:   send(WRITEACK,  $ts_i$ ) to  $w$

30: **upon** receive(READ,  $ts$ ) from reader  $r_j$  **do**

31:   **update**( $ts, r_j$ )

32:   send(READACK,  $ts_i, updated_i$ ) to  $r_j$

---

FIG. 4.1. Fast SWMR atomic register implementation with  $R < \frac{S}{t} - 2$

which relies on the sets  $updated_*$  to evaluate whether the highest received timestamp  $maxTS$  may be returned. Namely, if there is an integer  $a$  ( $1 \leq a \leq R + 1$ ), such that  $admissible(maxTS, rcvMsg, a)$  holds in line 16 (we simply say that  $maxTS$  is *admissible* (with degree  $a$ )), the `read` returns  $maxTS$  (line 17). Otherwise, if  $maxTS$  is not admissible, a `read` returns  $maxTS - 1$ . In any case,  $maxTS$  is cached locally, and is written back by the reader in its following invocation of `read` (line 12).

In the following, we give an intuition behind the predicate  $admissible()$  which is the heart of our fast implementation. The predicate is designed to guarantee that:

- (a)  $maxTS = k$  is admissible in `read`  $rd$  whenever  $wr_k$  precedes  $rd$  — this is vital for ensuring “read-write” atomicity, captured by Property SWA2, Section 3.1, and
- (b) if  $maxTS = k$  is admissible in `read`  $rd$ , then no  $rd'$  that follows  $rd$  returns a timestamp smaller than  $k$  — this is vital for ensuring “read-read” atomicity, captured by Property SWA4, Section 3.1.

First, we explain how our predicate guarantees (a). Consider the following partial run,  $pr_1$ . In  $pr_1$ , write  $wr_k$  ( $k \geq 1$ ) completes by writing  $k$  to all servers from some set  $\Sigma_1$  containing  $S - t$  servers. There are no writes in  $pr_1$  that follow  $wr_k$ . Moreover, `read`  $rd$  (by some reader  $r_i$ ), that follows  $wr_k$ , reads from set  $\Sigma_2$  of  $S - t$  servers that intersects in  $S - 2t$  servers with  $\Sigma_1$ , i.e.,  $rd$  misses  $t$  servers in  $\Sigma_1$ . By atomicity,  $rd$  must return  $k$  in  $pr_1$ , and it must do so without waiting for messages from servers from  $\overline{\Sigma_2}$  or the writer, since these may be faulty. In this case: (i) in line 15 of  $rd$ ,  $maxTS = k$ , and (ii) for every message  $m$  received by  $r_i$  in  $rd$  from servers from  $\Sigma_1 \cap \Sigma_2$ , we have  $m.ts = maxTS$  and  $\{w, r_i\} \subseteq m.updated$ . Since  $|\Sigma_1 \cap \Sigma_2| \geq S - 2t$ ,  $maxTS$  is admissible in  $rd$  with degree  $a = 2$ .

On the other hand, the key to guaranteeing (b) is the following invariant (hereafter,  $maxTS_{op}$  denotes  $maxTS$  computed in line 15 of the `read` operation  $op$ ):

LEMMA 1. *Let  $rd'$  be a complete read (by reader  $r_j$ ) that follows a complete read  $rd$  (by  $r_i$ ). If  $maxTS_{rd}$  is admissible with degree  $a_{rd} \leq R + 1$ , then:*

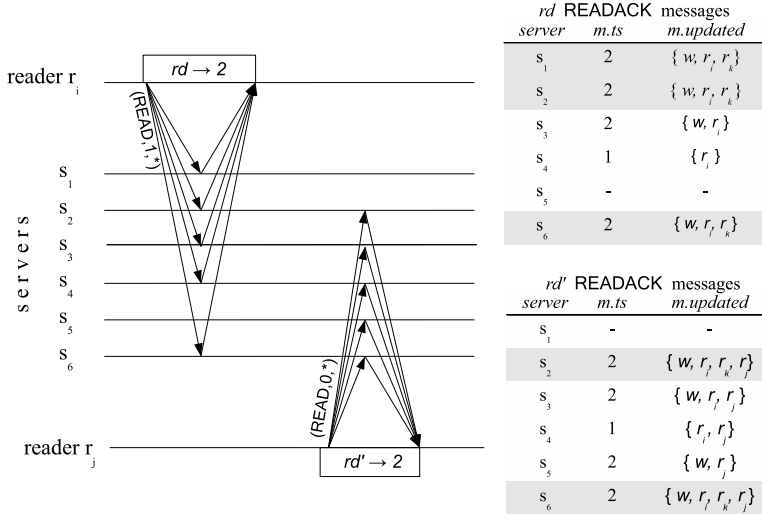
- $maxTS_{rd'} > maxTS_{rd}$ , or
- $maxTS_{rd'} = maxTS_{rd}$  and  $maxTS_{rd'}$  is admissible with degree  $a_{rd} + 1$  or degree 1 in  $rd'$ .

Here, we sketch the proof of Lemma 1 (the full correctness proof of our implementation can be found in Section 4.2), and illustrate it in Figure 4.2.

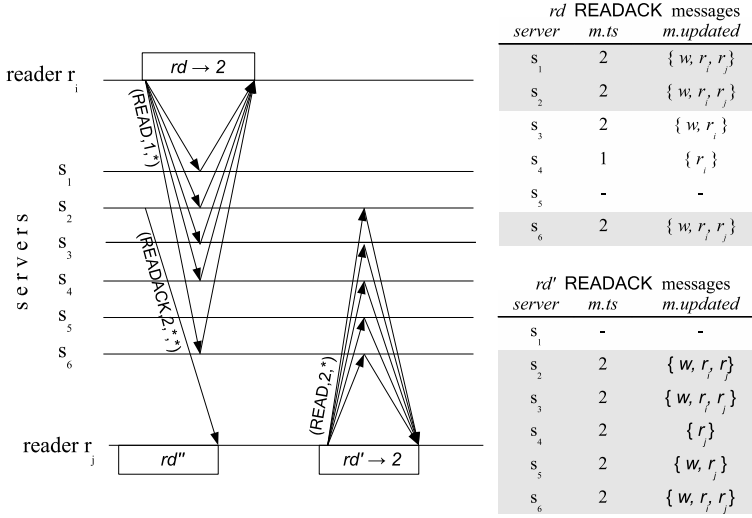
*Proof.* By the definition of the predicate *admissible* (line 8), there is a set of READACK messages  $\mu_{rd}$ , sent by servers from the set  $\Sigma_{rd}$  to reader  $r_i$ , such that, for every message  $m$  in  $\mu_{rd}$ ,  $m.ts = maxTS_{rd}$ ,  $|\Sigma_{rd}| = |\mu_{rd}| \geq S - a_{rd}t$  and  $|\Pi_{rd}| \geq a_{rd}$ , where  $\Pi_{rd} = \bigcap_{m \in \mu_{rd}} m.updated$ . Notice that, since  $a_{rd} \leq R + 1$  and  $R < \frac{S}{t} - 2$ , we have  $|\Sigma_{rd}| \geq t + 1$ . Moreover, since (a)  $rd'$  follows  $rd$ , (b)  $|\Sigma_{rd}| \geq t + 1$  and (c)  $rd'$  reads from  $S - t$  servers,  $rd'$  receives a READACK from at least 1 server from  $\Sigma_{rd}$ . Hence,  $maxTS_{rd'} \geq maxTS_{rd}$ .

If  $maxTS_{rd'} = maxTS_{rd} = k$ , we distinguish two cases:

Case (i), Fig. 4.2(a),  $r_j \notin \Pi_{rd}$  (note that this case is possible only if  $a_{rd} \leq R$ ). It is not difficult to see that  $k$  is admissible in  $rd'$  with degree  $a_{rd} + 1$ . Indeed,  $rd'$  will miss at most  $t$  servers from  $\Sigma_{rd}$ , receiving at least  $S - (a_{rd} + 1)t$  READACK messages containing the timestamp  $k$ , and the  $updated_*$  fields of these messages will be a su-



(a) In this case  $r_j \notin \Pi_{rd} = \bigcap_{m \in \mu_{rd}} m.updated = \{w, r_i, r_k\}$ . Since read  $rd'$  is guaranteed to READACK messages from at least 2 servers in  $\{s_1, s_2, s_6\}$  and these servers add  $r_j$  in their *updated* sets,  $maxTS_{rd'} = 2$  is admissible with degree 4 in  $rd'$  ( $\Pi_{rd} = \{w, r_i, r_k, r_j\}$ )



(b) In this case  $r_j \in \Pi_{rd} = \bigcap_{m \in \mu_{rd}} m.updated = \{w, r_i, r_j\}$ . Hence, all servers from  $\{s_1, s_2, s_6\}$  send a READACK message with  $ts = 2$  to  $r_j$  before  $rd'$  is invoked and  $maxTS_{rd''} = 2$  in some read  $rd''$  by  $r_j$  that precedes  $rd'$ . Hence, in  $rd'$ ,  $r_j$  sends READ message with  $ts = 2$  and  $maxTS_{rd'} = 2$  is admissible with degree 1 in  $rd'$  ( $\Pi_{rd} = \{r_j\}$ )

FIG. 4.2. Illustration of the proof of Lemma 1, for a run with  $R = 3$  readers,  $S = 6$  servers and  $t = 1$ . READACK messages that allow timestamp 2 to be admissible in reads  $rd$  and  $rd'$  are highlighted in grey. In both cases, in read  $rd$ ,  $maxTS_{rd} = 2$  is admissible with degree 3 (notice the READACK messages received from servers  $s_1, s_2$  and  $s_6$ ).

periset of  $\Pi_{rd} \cup \{r_j\}$ , hence each containing at least  $a_{rd} + 1$  clients.

Case (ii), Fig. 4.2(b),  $r_j \in \Pi_{rd}$  (which indicates that  $rd'$  is not the first read by  $r_j$ ). In this case, all servers from  $\Sigma_{rd}$ , at least  $t + 1$  of them, have sent a READACK message to  $r_j$  containing the timestamp  $k$  before  $rd'$  is invoked. At least one of those must have been received by  $r_j$  in read  $rd''$  that immediately precedes  $rd'$ . Hence,  $\max TS_{rd''} = k$ , and  $\max TS$  in line 12 of  $rd'$  equals  $k$ . Finally,  $S - t$  servers send READACK to  $rd'$  with the timestamp equal to  $k$  and the set *updated* that contains  $\{r_j\}$  (see lines 22-26), i.e.,  $k$  is admissible with degree 1 in  $rd'$ .  $\square$

Finally, to help distinguish READ and READACK messages from different reads of the same reader, we implicitly (for better readability of pseudocode) assume that every reader  $r_j$  maintains a local variable  $rdCnt_j$  that  $r_j$  increments at the beginning of every read invocation. Reader  $r_j$  includes  $rdCnt_j$  in each READ message (line 12) and servers reply including  $rdCnt_j$  in a READACK message (line 32). Servers cache the highest seen  $rdCnt_j$  for every reader  $r_j$ ; namely, before sending a READACK message to  $r_j$ , a server stores locally  $rdCnt_j$  and does not reply to READ messages from  $r_j$  that contain  $rdCnt'_j < rdCnt_j$ . Hence, once server  $s_i$  replies to read  $rd$  invoked by reader  $r_j$ ,  $s_i$  replies only to those reads of  $r_j$  that follow  $rd$ . To refer to this invariant, we say that servers *ignore old reads*.

This completes the brief description of the register implementation. We now describe how to modify the algorithm so as to associate values with timestamps. In the modified algorithm, in each write, the writer attaches two tags with the timestamp, containing the current value to be written and the value of the immediately preceding write. If the reader returns  $\max TS$  in the original algorithm, then it returns the current value attached to  $\max TS$  in the modified algorithm. If the reader returns  $\max TS - 1$  in the original algorithm, it returns the other tag attached to  $\max TS$  in the modified algorithm.

**4.2. Correctness of the Fast Implementation.** It is obvious that the read/write time-complexity is one communication round-trip. To show atomicity, we need to prove Properties SWA1-SWA4 of Section 3.1. In the proof, we use the following notation:

DEFINITION 4.1.

1.  $rcvMsg_{op}$  denotes the set of received READACK messages the reader collects in read operation  $op$  (lines 13-14);
2.  $\Sigma_{op}$  denotes the set of servers from which the reader received READACK messages in  $rcvMsg_{op}$  (in case  $op$  is a read), or the set of servers from which the writer received WRITEACK messages in line 6 of  $op$  (in case  $op$  is a write). Notice that, for every operation  $op$ ,  $|\Sigma_{op}| = S - t$ ;
3.  $\max TS_{op}^{old}$  denotes the value of  $\max TS$  in line 12 in read  $op$ . Moreover,  $\max TS_{op}$  denotes  $\max TS$  computed by the reader in line 15, in  $op$  (i.e., the highest timestamp in messages in  $rcvMsg_{op}$ ).
4.  $\mu_{op,a}$  denotes, in case  $\max TS_{op}$  is admissible with degree  $a$  in  $op$ , the subset of  $rcvMsg_{op}$ , such that (see line 8, definition of predicate admissible):

- (a)  $|\mu_{op,a}| \geq S - at$ ,
  - (b)  $\forall m \in \mu_{op,a}, m.ts = \max TS_{op}$ , and
  - (c)  $|\bigcap_{m \in \mu_{op,a}} m.updated| \geq a$ .
5.  $\Pi_{op,a}$  denotes the set of servers  $\bigcap_{m \in \mu_{op,a}} m.updated$ ; and
  6. Finally,  $\Sigma_{\mu_{op,a}}$  denotes the set of servers that sent messages in  $\mu_{op,a}$ .

We start with two simple observations that we use in the rest of the proof.

LEMMA 4.2. *If a server sets its local variable  $ts$  to  $x$  at time  $T$ , then the server never sets  $ts$  to a value that is lower than  $x$  after time  $T$ .*

*Proof.* By trivial server code inspection.  $\square$

LEMMA 4.3. *A read operation  $rd$  may only return either  $\max TS_{rd}$  or  $\max TS_{rd} - 1$ .*

*Proof.* By lines 16-19 and the definition of  $\max TS_{rd}$ .  $\square$

The following lemma captures the “writeback” mechanism of our algorithm and states, roughly, that read operations invoked by the same reader cannot violate atomicity. This lemma is crucial for proving Properties SWA1 and SWA2.

LEMMA 4.4. *A read  $rd$  cannot return a value smaller than  $\max TS_{rd}^{old}$ .*

*Proof.* Recall that  $\max TS_{rd}^{old}$  denotes  $\max TS$  sent in READ message in  $rd$  (line 12). By lines 23-24, every READACK message received by a reader in  $rd$  from some server  $s_j$  is with  $ts_j \geq \max TS_{rd}^{old}$ . Hence,  $\max TS_{rd} \geq \max TS_{rd}^{old}$ . There are the following two cases to consider. (1) If  $\max TS_{rd} > \max TS_{rd}^{old}$ , by Lemma 4.3, the return value is not smaller than  $\max TS_{rd}^{old}$ . (2) If  $\max TS_{rd} = \max TS_{rd}^{old}$ , then every READACK message in  $rcvMsg_{rd}$  has a timestamp equal to  $\max TS_{rd}^{old}$  and  $r_i \in updated$ . Since  $rd$  receives  $S - t$  READACK messages,  $\max TS_{rd}^{old}$  is admissible with degree  $a = 1$  ( $\mu_{rd,1} = rcvMsg_{rd}$ ). By lines 16-17,  $rd$  returns  $\max TS_{rd}^{old}$ .  $\square$

LEMMA 4.5. (**SWA1**) *If a read returns, it returns a non-negative timestamp.*

*Proof.* To prove the lemma, it is sufficient to show that there is no read  $rd$  in which  $\max TS_{rd}^{old} < 0$  (then, the lemma follows from Lemma 4.4).

To see this, assume by contradiction that there is a read  $rd$  by reader  $r_i$  in which  $\max TS_{rd}^{old} < 0$ . By lines 9-10, this is not the first read by  $r_i$ , i.e., there is a read  $rd'$  by  $r_i$  that (immediately) precedes  $rd$  such that  $\max TS_{rd'} < 0$ , i.e.,  $S - t$  servers sent READACK message in  $rd'$  with  $ts_* < 0$ . However, since server timestamps are initialized to 0, this contradicts Lemma 4.2.  $\square$

LEMMA 4.6. (**“read-write” atomicity (SWA2)**). *If a read  $rd$  returns timestamp  $l$  and  $rd$  follows write  $wr_k$ , then  $l \geq k$ .*

*Proof.* Denote by  $r_i$  the reader that invoked  $rd$  and let  $\Sigma' = \Sigma_{wr_k} \cap \Sigma_{rd}$ . Since  $|\Sigma_{wr_k}| = S - t$  and  $|\Sigma_{rd}| = S - t$ , we have  $|\Sigma'| \geq S - 2t$ .

When a server  $s_j$  in  $\Sigma_{wr_k}$  (and, hence, in  $\Sigma'$ ) replies to a WRITE message from  $wr_k$ , its  $ts_j$  is at least  $k$  (server timestamp is not smaller than  $k$  due to the condition in line 23). Since  $wr_k$  precedes  $rd$ , by Lemma 4.2, servers in  $\Sigma'$  reply with  $ts_* \geq k$  to  $rd$ . Hence,  $\max TS_{rd} \geq k$ . There are the following two cases to consider:

1.  $\max TS_{rd} > k$

By Lemma 4.3,  $rd$  does not return a timestamp lower than  $k$ .

2.  $\max TS_{rd} = k$

Let  $\mu'$  be the set of READACK messages sent by servers in  $\Sigma'$  to  $rd$ . By definition of  $\Sigma_{rd}$  and since  $\Sigma' \subseteq \Sigma_{rd}$ , we have  $\mu' \subseteq rcvMsg_{rd}$ . Since (a) every server  $s_j \in \Sigma'$  replies to  $rd$  with  $ts_j \geq k$ , (b)  $\mu' \subseteq rcvMsg_{rd}$  and (c)

$maxTS_{rd} = k$ , we have that every server  $s_j \in \Sigma'$  replies with  $ts_j = k$  to  $rd$  (and  $r_j$  receives these replies).

Moreover, since every server  $s_j \in \Sigma'$  replies  $ts_j = k$  to  $wr_k$  (since  $\Sigma' \subseteq \Sigma_{wr_k}$ ) before sending  $ts_j = k$  to  $rd$  (since  $wr_k$  precedes  $rd$ ), for every message  $m$  in  $\mu'$ ,  $w \in m.updated$ . Furthermore, since  $s_j$  replies with  $ts_j = k$  to  $rd$ , by line 26,  $r_i \in m.updated$ . Thus,  $\{w, r_i\} \subseteq \bigcap_{m \in \mu'} m.updated$ . As  $|\Sigma'| \geq S - 2t$ ,  $maxTS_{rd}$  is admissible in  $rd$  with degree  $a = 2$ . Hence,  $rd$  returns  $maxTS_{rd} = k$ .

□

The following lemma helps prove Property SWA3.

LEMMA 4.7. *If  $maxTS_{rd} \geq k$ , then  $rd$  does not precede  $wr_k$ .*

*Proof.* We focus on the case  $k \geq 1$ , since the proof for  $k = 0$  follows from the definition of  $wr_0$ . To prove the lemma, it is sufficient to show that no server sets its local timestamp to a value greater or equal to  $k$ , before  $wr_k$  is invoked. Assume, by contradiction, that there is such server  $s_i$  that is, moreover, the first server to set its local timestamp  $ts_i$  to  $l \geq k$  according to the global clock (at time  $T$ ), i.e., no server sets its local timestamp to  $l$  before time  $T$ .

It is obvious that  $wr_k$  and  $wr_l$  (which might be the same operations) are invoked after  $T$ . Hence,  $s_i$  must have set  $ts_i$  to  $l$  after receiving a READ message in a read  $rd'$  invoked by reader  $r_j$  in which  $maxTS_{rd'}^{old} = l$ . Since  $l \geq k \geq 1$ , there is a read  $rd''$  by  $r_j$  that immediately precedes  $rd'$  in which  $maxTS_{rd''} = l$ . Since  $rd''$  precedes  $rd'$ ,  $rd''$  completes before time  $T$ . By definition of  $maxTS_{rd''}$ , some server had set its local timestamp to  $l$  before  $rd''$  completed. A contradiction with the assumption that no server sets its local timestamp before time  $T$ . □

Lemma 4.7 has the following important corollary.

COROLLARY 4.8. *If  $maxTS_{rd} = k \geq 1$  then write  $wr_{k-1}$  completes before read  $rd$  completes.*

LEMMA 4.9. (**SWA3**) *If a read  $rd$  returns timestamp  $k$  ( $k \geq 1$ ), then  $rd$  does not precede  $wr_k$ .*

*Proof.* By Lemmas 4.3 and 4.7. □

We now proceed towards proving Property SWA4 of Section 3.1 (“read-read” atomicity). The following 2 auxiliary lemmas are related to the predicate *admissible* and to the sizes of the relevant subsets of the set *rcvMsg*.

LEMMA 4.10. *If  $maxTS_{rd}$  is admissible in  $rd$  with degree  $a$ , then  $\Sigma_{\mu_{rd,a}}$  contains at least  $t+1$  servers.*

*Proof.* By Definition 4.1 and inequalities  $a \leq R + 1$  and  $R < \frac{S}{t} - 2$ , we have  $|\Sigma_{\mu_{rd,a}}| \geq S - at > (R + 2)t - (R + 1)t = t$ . □

LEMMA 4.11. *Assume that  $maxTS_{rd}$  is admissible with degree  $a \in [1, R + 1]$  in some read  $rd$  and that a complete read  $rd'$  follows  $rd$ . Then, the number of servers in  $\Sigma_{\mu_{rd,a}} \cap \Sigma_{rd'}$  is at least  $S - (a + 1)t$ . Moreover,  $\Sigma_{\mu_{rd,a}} \cap \Sigma_{rd'}$  contains at least one server.*

*Proof.* Since  $|\Sigma_{\mu_{rd,a}}| = |\mu_{rd,a}| \geq S - at$  and  $|\Sigma_{rd'}| = S - t$ , it follows that  $|\Sigma_{rd'} \cap \Sigma_{\mu_{rd,a}}| \geq S - (a + 1)t$ . Moreover, since  $a \in [1, R + 1]$  and  $t < S/(R + 2)$ , we have  $S - (a + 1)t \geq 1$ . □

The following Lemma proves the key invariant used in the proof of Property SWA4 of Section 3.1.

LEMMA 4.12. *Assume that:*

1.  $maxTS_{rd}$  is admissible with degree  $a \in [1, R + 1]$  in some read  $rd$ ,

2. a complete read  $rd'$  by reader  $r_j$  follows  $rd$ ,
3. there is a set  $X \subseteq \Sigma_{\mu_{rd,a}}$  of at least  $t+1$  servers, such that for all  $s_i \in X$ ,  $s_i$  sends message  $m_i \in \mu_{rd,a}$  with  $r_j \in m_i.updated$ .

Then,  $rd'$  does not return a value smaller than  $maxTS_{rd}$ .

*Proof.* Since the messages in  $\mu_{rd,a}$  are sent before the completion of  $rd$  (and hence, before the invocation of  $rd'$ ) and since, there is a server  $s_i$  that sends  $m_i \in \mu_{rd,a}$  with  $r_j \in m_i.updated$ ,  $r_j$  has invoked at least one read before  $rd'$ . Let  $rd''$  be the last read by reader  $r_j$  which precedes  $rd'$ . Since  $|X| \geq t+1$  and  $|\Sigma_{rd''}| = S-t$ , there is at least one server  $s_k$  in  $X \cap \Sigma_{rd''}$ , such that the READACK message  $m$  sent by  $s_k$  is received by  $r_j$  in  $rd''$ . In the following paragraph, we show that  $m.ts \geq maxTS_{rd}$ .

By contradiction, assume  $m.ts < maxTS_{rd}$ . Since servers ignore old reads, there is a read  $rd_\alpha$  by  $r_j$ , such that  $rd_\alpha$  follows  $rd''$  and  $s_k$  sends a READACK message  $m_\alpha$  to  $rd_\alpha$ , before  $s_k$  sends  $m_k \in \mu_{rd,a}$ , i.e., before  $rd'$  is invoked. Hence,  $rd''$  is not the last read by reader  $r_j$  which precedes  $rd'$ . A contradiction.

Since  $m.ts \geq maxTS_{rd}$  and  $m \in rcvMsg_{rd''}$ , we have  $maxTS_{rd''} \geq maxTS_{rd}$ . Since  $rd'$  follows  $rd''$ , it follows that  $r_j$  in  $rd'$  sends READ messages with  $ts \geq maxTS_{rd}$ . By Lemma 4.4,  $rd'$  returns a timestamp greater than or equal to  $maxTS_{rd}$ .  $\square$

LEMMA 4.13. **“read-read” atomicity (SWA<sub>4</sub>)**. *If reads  $rd_1$  and  $rd_2$  return timestamps  $ret_{rd_1}$  and  $ret_{rd_2}$ , respectively, and if  $rd_2$  follows  $rd_1$ , then  $ret_{rd_2} \geq ret_{rd_1}$ .*

*Proof.* Without loss of generality we can assume (to facilitate the notation) that read  $rd_1$  (resp.,  $rd_2$ ) is invoked by reader  $r_1$  (resp.,  $r_2$ ). Suppose  $r_1 = r_2$ . Then, in the read that immediately follows  $rd_1$ ,  $r_1$  sends a READ message with  $ts \geq ret_{rd_1}$ , and hence, by Lemma 4.4, the read returns a value greater than or equal to  $ret_{rd_1}$ . Using Lemma 4.4 and a simple induction, we can derive that any read by  $r_1$  which follows  $rd_1$  (including  $rd_2$ ) returns  $ts \geq ret_{rd_1}$ . Therefore, in the rest of the proof we assume that  $r_1 \neq r_2$ . We distinguish the following two cases:

1.  $maxTS_{rd_1}$  is not admissible in  $rd_1$ .

It follows that  $ret_{rd_1} = maxTS_{rd_1} - 1$ . By Lemma 4.5,  $ret_{rd_1} \geq 0$  and, hence,  $maxTS_{rd_1} \geq 1$ . By Corollary 4.8,  $wr_{ret_{rd_1}}$  completes before  $rd_1$  is completed. Since  $rd_1$  precedes  $rd_2$ , it follows that  $wr_{ret_{rd_1}}$  precedes  $rd_2$ . By Lemma 4.6,  $rd_2$  returns  $ret_{rd_2} \geq ret_{rd_1}$ .

2.  $maxTS_{rd_1}$  is admissible in  $rd_1$ .

It follows that  $ret_{rd_1} = maxTS_{rd_1}$ , and there is some  $a \in [1, R+1]$  such that  $ret_{rd_1}$  is admissible in  $rd_1$  with degree  $a$ . By Lemma 4.11, there is at least one server  $s_i \in \Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$ . Since  $rd_1$  precedes  $rd_2$ ,  $s_i$  first replies with  $ts_i = ret_{rd_1}$  to  $rd_1$  before  $s_i$  replies to  $rd_2$ . Finally, by Lemma 4.2, it follows that  $s_i$  replies to  $rd_2$  with  $ts_i \geq ret_{rd_1}$ , i.e.,  $maxTS_{rd_2} \geq ret_{rd_1}$ .

We distinguish the following three exhaustive cases:

- (a)  $maxTS_{rd_2} > ret_{rd_1}$

By Lemma 4.3, we have  $ret_{rd_2} \geq ret_{rd_1}$ .

- (b)  $maxTS_{rd_2} = ret_{rd_1}$  and  $maxTS_{rd_2}$  is admissible in  $rd_2$

By lines 16-17,  $ret_{rd_2} = maxTS_{rd_2} = ret_{rd_1}$ .

- (c)  $maxTS_{rd_2} = ret_{rd_1}$  and  $maxTS_{rd_2}$  is not admissible in  $rd_2$

We show that this case is impossible by exhibiting appropriate contradictions.

In this case, by lines 16-19,  $ret_{rd_2} = maxTS_{rd_2} - 1 = ret_{rd_1} - 1 =$

$\max TS_{rd_1} - 1$ . By Lemma 4.11, there is at least one server in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$ . Since  $rd_1$  precedes  $rd_2$  and servers in  $\Sigma_{\mu_{rd_1,a}}$  reply with  $ts_* = \text{ret}_{rd_1}$  to  $rd_1$ , by Lemma 4.2, servers in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$  reply to  $rd_2$  with  $ts_* \geq \text{ret}_{rd_1}$ . Since  $\text{ret}_{rd_2} + 1 = \text{ret}_{rd_1} = \max TS_{rd_2}$ , every server in  $\Sigma_{rd_2} \cap \Sigma_{\mu_{rd_1,a}}$  replies to  $rd_2$  with  $ts = \text{ret}_{rd_1} = \text{ret}_{rd_2} + 1$ . There are the following two cases to consider:

i.  $a \leq R$

In this case, by Lemma 4.11, the number of servers in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$  is greater than  $t$ . Let  $\mu_1$  be the set of READACK messages sent from servers in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$  to  $rd_1$ . There are two cases to consider:

A.  $r_2 \notin \bigcap_{m \in \mu_1} m.\text{updated}$

Denote  $\bigcap_{m \in \mu_1} m.\text{updated}$  by  $\Pi_1$ . Notice that, by definitions of  $\mu_1$  and  $\mu_{rd_1,a}$ ,  $\mu_1 \subseteq \mu_{rd_1,a}$ . Hence, we have  $\Pi_1 \supseteq \Pi_{rd_1,a}$  and  $|\Pi_1| \geq a$ .

Let  $\mu_2$  be the set of messages received by  $rd_2$  from servers in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$ . For any server  $s_i \in \Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$ , let  $m_1$  and  $m_2$  be the messages sent by  $s_i$  in  $\mu_1$  and  $\mu_2$  respectively. Since we know that  $m_1.ts_i = m_2.ts_i = \text{ret}_{rd_1}$  and since  $m_1$  is sent before  $m_2$ , we have  $m_1.\text{updated} \subseteq m_2.\text{updated}$ . Hence,  $\Pi_1 \subseteq \bigcap_{m \in \mu_2} m.\text{updated}$ . Since every server which replies to  $r_2$  in  $rd_2$ , adds  $r_2$  to its *updated* set before replying to  $r_2$ ,  $r_2 \in \bigcap_{m \in \mu_2} m.\text{updated}$ .

Since  $r_2 \notin \Pi_1$ ,  $|\bigcap_{m \in \mu_2} m.\text{updated}| \geq |Pi_1| + 1 \geq a + 1$ . Since (a) the number of messages in  $\mu_2$  equals the number of servers in  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$  and (b)  $a + 1 \leq R + 1$ , by Lemma 4.11 and the definition of predicate *admissible* (line 8), we have that  $\text{ret}_{rd_1} = \text{ret}_{rd_2} + 1$  is admissible in  $rd_2$  with degree  $a + 1$ . Hence, the timestamp returned by  $rd_2$  is  $\text{ret}_{rd_2} + 1$ , a contradiction (with the assumption that  $rd_2$  returns  $\text{ret}_{rd_2}$ ).

B.  $r_2 \in \bigcap_{m \in \mu_1} m.\text{updated}$

Denote by  $X$  the set  $\Sigma_{\mu_{rd_1,a}} \cap \Sigma_{rd_2}$ . By definition of  $\mu_1$ , messages in  $\mu_1$  are sent by processes in  $X$ . By Lemma 4.11 and since  $a \leq R$ , it follows that the number of servers in  $X$  is greater than  $t$ . Hence, by Lemma 4.12,  $\text{ret}_{rd_2} \geq \max TS_{rd_1}$ . A contradiction with  $\text{ret}_{rd_2} = \max TS_{rd_1} - 1$ .

ii.  $a = R + 1$

Since  $|\{w, r_1, \dots, r_R\}| = R + 1$  and  $|\bigcap_{m \in \mu_{rd_1,a}} m.\text{updated}| \geq a = R + 1$ , we have  $r_2 \in \bigcap_{m \in \mu_{rd_1,a}} m.\text{updated}$ . By Lemma 4.10,  $\Sigma_{\mu_{rd_1,a}}$  contains at least  $t + 1$  servers. Replacing  $X$  with  $\Sigma_{\mu_{rd_1,a}}$  in Lemma 4.12, it follows that  $\text{ret}_{rd_2} \geq \max TS_{rd_1}$ . A contradiction with  $\text{ret}_{rd_2} = \max TS_{rd_1} - 1$ .

□

Finally, we combine the above lemmas to prove the correctness of our fast implementation.

**THEOREM 4.14** (Fast atomic register). *The algorithm of Figure 4.1 is a fast implementation of an atomic SWMR register.*

*Proof.* Atomicity follows from Lemmas 4.5, 4.6, 4.9 and 4.13. Moreover, it is



obvious that our implementation satisfies Termination (conditions in lines 6 and 13 are non-blocking). Finally, our implementation is fast: all operations involve a single communication round-trip between a client and servers.  $\square$

**5. Lower Bound.** The following proposition states that the resilience required by our fast implementation is indeed necessary.

**PROPOSITION 5.1.** *Let  $t \geq 1$  and  $R \geq 2$ . If  $R \geq \frac{S}{t} - 2$ , then there is no fast atomic register implementation.*

**Preliminaries.** Recall first that  $w$  denotes the writer,  $r_i$  for  $1 \leq i \leq R$  denote the readers, and  $s_i$  for  $1 \leq i \leq S$  denote the servers. Suppose by contradiction that  $R \geq \frac{S}{t} - 2$  and that there is a fast implementation  $I$  of an atomic register. Given that  $t \geq S/(R+2)$ , we can partition the set of servers into  $R+2$  subsets, which we call *blocks*, denoted by  $B_i$  ( $1 \leq i \leq R+2$ ), each of size less than or equal to  $t$ .

For instance, one such partition is: for  $1 \leq i \leq R+1$ ,  $B_i = \{s_j \mid (\lfloor \frac{S}{R+2} \rfloor (i-1) + 1) \leq j \leq (\lfloor \frac{S}{R+2} \rfloor i)\}$ , and  $B_{R+2} = \{s_j \mid (\lfloor \frac{S}{R+2} \rfloor (R+1)) \leq j \leq S\}$ . However, if  $R > S - 2$  then the above partitioning is not possible. In that case we consider a system where the number of readers is  $S - 2$  and the set *readers* is  $\{r_1, \dots, r_{S-2}\}$ , and show the impossibility. The impossibility still holds if we add more readers to this system (i.e.,  $R > S - 2$ ).

Notice that our model does not require that a message sent by a faulty process is received by the receiver. Hence, in any (partial) run, it is possible any message sent by a faulty process remains in transit (i.e., that such a message is never received by the receiver). In our proof we construct (partial) runs in which, unless explicitly stated otherwise, all messages sent by faulty processes are in transit.

We say that an *incomplete operation  $op$  skips* a set of blocks  $BS$  in a partial run, where  $BS \subseteq \{B_1, \dots, B_{R+2}\}$ , if (1) no server in any block  $B_i \in BS$  receives any READ or WRITE message from  $op$  in that partial run, (2) all other servers receive the READ or the WRITE message from  $op$  and reply to that message, and (3) *all these reply messages are in transit*. We say that a *complete operation  $op$  skips* a block  $B_i$  in a partial run, if (1) no server in  $B_i$  receives any READ or WRITE message from  $op$  in that partial run, (2) all servers that are not in  $B_i$  receive the READ or WRITE message from  $op$  and reply to that message and (3) the invoking process *receives all these reply messages and returns from the invocation*.

**Block diagrams.** To depict our proof, we use block diagrams (see Fig. 5.1). We depict an operation  $op$  through a set of rectangles, (generally) arranged in a single column. In the column corresponding to some operation  $op$ , we draw a rectangle in the  $i^{th}$  row, if all servers in block  $B_i$  have received the READ or WRITE message from  $op$  and have sent reply messages, i.e., we draw a rectangle in the  $i^{th}$  row if  $op$  does not skip  $B_i$ .

We illustrate a particular instance of the proof in Figure 5.2 and Figure 5.3, where  $R = 3$  and the set of servers are partitioned into five blocks,  $B_1$  to  $B_5$ .

We now proceed to the proof of Proposition 5.1.

*Proof.* To show a contradiction, we construct a partial run of the fast implementation  $I$  that violates atomicity: a partial run in which some read returns 1 and a subsequent read returns an older value, namely, the initial value of the register,  $\perp$ .

*Partial writes.* Consider a partial run  $wr$  in which  $w$  completes  $\text{write}(1)$  on the register. The operation skips  $B_{R+2}$ . We define a series of partial runs each of which can be

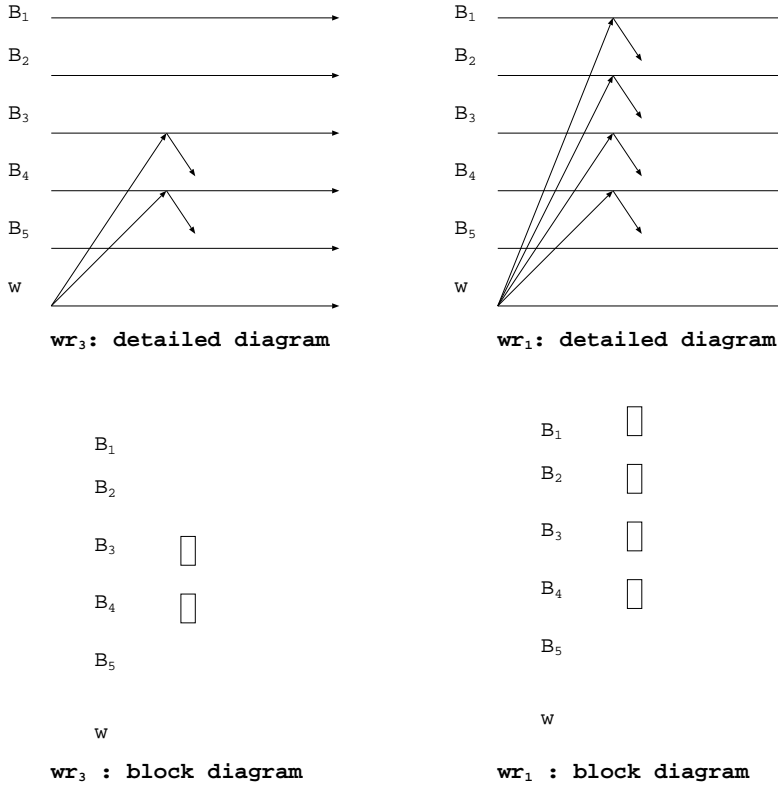


FIG. 5.1. Block diagrams

extended to  $wr$ . Let  $wr_{R+2}$  be the partial run in which  $w$  has invoked the write and has sent the WRITE message to all processes, and all WRITE messages are in transit. For  $1 \leq i \leq R+1$ , we define  $wr_i$  as the partial run which contains an incomplete write(1) that skips  $\{B_{R+2}\} \cup \{B_j | 1 \leq j \leq i-1\}$ . We make the following simple observations: (1) for  $1 \leq i \leq R$ ,  $wr_i$  and  $wr_{i+1}$  differ only at servers in  $B_i$ , (2)  $wr$  is an extension of  $wr_1$ , such that, in  $wr$ ,  $w$  receives the replies (that are in transit in  $wr_1$ ) and the write completes, and hence, (3)  $wr$  and  $wr_1$  differ only at  $w$ . A sample partial writes,  $wr_1$  and  $wr_3$  are presented in Figure 5.1.

*Appending reads.* Partial run  $pr_1$  extends  $wr$  by appending a complete read by  $r_1$  that skips block  $B_1$ . By atomicity, the read returns 1. Observe that  $r_1$  cannot distinguish  $pr_1$  from some partial run  $\Delta pr_1$ , that extends  $wr_2$  by appending a complete read by  $r_1$  that skips  $B_1$ . To see why, notice that  $wr$  and  $wr_2$  differ at  $w$  and at block  $B_1$ , and  $r_1$  does not receive any message from these processes in both runs. Thus  $r_1$ 's read returns 1 in  $\Delta pr_1$ .

Starting from  $\Delta pr_1$ , we iteratively define the following partial runs for  $2 \leq i \leq R$ . Partial run  $pr_i$  extends  $\Delta pr_{i-1}$  by appending a complete read by  $r_i$  that skips  $B_i$ . Partial run  $\Delta pr_i$  is constructed by deleting from  $pr_i$ , all steps of the servers in block  $B_i$ . Since the last read in  $pr_i$  by reader  $r_i$  skips block  $B_i$ ,  $r_i$  cannot distinguish  $pr_i$  from  $\Delta pr_i$ . More precisely, partial run  $\Delta pr_i$  extends  $wr_{i+1}$  by appending the following  $i$

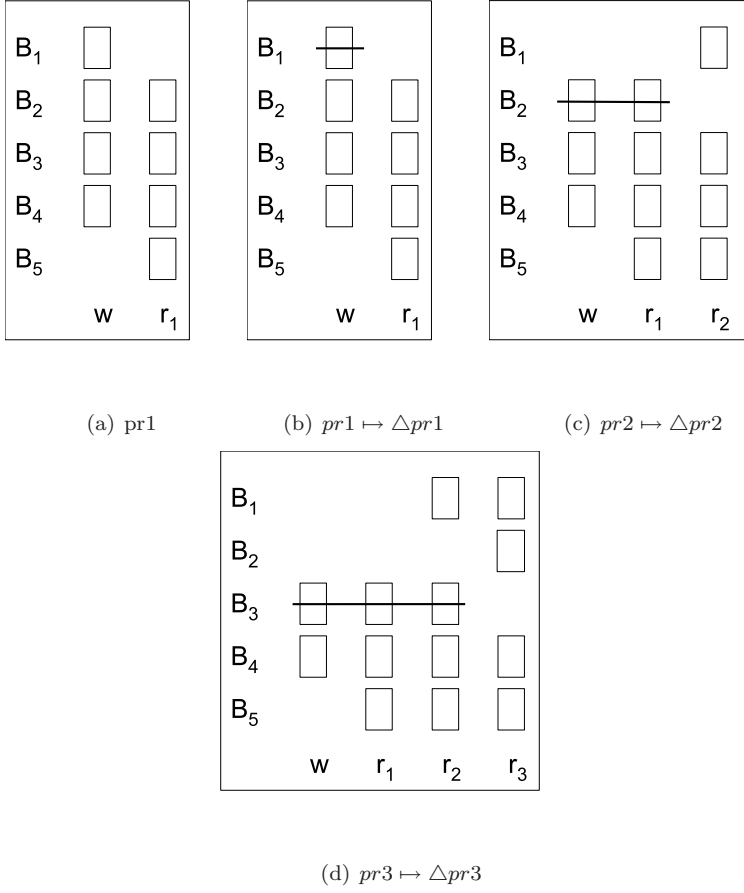
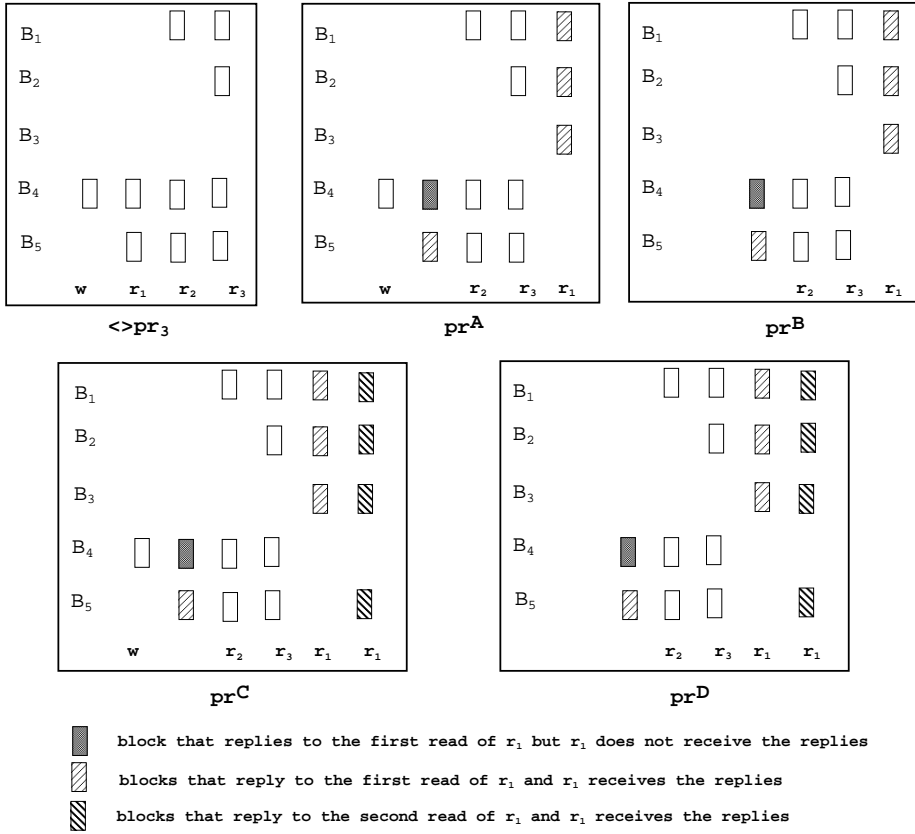


FIG. 5.2. Partial runs:  $pr_i$  (including the crossed out rectangles) and  $\Delta pr_i$  (excluding the crossed out rectangles)

reads one after the other: for  $1 \leq h \leq i$ ,  $r_h$  does a read that skips  $\{B_j | h \leq j \leq i\}$ . Here, the first  $i - 1$  appended reads are incomplete whereas the last one is complete. Figure 5.2 depicts block diagrams of  $pr_i$  and  $\Delta pr_i$  with  $R = 3$ . (The deletion of steps to obtain  $\Delta pr_i$  from  $pr_i$  is shown by crossing out the rectangles corresponding to the deleted steps.)

Reader  $r_1$ 's read in  $\Delta pr_1$  returns 1. Since  $pr_2$  extends  $\Delta pr_1$ , by atomicity,  $r_2$ 's read in  $pr_2$  returns 1. However, as  $r_2$  cannot distinguish  $pr_2$  from  $\Delta pr_2$ ,  $r_2$ 's read in  $\Delta pr_2$  returns 1. In general, since  $pr_i$  extends  $\Delta pr_{i-1}$ , and  $r_i$  cannot distinguish  $pr_i$  from  $\Delta pr_i$  (for all  $i$  such that  $2 \leq i \leq R$ ), it follows from a trivial induction that  $r_i$ 's read in  $\Delta pr_i$  returns 1. In particular,  $r_R$  reads 1 in  $\Delta pr_R$ .

*Partial run  $pr^A$ .* Consider the partial run  $\Delta pr_R$ :  $wr_{R+1}$  extended by appending  $R$  reads by each reader  $r_h$  ( $1 \leq h \leq R$ ) such that  $r_h$ 's read skips  $\{B_j | h \leq j \leq R\}$ . The read by  $r_1$  is incomplete in  $\Delta pr_R$ : only servers in  $B_{R+1}$  and  $B_{R+2}$  send replies to  $r_1$ , and those reply messages are in transit. Observe that, in  $\Delta pr_R$ , only the servers in  $B_{R+1}$  receive the WRITE message from the write(1) operation. Consider the following partial run  $pr^A$  which extends  $\Delta pr_R$  as follows. After  $\Delta pr_R$ , (1)  $r_1$  receives the replies

FIG. 5.3. Partial runs:  $pr^A$ ,  $pr^B$ ,  $pr^C$  and  $pr^D$ 

of its READ messages from  $B_{R+2}$  (that were in transit in  $\Delta pr_R$ ), (2) the servers in  $B_1$  to  $B_R$  receive the READ message from  $r_1$  (that were in transit in  $\Delta pr_R$ ) and reply to  $r_1$ , (3) reader  $r_1$  receives these replies from servers in  $B_1$  to  $B_R$ , and then  $r_1$  returns from the read invocation. (Notice that,  $r_1$  received replies from  $R + 1$  blocks, and so, must return from the read.) However,  $r_1$  does not receive the replies from servers in  $B_{R+1}$  (that were in transit in  $\Delta pr_R$ ). Figure 5.3 depicts block diagrams for  $pr^A$  with  $R = 3$ .

*Partial run  $pr^B$ .* Consider another partial run  $pr^B$  with the same communication pattern as  $pr^A$ , except that `write(1)` is not invoked at all, and hence, servers in  $B_{R+1}$  do not receive any WRITE message (Figure 5.3). Clearly, only servers in  $B_{R+1}$ , the writer, and the readers  $r_2$  to  $r_R$  can distinguish  $pr^A$  from  $pr^B$ . Reader  $r_1$  cannot distinguish the two partial runs because it does not receive any message from the servers in  $B_{R+1}$ , the writer, or other readers. By atomicity,  $r_1$ 's read returns (the initial value of the register)  $\perp$  in  $pr^B$  because there is no `write(*)` invocation in  $pr^B$ , and hence,  $r_1$ 's read returns  $\perp$  in  $pr^A$  as well.

*Partial runs  $pr^C$  and  $pr^D$ .* Notice that, in  $pr^A$ , even though  $r_1$ 's read returns  $\perp$  after  $r_R$ 's read returns 1,  $pr^A$  does not violate atomicity, because the two reads are

concurrent. We construct two more partial runs: (1)  $pr^C$  is constructed by extending  $pr^A$  with another complete read by  $r_1$ , which skips  $B_{R+1}$ , and (2)  $pr^D$  is constructed by extending  $pr^B$  with another complete read by  $r_1$ , which skips  $B_{R+1}$  (Figure 5.3). Since  $r_1$  cannot distinguish  $pr^A$  from  $pr^B$ , and  $r_1$ 's second read skips  $B_{R+1}$  (i.e., the servers which can distinguish  $pr^A$  from  $pr^B$ ), it follows that  $r_1$  cannot distinguish  $pr^C$  from  $pr^D$  as well. Since there is no  $\text{write}(\ast)$  invocation in  $pr^D$ ,  $r_1$ 's second read returns  $\perp$  in  $pr^D$ , and hence,  $r_1$ 's second read in  $pr^C$  returns  $\perp$ . Since  $pr^C$  is an extension of  $pr^A$ ,  $r_R$ 's read in  $pr^C$  returns 1. Thus, in  $pr^C$ ,  $r_1$ 's second read returns  $\perp$  and follows  $r_R$ 's read which returns 1. Clearly, partial run  $pr^C$  violates atomicity.  $\square$

**6. Arbitrary Failure Model.** In this section we consider fast implementations that tolerate arbitrary failures of servers and readers, but the writer can fail only by crashing (intuitively, in our single-writer setting, arbitrarily faulty writer could make the shared data structure useless). An arbitrary failure can either correspond to a crash or a malicious behavior. A process is malicious if it deviates from the algorithm assigned to it in a way that is different from simply stopping all activities (crashing). We allow for *any* number of arbitrarily faulty readers and distinguish two resilience thresholds for server failures:  $b$  and  $t$  [19]. Just as in the crash-stop model, a maximum number of  $t$  servers can fail. However, out of these  $t$  servers, at most  $b$  can be malicious. We therefore always have  $b \leq t$ . In the literature, the special case where  $b = t$  is usually considered. However, by considering  $b$  and  $t$  separately we can directly generalize the results in the previous sections (i.e. with  $b = 0$ ).

In our arbitrary failure model, malicious processes can deviate arbitrarily from automata assigned to them, with the restriction with respect to creation of digital signatures of other processes as detailed below. In the remainder of this Section, we say that a process is faulty if it fails by crashing or if it is malicious (otherwise, a process is called correct). We also say that a process is non-malicious if it is correct or fails by crashing.

In this paper, we assume that a process can produce cryptographic digital signatures (e.g., [28]). The functionality of the digital signature scheme provides two operations:  $\sigma$  for signing and  $Ver$  for signature verification. The invocation of  $\sigma$  takes a process ID, say  $p$  and a bit string  $m$  as parameters and returns a bit string  $sig$ , called *signature*. The verification operation  $Ver$  takes a process ID  $p$ , and two bit strings  $m$  and  $sig$  as parameters and returns a boolean. The verification function has the property that  $Ver(p, m, sig)$  invoked by a benign process evaluates to *true* if and only if process  $p$  executed  $\sigma(p, m)$  in some previous step. Furthermore, no process (including Byzantine ones) other than  $p$  may invoke  $\sigma(p, m)$  (we say signatures are *unforgeable*); hence, alternatively, we also write  $\sigma(p, m)$  as  $\sigma_p(m)$ .

Finally, given that we allow for arbitrarily faulty readers, we require atomicity to hold in a given run only on the subset of non-malicious read/write operations invoked by non-malicious clients (i.e., when read invocations/responses at malicious readers are removed from a run). For simplicity of presentation, we refer to an operation invoked by a non-malicious client as to a *non-malicious operation*.

**6.1. A Fast Implementation.** We describe in this section a fast implementation in the arbitrary failure model assuming  $S > (R+2)t + (R+1)b$  which is equivalent to  $R < \frac{S+b}{t+b} - 2$  (Figure 6.1). The algorithm is similar to the one presented in Section 4 except for a few key differences. First of all, the writer digitally signs each value it sends to servers. Apart from the addition of digital signatures, the write mechanism is unchanged and the writer waits for the response of  $S - t$  servers. Server code related

---

0: **at the writer**  $w$ :

1: **procedure** initialization:

2:    $ts \leftarrow 0$

3: **procedure** write()

4:    $ts \leftarrow ts + 1$

5:   send (WRITE,  $ts$ ,  $\sigma_w(ts)$ ) to all servers

6:   **wait until** receive (WRITEACK,  $ts$ ) from  $S - t$  servers

7:   return(OK)

---

**at each reader**  $r_i$ :

8:  $admissible(TS, Msg, a) \equiv \exists \mu \subseteq Msg, \forall m \in \mu :$   
 $(m.ts = TS) \wedge (|\mu| \geq S - at - (a - 1)b) \wedge (|\bigcap_{m' \in \mu} m'.updated| \geq a)$

9: **procedure** initialization:

10:    $maxTS \leftarrow 0$ ;  $sig \leftarrow \perp$

11: **procedure** read()

12:   send(READ,  $maxTS$ ,  $sig$ ) to all servers

13:   **wait until** receive (READACK,  $ts'$ ,  $sig'$ ,  $updated'$ ) from  $S - t$  servers, such that:  
 $ts' \geq maxTS$  **and**  $r_i \in updated'$  **and**  $Ver(w, ts', sig')$

14:    $rcvMsg \leftarrow \{m | r_i \text{ received (READACK, *, *, *) in line 13}\}$

15:    $maxTS \leftarrow \mathbf{Max}\{ts' \mid (READACK, ts', *, *) \in rcvMsg\}$

16:    $sig \leftarrow sig_{maxTS} : (READACK, maxTS, sig_{maxTS}, *) \in rcvMsg$

17:   **if** there is  $a \in [1, R + 1]$ :  $admissible(maxTS, rcvMsg, a)$  **then**

18:     return( $maxTS$ )

19:   **else**

20:     return( $maxTS - 1$ )

---

**at each server**  $s_i$ :

21: **procedure** initialization:

22:    $ts_i \leftarrow 0$ ;  $updated_i \leftarrow \emptyset$ ;  $sig_i \leftarrow \perp$

23: **procedure** update( $ts$ ,  $sig$ ,  $c$ )

24:   **if**  $ts > ts_i$  **then**

25:      $ts_i \leftarrow ts$ ;  $sig_i \leftarrow sig$ ;  $updated_i \leftarrow \{c\}$

26:   **else**

27:      $updated_i \leftarrow updated_i \cup \{c\}$

28: **upon** receive(WRITE,  $ts$ ,  $sig$ ) from writer  $w$  **and**  $Ver(w, ts, sig)$  **do**

29:   **update**( $ts$ ,  $sig$ ,  $w$ )

30:   send(WRITEACK,  $ts_i$ ) to  $w$

31: **upon** receive(READ,  $ts$ ,  $sig$ ) from reader  $r_j$  **and**  $Ver(w, ts, sig)$  **do**

32:   **update**( $ts$ ,  $sig$ ,  $r_j$ )

33:   send(READACK,  $ts_i$ ,  $sig_i$ ,  $updated_i$ ) to  $r_j$

---

FIG. 6.1. Fast atomic storage implementation with  $S \geq (R + 2)t + (R + 1)b + 1$

to write operation is also unchanged, except that servers store the digital signature of every value they store in addition to the value itself.

Our read procedure begins with servers issuing a READ message containing the highest timestamp encountered in the previous read invocation (line 12) along with the respective signature of the writer. In a way, the reader writes back this timestamp, signed by the writer, to all servers. During the first read invocation, the reader issues a read message with the default timestamp 0, which is also the initial timestamp at servers and writer. We assume that this initial value is known to all readers (and hence, needs not be digitally signed by the writer). Then, the reader collects responses from  $S - t$  servers containing the latest timestamps encountered by the servers (including the one being written back by the reader); all the timestamps need to be accompanied with the verifiable signature of the writer (line 13). The reader then selects the highest such timestamp,  $maxTS$  (line 15). Moreover, the reader stores the corresponding writer's signature into variable  $sig$  (line 16). The pair  $(maxTS, sig)$  will be written back by the reader in its next read invocation (lines 12, 23-27 and 31-33).

Apart from using digital signatures as described above, the mechanism of the read procedure is very similar to our crash-tolerant algorithm of Figure 4.1 (Sec. 4.1). The additional difference is related to predicate *admissible* in line 8, which checks if the latest value has been seen by a sufficient number of servers, and which is slightly modified. Namely, in order for  $maxTS$  (i.e., the highest timestamp received in a read) to be admissible with degree  $a$ ,  $maxTS$  must have been reported to the reader by at least  $S - at - (a - 1)b$  servers. This is to be contrasted with at least  $S - at$  reports needed in the crash-only case (notice here that the number of servers  $S$  is not identical in the crash-only and the arbitrary failure cases, being higher in the latter). To see why our algorithm requires  $S - at - (a - 1)b$  confirmations from different servers, consider the case of a write with timestamp  $ts$  that is followed by a read. In the first partial run  $pr_1$ , the write completes by writing  $ts$  at  $S - t$  servers, out of which at least  $S - t - b$  are non-malicious; denote this set of servers by  $S_1$ . Subsequently, a reader reads from a set  $S_2$  (of  $S - t$  servers) that overlaps in  $S - 2t - b$  (non-malicious) servers with  $S_1$ , i.e., the reader misses  $t$  servers in  $S_1$ . By atomicity, the read returns  $ts$ . In the second partial run  $pr_2$ , with a failure pattern different from  $pr_1$ , the write is incomplete and the writer writes  $ts$  only to  $S - 2t - b$  servers (possibly malicious) in  $S_1 \cap S_2$ . A subsequent reader that reads from  $S_2$  cannot distinguish  $pr_1$  from  $pr_2$ , and returns  $ts$ . If we extend each partial run with another read by a distinct reader that misses  $t$  servers from  $S_1 \cap S_2$ , and accounting for the possibility that another  $b$  servers are malicious, it is easy to see that the new read has to return  $ts$ , even if it sees  $ts$  at  $S - 3t - 2b$  servers that have already replied to both the write and the first read. This can be extrapolated further depending on the number of the readers in the system. Hence the need for as few as  $S - at - (a - 1)b$  different confirmations for a timestamp to be admissible (with degree  $a$ ).

We now prove the correctness of the fast implementation depicted in Figure 6.1.

**6.2. Correctness of the Fast Implementation.** The skeleton of the proof follows the proof of our crash-tolerant algorithm (Sec. 4.2); differences in two proofs account for counteracting possible actions of malicious processes, the use of digital signatures and modifications of predicate *admissible*. In the following, we omit correctness proofs of those lemmas that can be trivially obtained from the proof of their counterparts from Section 4.2, by inserting the attribute “non-malicious” before every occurrence of “server”, “reader” and “read” and by replacing every reference to

Lemma/Corollary 4.x, by Lemma/Corollary 6.x. However, for completeness, we repeat the statement of each of the lemmas in the arbitrary failure context. In our proof, we maintain the assumption that non-malicious servers ignore old reads as explained in Sec. 4.1.

First we modify some notation of the Definition 4.1 as follows (other notation from Def. 4.1 remains):

DEFINITION 6.1.

- $\mu_{op,a}$  denotes, in case  $maxTS_{op}$  is admissible with degree  $a$  in  $op$ , the subset of  $rcvMsg_{op}$ , such that (see line 8, Fig. 6.1):
  - (a)  $|\mu_{op,a}| \geq S - at - (a - 1)b$ ,
  - (b)  $\forall m \in \mu_{op,a}, m.ts = maxTS_{op}$  and
  - (c)  $|\bigcap_{m \in \mu_{op,a}} m.updated| \geq a$ .

LEMMA 6.2. *If a non-malicious server  $s_i$  sets its local variable  $ts_i$  to  $x$  at time  $T$ , then  $s_i$  never sets  $ts_i$  to a value that is lower than  $x$  after time  $T$ .*

LEMMA 6.3. *A read operation  $rd$  may only return either  $maxTS_{rd}$  or  $maxTS_{rd} - 1$ .*

LEMMA 6.4. *A non-malicious read  $rd$  cannot return a value smaller than  $maxTS_{rd}^{old}$ .*

*Proof.* By line 24, every READACK message received by a non-malicious reader in  $rd$  from a non-malicious server is with a timestamp at least  $maxTS_{rd}^{old}$ . The reader awaits for  $S - t$  READACK messages before returning a value. Moreover, reader discards all READACK messages that have a timestamp less than  $maxTS_{rd}^{old}$  (line 13), as those READACK messages are clearly from malicious servers. Eventually, since we assume at most  $t$  server failures,  $rd$  receives READACK messages from  $S - t$  non-malicious servers that satisfy conditions in line 13. Clearly,  $maxTS_{rd} \geq maxTS_{rd}^{old}$ . There are the following two cases to consider. (1) If  $maxTS_{rd} > maxTS_{rd}^{old}$ , then, by Lemma 6.3, the return value is not smaller than  $maxTS_{rd}^{old}$ . (2) If  $maxTS_{rd} = maxTS_{rd}^{old}$ , then every READACK message in  $rcvMsg_{rd}$ , at least  $S - t$  of them, has timestamp equal to  $maxTS_{rd}^{old}$  and has  $r_i \in updated$  (possibly different READACK messages from malicious servers are discarded in line 13). Hence, predicate *admissible* holds for  $maxTS_{rd}$  with degree  $a = 1$  and  $rd$  returns  $maxTS_{rd}^{old}$ .  $\square$

LEMMA 6.5. **(SWA1)** *If a non-malicious read returns, it returns a non-negative timestamp.*

*Proof.* To prove the lemma, it is sufficient to show that there is no non-malicious read  $rd$  in which  $maxTS_{rd}^{old} < 0$  (then, the lemma follows from Lemma 4.4).

To see this, assume by contradiction that there is a read  $rd$  by non-malicious reader  $r_i$  in which  $maxTS_{rd}^{old} < 0$ . Moreover, without loss of generality, we can fix  $rd$  such that there is no read  $rd'$  by  $r_i$  such that  $rd'$  precedes  $rd$  and  $maxTS_{rd'}^{old} < 0$ . By lines 9-10, this  $rd$  is not the first read by  $r_i$ , i.e., there is a read  $rd'$  by  $r_i$  that (immediately) precedes  $r_i$  such that  $maxTS_{rd'} < 0$  and  $maxTS_{rd'}^{old} \geq 0$ . However, this contradicts the condition in line 13 that requires  $maxTS_{rd'} \geq maxTS_{rd'}^{old}$ .  $\square$

LEMMA 6.6. **“read-write” atomicity (SWA2).** *If a non-malicious read  $rd$  returns timestamp  $l$  and  $rd$  follows write  $wr_k$ , then  $l \geq k$ .*

*Proof.* Denote by  $r_i$  the non-malicious reader that invoked  $rd$  and let  $\Sigma' = \Sigma_{wr_k} \cap \Sigma_{rd}$  and let  $\Sigma^{NM}$  be the subset of  $\Sigma'$  that contains only non-malicious (NM) servers. Obviously,  $|\Sigma^{NM}| \geq S - 2t - b$ .



When a non-malicious server  $s_j$  in  $\Sigma_{wr_k}$  (and, hence, in  $\Sigma^{NM}$ ) replies to a WRITE message from  $wr_k$ , its  $ts_j$  is at least  $k$  (the timestamp is not smaller than  $k$  due to the condition in line 24). Since  $wr_k$  precedes  $rd$ , by Lemma 6.2, servers in  $\Sigma'$  reply with  $ts_* \geq k$  to  $rd$ . Hence,  $maxTS_{rd} \geq k$ . There are the following two cases to consider:

1.  $maxTS_{rd} > k$

By Lemma 6.3,  $rd$  does not return a timestamp lower than  $k$ .

2.  $maxTS_{rd} = k$

Let  $\mu^{NM}$  be the set of READACK messages sent by servers in  $\Sigma^{NM}$  to  $rd$ . By definition of  $\Sigma_{rd}$  and since  $\Sigma^{NM} \subseteq \Sigma_{rd}$ , we have  $\mu^{NM} \subseteq rcvMsg_{rd}$ . Since (a) every server  $s_j \in \Sigma^{NM}$  replies to  $rd$  with  $ts_j \geq k$ , (b)  $\mu^{NM} \subseteq rcvMsg_{rd}$  and (c)  $maxTS_{rd} = k$ , we have that every server  $s_j \in \Sigma^{NM}$  replies with  $ts_j = k$  to  $rd$  (and  $r_j$  receives these replies).

Moreover, since every server  $s_j \in \Sigma^{NM}$  replies  $ts = k$  to  $wr_k$  (since  $\Sigma^{NM} \subseteq \Sigma_{wr_k}$ ) before sending  $ts = k$  to  $rd$  (since  $wr_k$  precedes  $rd$ ), for every message  $m$  in  $\mu^{NM}$ ,  $w \in m.updated$ . Furthermore, since  $s_j$  replies with  $ts_j = k$  to  $rd$ , by line 27,  $r_i \in m.updated$ . Thus,  $\{w, r_i\} \subseteq \bigcap_{m \in \mu'} m.updated$ . As  $|\Sigma^{NM}| \geq S - 2t - b$ ,  $maxTS_{rd}$  is admissible in  $rd$  with degree  $a = 2$ . Hence,  $rd$  returns  $maxTS_{rd} = k$ .

□

LEMMA 6.7. *If  $maxTS_{rd} \geq k$  in a non-malicious read  $rd$ , then  $rd$  does not precede  $wr_k$ .*

*Proof.* In case  $k = 0$ , the lemma follows directly from the definition of  $wr_0$ . In case  $k \geq 1$ , the proof follows directly from the unforgeability of writer's signatures and the fact that the writer does not sign any value greater or equal to  $k$  before it invokes  $wr_k$ . Hence, no timestamp  $k' \geq k$  can pass the signature verification check in line 13 before  $wr_k$  is invoked. □

COROLLARY 6.8. *If  $maxTS_{rd} = k \geq 1$  in a non-malicious read, then write  $wr_{k-1}$  completes before read  $rd$  completes.*

LEMMA 6.9. **(SWA3)** *If a non-malicious read  $rd$  returns timestamp  $k$  ( $k \geq 1$ ), then  $rd$  does not precede  $wr_k$ .*

LEMMA 6.10. *If  $maxTS_{rd}$  is admissible with degree  $a$  in non-malicious read  $rd$ , then  $\Sigma_{\mu_{rd,a}}$  contains at least  $t+1$  non-malicious servers.*

*Proof.* By Definitions 4.1 and 6.1, and inequalities  $a \leq R + 1$  and  $R < \frac{S+b}{t+b} - 2$ , we have  $|\Sigma_{\mu_{rd,a}}| \geq S - at - (a-1)b > (R+2)t + (R+1)b - (R+1)t - Rb > t+b$ . Since we assume at most  $b$  malicious servers,  $\Sigma_{\mu_{rd,a}}$  contains at least  $t+1$  non-malicious servers. □

LEMMA 6.11. *Assume that  $maxTS_{rd}$  is admissible with degree  $a \in [1, R+1]$  in some non-malicious read  $rd$  and that a complete non-malicious read  $rd'$  follows  $rd$ . Then, the number of non-malicious servers in  $\Sigma_{\mu_{rd,a}} \cap \Sigma_{rd'}$  is at least  $S - (a+1)t - ab$ . Moreover,  $\Sigma_{\mu_{rd,a}} \cap \Sigma_{rd'}$  contains at least one non-malicious server.*

*Proof.* Since  $|\Sigma_{\mu_{rd,a}}| = |\mu_{rd,a}| \geq S - at - (a-1)b$  and  $|\Sigma_{rd'}| = S - t$ , it follows that  $|\Sigma_{rd'} \cap \Sigma_{\mu_{rd,a}}| \geq S - (a+1)t - (a-1)b$ . Furthermore, since at most  $b$  servers are malicious,  $\Sigma_{\mu_{rd,a}} \cap \Sigma_{rd'}$  contains at least  $S - (a+1)t - ab$  servers.

Moreover, since  $a \in [1, R+1]$  and  $R < \frac{S+b}{t+b} - 2$ , we have  $S - (a+1)t - ab \geq 1$ . □

LEMMA 6.12. *Assume that:*

1.  $maxTS_{rd}$  is admissible with degree  $a \in [1, R+1]$  in some non-malicious read  $rd$ ,

2. a complete non-malicious read  $rd'$  by reader  $r_j$  follows  $rd$ ,
3. there is a set  $X \subseteq \Sigma_{\mu_{rd,a}}$  of at least  $t+1$  non-malicious servers, such that for

all  $s_i \in X$ ,  $s_i$  sends message  $m_i \in \mu_{rd,a}$  with  $r_j \in m_i.updated$ .

Then,  $rd'$  does not return a value smaller than  $\max TS_{rd}$ .

LEMMA 6.13. (**“read-read” atomicity (SWA4)**). *If non-malicious reads  $rd_1$  and  $rd_2$  return timestamps  $ret_{rd_1}$  and  $ret_{rd_2}$ , respectively, and if  $rd_2$  follows  $rd_1$ , then  $ret_{rd_2} \geq ret_{rd_1}$ .*

THEOREM 6.14. *The algorithm of Figure 6.1 is a fast implementation of an atomic SWMR register in the arbitrary failure model.*

*Proof.* Atomicity follows from Lemmas 6.5, 6.6, 6.9 and 6.13. Moreover, it is obvious that our implementation satisfies Termination — conditions in lines 6 and 13 are non-blocking since we assume at least  $S-t$  correct (and non-malicious) servers. Finally, our implementation is fast: all operations involve a single communication round-trip between a client and servers.  $\square$

**6.3. Optimality.** The following proposition states that the resilience required by our fast implementation is indeed necessary.

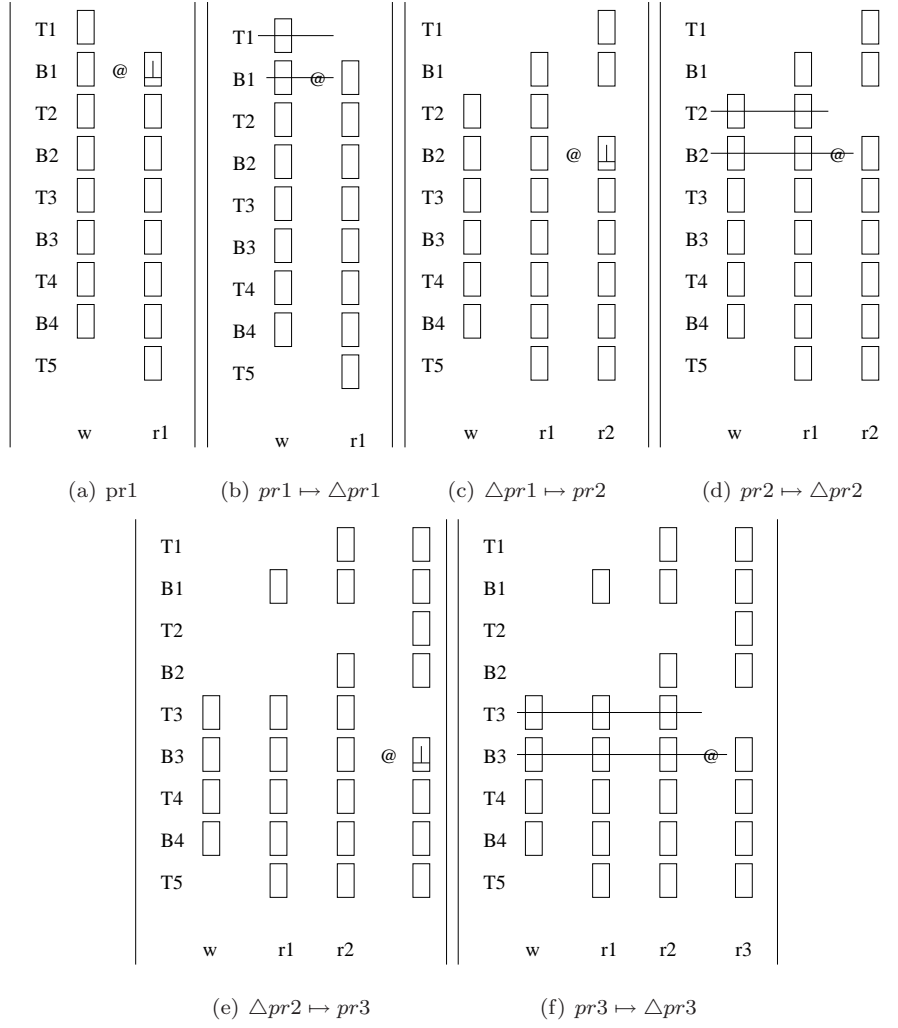
PROPOSITION 6.15. *Let  $t \geq 1$ ,  $b \geq 0$  and  $R \geq 2$ . If  $(R+2)t + (R+1)b \geq S$ , then there is no fast atomic register implementation.*

This proof is similar to the one in Section 5: we suppose by contradiction that  $(R+2)t + (R+1)b \geq S$  and that there is a fast implementation  $I$  of an atomic register (even if  $I$  makes use of digital signatures). We construct a partial run of the fast implementation  $I$  that violates atomicity: a partial run in which some **read** returns 1 and a subsequent **read** returns an older value, namely, the initial value of the register,  $\perp$ . This run is different from the one in the previous proof.

*Proof.* Given that  $(R+2)t + (R+1)b \geq S$ , we can partition the set of servers into  $2R+3$  subsets, which we call *blocks*, denoted by  $T_i$  ( $1 \leq i \leq R+2$ ) and  $B_j$  ( $1 \leq j \leq R+1$ ), such that each of the blocks  $T_i$  (resp.,  $B_j$ ) is of size less than or equal to  $t$  (resp.,  $b$ ). We illustrate a particular instance of the proof in Figure 6.2 and Figure 6.3, where  $R=3$  and the set of servers are partitioned into nine blocks,  $T_1$  to  $T_5$  and  $B_1$  to  $B_4$ . In these figures, we denote the arbitrary failure of  $B_i$  by  $\textcircled{a}$ .

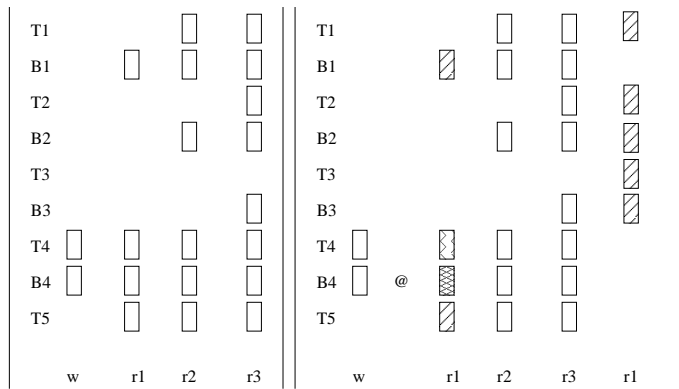
*Partial writes.* Consider a partial run  $wr$  in which  $w$  completes **write**(1). The operation skips  $T_{R+2}$ . We define a series of partial executions each of which can be extended to  $wr$ . Let  $wr_{R+2}$  be the partial run in which  $w$  has invoked the **write** and has sent the **WRITE** message to all processes, and all **WRITE** messages are in transit. For  $1 \leq i \leq R+1$ , we define  $wr_i$  as the partial run which contains an incomplete operation **write**(1) that skips  $\{T_{R+2}\} \cup \{T_j | 1 \leq j \leq i-1\} \cup \{B_j | 1 \leq j \leq i-1\}$ . We make the following simple observations: (1) for  $1 \leq i \leq R$ ,  $wr_i$  and  $wr_{i+1}$  differ only at servers in  $T_i \cup B_i$ , (2)  $wr$  is an extension of  $wr_1$ , such that, in  $wr$ ,  $w$  receives the replies (that are in transit in  $wr_1$ ) and the **write** completes, and hence, (3)  $wr$  and  $wr_1$  differ only at  $w$ .

*Appending reads.* Partial run  $pr_1$  extends  $wr$  by having block  $B_1$  failing upon completion of **write**(1) and appending a complete **read** by  $r_1$  that skips block  $T_1$  (Fig. 6.2(a)).  $B_1$  fails in such a way that it behaves as if it never received any **write** message (i.e., a message from operation **write**(1)). We say that  $B_1$  fails and loses its memory. Observe that  $r_1$  cannot distinguish  $pr_1$  from some partial run  $\Delta pr_1$ , that extends  $wr_2$  by appending a complete **read** by  $r_1$  that skips  $T_1$ . To see why, notice that (a)  $wr$  and  $wr_2$  differ at  $w$  and at blocks  $T_1$  and  $B_1$ , (b)  $r_1$  does not receive any message from writer

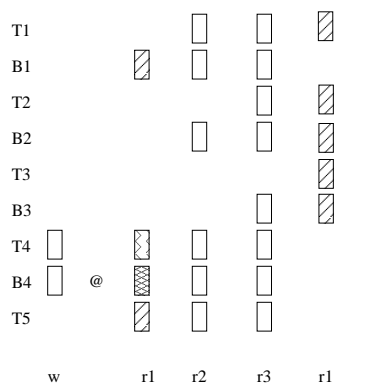
FIG. 6.2. Partial runs  $pr_i$  and  $\Delta pr_i$ 

$w$  and block  $T_1$  in both executions and (c)  $r_1$  received the same message from block  $B_1$  in both executions. By wait-freedom property and since  $w$  can fail by crashing,  $r_1$ 's read in  $\Delta pr_1$  must return some value  $x$ , since it cannot wait for the completion of the writer's invocation, nor a message from  $w$ . Since  $r_1$  cannot distinguish  $\Delta pr_1$  from  $pr_1$ ,  $r_1$  returns the same value  $x$  in  $pr_1$  as well, and by atomicity, in  $pr_1$ ,  $x$  must equal 1. Therefore, in  $\Delta pr_1$ ,  $r_1$  also returns 1.

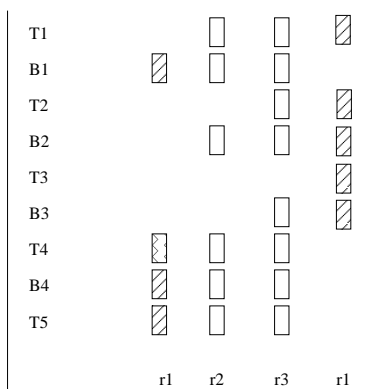
Starting from  $\Delta pr_1$ , we iteratively define the following partial executions for  $2 \leq i \leq R$ . Partial run  $pr_i$  extends  $\Delta pr_{i-1}$  by: (1) block  $B_i$  failing in such a way that it behaves as if it never received any message (loses memory) and (2) appending a complete read by  $r_i$  that skips  $T_i$ . Partial run  $\Delta pr_i$  is constructed by deleting from  $pr_i$  all steps of the servers in block  $T_i$  and all steps of servers in block  $B_i$  up to the instant in which  $B_i$  lost its memory (including that particular step). Since the last read in  $pr_i$  by reader  $r_i$  skips block  $T_i$ ,  $r_i$  cannot distinguish  $pr_i$  from  $\Delta pr_i$ , as in both executions  $r_i$  receives the same messages from  $B_i$ . More precisely, partial run  $\Delta pr_i$



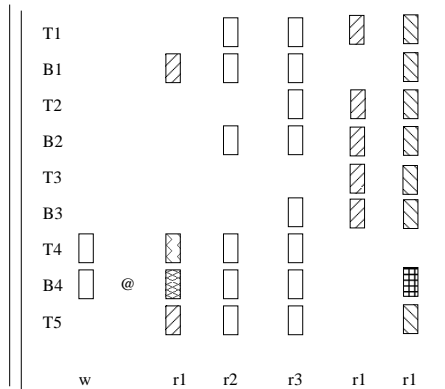
(a)  $\Delta pr^3$



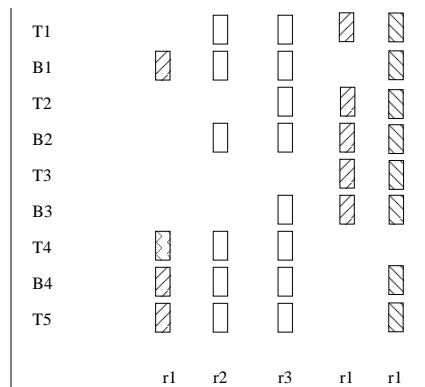
(b)  $pr^A$



(c)  $pr^B$



(d)  $pr^C$



(e)  $pr^D$

- non-malicious block that replies to the first read of r1, but r1 does not receive replies
- malicious block that replies to the first read of r1 and r1 receives the replies
- non-malicious block that replies to the first read of r1 and r1 receives the replies
- malicious block that replies to the second read of r1 and r1 receives the replies
- non-malicious block that replies to the second read of r1 and r1 receives the replies

(f) Legend

FIG. 6.3. Partial executions:  $pr^A$ ,  $pr^B$ ,  $pr^C$  and  $pr^D$

extends  $wr_{i+1}$  by appending the following  $i$  reads one after the other: for  $1 \leq h \leq i-1$ ,  $r_h$  does a read that skips  $\{T_j | h \leq j \leq i\} \cup \{B_j | h+1 \leq j \leq i\}$  and  $r_i$  does a (complete) read that skips  $T_i$ . Here, the first  $i-1$  appended reads are incomplete whereas the last one is complete. Figure 6.2 depicts block diagrams of  $pr_i$  and  $\Delta pr_i$  with  $R=3$ . (The deletion of steps to obtain  $\Delta pr_i$  from  $pr_i$  is shown by crossing out the rectangles corresponding to the deleted steps.)

Reader  $r_1$ 's read in  $\Delta pr_1$  returns 1. By wait-freedom, in  $\Delta pr_2$   $r_2$  must return some value, say  $x_2$ . However, since  $r_2$  cannot distinguish  $pr_2$  from  $\Delta pr_2$ ,  $r_2$  must return a value  $x_2$  in  $pr_2$  as well. Since  $pr_2$  extends  $\Delta pr_1$ , by atomicity,  $r_2$ 's read in  $pr_2$  must return  $x_2 = 1$ . Therefore,  $r_2$ 's read in  $\Delta pr_2$  returns 1. In general, since  $pr_i$  extends  $\Delta pr_{i-1}$ , and  $r_i$  cannot distinguish  $pr_i$  from  $\Delta pr_i$  (for all  $i$  such that  $2 \leq i \leq R$ ), in which it must return a value, it follows by trivial induction that  $r_i$ 's read in  $\Delta pr_i$  returns 1. In particular,  $r_R$  reads 1 in  $\Delta pr_R$ . Moreover, note that in  $\Delta pr_R$  no object is faulty.

*Partial run  $pr^A$ .* Consider again partial run  $\Delta pr_R$ , i.e., partial run  $wr_{R+1}$  extended by appending  $R$  reads by each reader  $r_h$  ( $1 \leq h \leq R-1$ ) such that  $r_h$ 's read skips  $\{T_j | h \leq j \leq R-1\} \cup \{B_j | h+1 \leq j \leq R-1\}$ , whereas a read by reader  $r_R$  skips  $T_R$  only. The read by  $r_1$  is incomplete in  $\Delta pr_R$ : only servers in  $B_1, T_{R+1}, B_{R+1}$  and  $T_{R+2}$  send replies to  $r_1$ , and those reply messages are in transit. Observe that, in  $\Delta pr_R$ , only servers in  $T_{R+1}$  and  $B_{R+1}$  receive the WRITE message from  $\text{write}(1)$ . Consider the following partial run  $pr^A$  which differs from  $\Delta pr_R$  in the following:

1. Upon reception of message from  $\text{write}(1)$  operation,  $B_{R+1}$  fails arbitrarily in such a way that, from that point on, it sends replies to all processes but  $r_1$  as if it was not faulty, and to  $r_1$  as if it never received a  $\text{write}(1)$  message. Moreover, after completion of read by  $r_R$ ,
2.  $r_1$  receives the READACK messages from  $T_{R+2}$  and  $B_1$  (that were in transit in  $\Delta pr_R$ ) and  $B_{R+1}$  (i.e., from the Byzantine faulty objects),
3. servers in  $T_1$  to  $T_R$  and  $B_2$  to  $B_R$  receive the READ message from  $r_1$  (that were in transit in  $\Delta pr_R$ ) and reply to  $r_1$ , and
4. reader  $r_1$  receives these replies from servers in  $T_1$  to  $T_R$  and  $B_2$  to  $B_R$ , and then  $r_1$  returns from the read invocation.

Notice that  $r_1$  received replies from all blocks but  $T_{R+1}$ , and so, must return from the read; however,  $r_1$  does not receive the replies from servers in  $T_{R+1}$ , i.e., from the only benign servers whose state was modified by  $\text{write}(1)$ .

*Partial run  $pr^B$ .* Consider another partial run  $pr^B$  with the same communication pattern as  $pr^A$ , except that  $\text{write}(1)$  is not invoked at all and block  $B_{R+1}$  is not faulty. Hence, servers in  $T_{R+1}$  do not receive any WRITE message (Figure 6.3). Clearly, only servers in  $T_{R+1}, B_{R+1}$ , the writer, and the readers  $r_2$  to  $r_R$  can distinguish  $pr^A$  from  $pr^B$ . Reader  $r_1$  cannot distinguish the two partial executions because it does not receive any message from the servers in  $T_{R+1}$ , the writer, or other readers and it receives the same message from the servers in  $B_{R+1}$  in both executions. By atomicity,  $r_1$ 's read returns (the initial storage value)  $\perp$  in  $pr^B$  because there is no  $\text{write}(\ast)$  invocation in  $pr^B$ , and hence,  $r_1$ 's read returns  $\perp$  in  $pr^A$  as well.

*Partial executions  $pr^C$  and  $pr^D$ .* Notice that, in  $pr^A$ , even though  $r_1$ 's read returns  $\perp$  after  $r_R$ 's read returns 1,  $pr^A$  does not violate atomicity, because the two reads are concurrent. We construct two more partial executions: (1)  $pr^C$  is constructed by

extending  $pr^A$  with another complete read by  $r_1$ , which skips  $T_{R+1}$  (as in  $pr^A$ , in  $pr^C$   $B_{R+1}$  always replies to  $r_1$  as if it never received any WRITE message), and (2)  $pr^D$  is constructed by extending  $pr^B$  with another complete read by  $r_1$ , which skips  $T_{R+1}$  (Figure 6.3). Since  $r_1$  cannot distinguish  $pr^A$  from  $pr^B$ , and  $r_1$ 's second read skips  $T_{R+1}$  (i.e., servers which can distinguish  $pr^A$  from  $pr^B$ ), it follows that  $r_1$  cannot distinguish  $pr^C$  from  $pr^D$  as well. Since there is no  $write(*)$  invocation in  $pr^D$ ,  $r_1$ 's second read returns  $\perp$  in  $pr^D$ , and hence,  $r_1$ 's second read in  $pr^C$  returns  $\perp$ . Since  $pr^C$  is an extension of  $pr^A$ ,  $r_R$ 's read in  $pr^C$  returns 1. Thus, in  $pr^C$ ,  $r_1$ 's second read returns  $\perp$  and follows  $r_R$ 's read which returns 1. Clearly, partial run  $pr^C$  violates atomicity.  $\square$

**7. Multiple Writers.** In the impossibility proof below, we use two simple properties of MWMR atomic register which can be easily deduced from atomicity (see Section 3.1):

- (Property **P1**) in any partial run, if a write  $wr$  that writes  $v$ , precedes some read  $rd$ , and all other writes precede  $wr$ , then if  $rd$  returns, it returns  $v$ , and
- (Property **P2**) in any partial run, if there are two reads such that all writes precede both reads, then the reads do not return different values.

The proposition below states that there cannot exist a fast multi-writer atomic register implementation (in the following,  $W$  denotes the number of writers). The proof is written for the crash-stop model. But by extension the impossibility directly applies to the arbitrary failure model.

**PROPOSITION 7.1.** *Let  $t \geq 1$ ,  $R \geq 2$  and  $W \geq 2$ . Any atomic register implementation has a run in which some complete read or write is not fast.*

*Proof.* It is sufficient to show the impossibility in a system where  $W = R = 2$ , and  $t = 1$ . Let the writers be  $w_1$  and  $w_2$ , and the readers be  $r_1$  and  $r_2$ . Let  $s_1$  to  $s_S$  be the servers. Suppose by contradiction that there is a fast implementation of an atomic register in this system. To show the desired contradiction, we construct a series of runs, each consisting of two writes followed by a read.

Since the writer, any number of readers, and up to  $t$  servers might crash in our model, the invoking process can only wait for reply messages from  $S - t$  servers. Given that we assume a fast implementation, on receiving a READ (or a WRITE) message, the servers cannot wait for messages from other processes, before replying to the READ (or the WRITE) message. We can thus construct partial runs of a fast implementation such that only READ (or WRITE) messages from the invoking processes to the servers, and the replies from servers to the invoking processes, are delivered in those partial runs. All other messages remain in transit. In particular, no server receives any message from other servers, and no invoking process receives any message from other invoking processes. In our proof, we only construct such partial runs.

We say that a complete operation  $op$  *skips* a server  $s_i$  in a partial run if every server distinct from  $s_i$  receives the READ or the WRITE message from  $op$  and replies to that message,  $op$  receives those replies and returns, and all other messages are in transit. In other words, only  $s_i$  does not receive READ or WRITE message from  $op$ . Since  $t = 1$ , any complete operation may skip at most one server. If a complete operation does not skip any servers, we say that the operation is *skip-free*.

Consider a partial run  $run_1$  constructed with the following three non-concurrent operations: (1) a skip-free write(2) by  $w_2$ , that precedes (2) a skip-free write(1) by  $w_1$ , that in turn precedes (3) a skip-free read() by  $r_1$ . From property P1, the read returns 1.

We now construct a similar partial run  $run_2$  in which the order of the two writes are interchanged: (1) a skip-free  $write(1)$  by  $w_1$ , that precedes (2) a skip-free  $write(2)$  by  $w_2$ , that in turn precedes (3) a skip-free  $read()$  by  $r_1$ . From property P1, the read returns 2.

Consider a series of partial runs  $run^i$ , where  $i$  varies from 1 to  $S + 1$ . We define  $run^1$  to be  $run_1$ . We iteratively define the remaining partial runs. We define  $run^{i+1}$  to be identical to  $run^i$  except in the following:  $s_i$  receives the WRITE message (and replies to that message) from  $w_1$  before the message from  $w_2$  (i.e., the replies of  $s_i$  are sent in the opposite order in  $run^{i+1}$  from that in  $run^i$ ). Since servers do not receive any message from other servers in the partial runs we construct, the only server that can distinguish  $run^i$  from  $run^{i+1}$  is  $s_i$ . Also  $w_1$ ,  $w_2$  and  $r_1$  can distinguish the two partial runs. It is easy to see that no server can distinguish  $run^{S+1}$  from  $run_2$ , and hence,  $r_1$  can not distinguish between the two runs as well. Thus  $r_1$  returns 2 in  $run^{S+1}$ , and  $run^{S+1}$  and  $run_2$  differ only at  $w_1$  and  $w_2$ . Since  $r_1$  returns 1 in  $run^1$ , 2 in  $run^{S+1}$ , and either 1 or 2 in  $run^i$  ( $2 \leq i \leq S$ ), there are two partial runs,  $run^{i1}$  and  $run^{i1+1}$ , such that  $1 \leq i1 \leq S$  and the read by  $r_1$  returns 1 in  $run^{i1}$  and returns 2 in  $run^{i1+1}$ .

Consider a partial run  $run'$  which extends  $run^{i1}$  with a read by  $r_2$  that skips  $s_{i1}$ . From property P2, it follows that  $r_2$  returns 1. Similarly we construct a partial run  $run''$  which extends  $run^{i1+1}$  with a read by  $r_2$  that skips  $s_{i1}$ . Recall that, only  $w_1$ ,  $w_2$ ,  $r_1$  and  $s_{i1}$  can distinguish  $run^{i1}$  from  $run^{i1+1}$ . Since  $r_2$  skips  $s_{i1}$  in both  $run'$  and  $run''$ ,  $r_2$  cannot distinguish the two partial runs. Thus  $r_2$  returns 1 in  $run''$ . However,  $r_1$  returns 2 in  $run^{i1+1}$ , and hence, in returns 2 in  $run''$  as well. Clearly,  $run^{i1+1}$  violates property P2.  $\square$

To see why the above proof does not apply to the single writer case, observe that in most partial runs in the above proof, the two writes are concurrent. However, in our system model, a process can invoke at most one operation at a time. Thus we cannot construct partial runs with concurrent writes in the single-writer case.

**8. Related Work.** A seminal SWMR crash-tolerant atomic register implementation assuming a majority of correct processes, known as ABD, was presented by Attiya, Bar-Noy and Dolev in [4]. In ABD, all write operations are fast; however, read operations always take two communication round trips between a client and servers. In this paper, we show that having fast read operation in a SWMR atomic implementation is possible, yet it comes with a somewhat steep price — a limited number of readers.

In [20, 10], ABD was extended to quorum system-based implementations of MWMR atomic register. In both these MWMR implementations, a read or write operation requires at least two rounds trips. In [23], Lynch and Shvartsman implement a MWMR register in a dynamic system, where processes can join or leave the set of servers implementing the register. However, even in executions where no process joins or leaves the set of server, a read or write operation in [23] requires at least two round-trips. Thus, the time-complexities of these implementations are consistent with our result on the impossibility of fast MWMR implementations when servers may fail.

Our results adapt the classical theorem “atomic reads must write”, stated in a shared-memory context [18, 5], to a message-passing context. In particular, to simulate a multi-reader atomic register from single-reader atomic registers, at least one of the readers must write into some single-reader register [5]. A similar result appears in the context of atomic register implementations over weaker *regular* ones

[18]. Namely, in such atomic register implementations, a process that reads a value  $v$  also needs to write it, in order to make sure that no other process will subsequently read an *older* value  $v'$ , which is possible when reading from regular registers.

Assuming a message-passing system, Fan and Lynch show [11] that every atomic read must modify the state of at least  $t$  servers, which might be interpreted as a need for a second communication round-trip. However, in such a system, any message received by a server can potentially modify the server's state. Hence, even in one round-trip, a read can modify at least  $S - t > t$  servers (assuming a majority of correct servers).

There is a prolific line of research in Byzantine fault tolerant atomic register implementations in message-passing systems, e.g., [24, 25, 6, 14, 3, 15], with a typical focus on providing optimal resilience (in our model, this amounts to  $S = 2t + b + 1$  servers [25]). The work of Malkhi and Reiter [24] casts the ABD algorithm to the MWMM Byzantine context, featuring both two round-trip writes and two round-trip reads, using writer's digital signatures, which we also use in the Byzantine-tolerant version of our implementation. A MWMM implementation by Martin et al. [25] introduces the "Listeners" pattern in which readers, roughly speaking, subscribe to updates from servers. Like in [25], in our implementation readers modify the servers' states, but receive no updates since this would, intuitively, violate the requirement for a fast implementation. Cachin and Tessaro propose in [6] a Byzantine-tolerant variant of Rabin's information dispersal algorithm [27] to minimize the storage blowup inherent to data replication. To this end, a MWMM implementation of [6] relies on communication among servers which, in a sense, prohibits fast operations. SWMM implementations that allow fast "best-case" read/write operations, i.e., operations that execute in synchronous periods, with few failures and no read/write concurrency, were presented in [14, 15]. In contrast, in this paper we consider the problem of allowing *all* operations to be fast while assuming the general, unrestricted asynchronous system. Not surprisingly, our limitations related to the number of readers are incurred by worst-case interleaving among different, concurrent operations, with roots in asynchrony and (possible) failures. Note that [14, 15] as well as the MWMM implementation of Aiyer et al. [3], renounce digital signatures. In this light, it is important to note that the existence of fast Byzantine-resilient atomic register implementations that do not rely on digital signatures remains an open problem.

After the appearance of the preliminary version of this work [8], several papers extended the notion of fast implementations. For SWMM implementations where the servers can only fail by crashing, Georgiu et al. propose in [13] how to circumvent our fast implementation lower bound ( $R < \frac{S}{t} - 2$ ) by permitting some reads that are not fast (called *slow* reads). More specifically, in the *semi-fast* implementation of [13], the readers are grouped into virtual nodes where readers in the same node possess the same virtual identifiers. Then, as long as there are at most  $\frac{S}{t} - 2$  virtual identifiers in the system (irrespective of the number of readers), most read operations are fast (and there is at most one slow read operation per write operation). In [12], the same authors investigate quorum system-based fast and semi-fast implementations. The paper shows that for *robust* quorum systems (i.e., quorum systems that remains available when one of the servers fails) and in presence of arbitrary number of readers, it is impossible to implement fast or semi-fast SWMM registers. The paper then presents a *weak-semifast* implementation that allows multiple slow read operations per write operation. In a recent work [9], Englert et al. investigate the possibility of MWMM implementations where most operations are fast, by assigning some additional



responsibilities at server for ordering operations.

Thus, our results in this paper have initiated a line of work that investigates the trade-off between the efficiency of atomic register implementations and the bounds on the number of readers and writers. This is not surprising since: a) atomic read/write registers are seen as a fundamental abstraction in building practical distributed storage and file systems (see e.g., [30, 29]) and b) our results demonstrate a fundamental limitation on the number of readers that an asynchronous fast SWMR atomic implementations can support, as well as the impossibility of fast MWMR atomic implementations. In a sense, the line of research that stems from our work seeks for practically applicable atomic register implementations by circumventing our results, while possibly allowing for some operations to be fast.

**9. Summary.** This paper establishes the exact conditions required for a fast implementation (an implementation in which all operations complete in a single round-trip) of an atomic read-write data structure, also called a register.

In the case of multiple writers, we proved that a fast implementation is impossible even if only one server can fail, and it can only do by crashing.

In the case of a single-writer where  $t$  out of  $S$  servers can fail by crashing, the number of readers must be smaller than  $S/t - 2$ . In the general arbitrary failure model, this number must be smaller than  $(S + b)/(t + b) - 2$  where up to  $b$  out of  $t$  servers can be malicious.

Finally, it would be interesting to look into optimal register implementations with respect to other complexity metrics (e.g., message complexity). This is left as future work.

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