Optimizing the Noisy MIMO Interference Channel at High SNR

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Abstract—Centralized algorithms for weighted sum rate (WSR) maximization for the K-user frequency-flat MIMO Interference Channel (MIMO IFC) with full channel state information (CSI) are considered. Maximization of WSR is desirable since it allows the system to cover all the rate tuples on the rate region boundary for a given MIMO IFC. First, we propose an iterative algorithm to design optimal linear transmitters and receivers. The transmitters and receivers are optimized to maximize the WSR of the MIMO IFC. Subsequently, we study the problem of WSR maximization in the High SNR regime. Starting from the High SNR approximation of the WSR we observe that the optimization problem in High SNR becomes, in a first instance, an exploration of the (discrete) pre-log region. Once the optimal pre-log distribution if found, for a given set of weights, the WSR optimization becomes the maximization of the High SNR Rate offset. To avoid the many local optima indicated by this analysis, the use of Deterministic Annealing in 1/SNR is suggested.

Index Terms—MIMO, MMSE, weighted sum rate, Interference Channel, linear transmitter, linear receiver, interference alignment, deterministic annealing

I. INTRODUCTION

To achieve higher system capacity in modern cellular communication standards a frequency reuse factor of 1 is used. This increment in system performances determines, on the other hand, a drastical reduction of the capacity of the cell-edge users due to the fact that this aggressive frequency reuse factor increases the inter-cell interference.

To handle this problem current communication systems include different interference management solutions. Even if interference coming from out-of-cell transmission can be reduced using careful planning these techniques are sometimes not enough to guarantee high performance to cell-edge user. For that major standardization bodies are now including explicit interference coordination strategies in next generation cellular communication standards. A systematic study of the performance of cellular communication systems where each cell communicates multiple streams to its users while enduring/causing interference from/to neighboring cells due to transmission over a common shared resource comes under the purview of MIMO interference channels (MIMO IFC). A K-user MIMO-IFC models a network of K transmit-receive pairs where each transmitter communicates multiple data streams to its respective receiver. In doing so, it generates interference at all other receivers. While the interference channel has been the focus of intense research over the past few decades, its

capacity in general remains an open problem and is not well understood even for simple cases. In [1] they show that even for the 2−users system, the most studied case, to achieve the system capacity within one bit very complicated transmission schemes are required.

Recently, it was shown that the concept of interference alignment (IA) [2], allows each receiver to suppress more interfering streams than it could otherwise cancel in interference channels. This can be done using more simple linear transmitter and receiver filter. This makes IA a very attractive solution in practical systems. The focus of this paper is on the K-user frequency-flat MIMO IFC. (In a frequency-flat MIMO IFC, the total number of streams contributing to the input signal at each receiver are, in general, greater than the number of antennas available at the transmitter or at the receiver. This would lead one to believe that, at least in the high-SNR regime, the network (comprising of K user pairs) performance can be maximized (i.e, the sum-rate can be maximized) using IA since aligning the streams at the transmitter will now allow the maximization of the capacity pre-log factor in a K-user IFC. A distributed algorithm that exploits the reciprocity of the MIMO IFC to obtain the transmit and receiver filters in a K-user MIMO IFC was proposed in [3]. It is was shown there that IA is a suboptimal strategy at finite SNRs. In the same paper, the authors propose a signal-to-interference-plus-noise-ratio (SINR) maximizing algorithm which outperforms the IA in finite SNRs and converges to the IA solution in the high SNR regime. However, this approach can be shown to be suboptimal for multiple stream transmission since it allocates equal power to all streams. Moreover, the convergence of this iterative algorithm has not been proved. Thus an optimal solution for MIMO IFC at finite SNR remains an open problem.
Some early work on the MIMO IFC was reported in [4] by Ye and Blum for the asymptotic cases when the interference to noise ratio (INR) is extremely small or extremely large. It was shown there that a "greedy approach" where each transmitter attempts to maximize its individual rate regardless of its effect on other un-intended receivers is provably suboptimal.There have been some attempts to port the solution concepts of the MIMO BC and MIMO MAC to the MIMO IFC. For instance, the problem of joint transmitter and receiver design to minimize the sum-MSE of a multiuser MIMO uplink was considered in [5] where iterative algorithms that jointly opti-

mize precoders and receivers were proposed. Subsequently [6] applied this algorithm to the MIMO IFC where each user transmits a single stream and a similar iterative algorithm to maximize the sum rate was proposed in [7].

II. SIGNAL MODEL
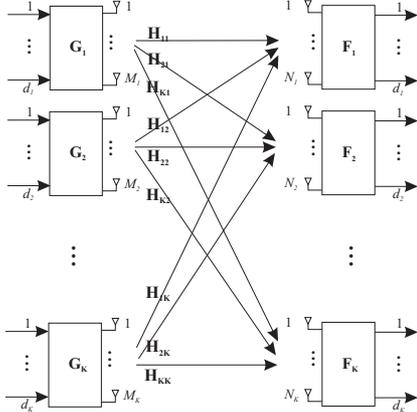
Fig. 1: MIMO Interference Channel

Fig. 1 depicts a K-user MIMO interference channel with K transmitter-receiver pairs. The k-th transmitter and its corresponding receiver are equipped with M_k and N_k antennas respectively. The k-th transmitter generates interference at all $l \neq k$ receivers. Assuming the communication channel to be frequency-flat, the $\mathbb{C}^{N_k \times 1}$ received signal $\mathbf{y}_k$ at the k-th receiver, can be represented as

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{x}_k + \sum_{\substack{l=1 \\ l \neq k}}^{K} \mathbf{H}_{kl}\mathbf{x}_l + \mathbf{n}_k \tag{1}$$

where $\mathbf{H}_{kl} \in \mathbb{C}^{N_k \times M_l}$ represents the channel matrix between the l-th transmitter and k-th receiver, $\mathbf{x}_k$ is the $\mathbb{C}^{M_k \times 1}$ transmit signal vector of the k-th transmitter and the $\mathbb{C}^{N_k \times 1}$ vector $\mathbf{n}_k$ represents (temporally white) AWGN with zero mean and covariance matrix $\mathbf{R}_{n_k n_k}$. Each entry of the channel matrix is a complex random variable drawn from a continuous distribution. It is assumed that each transmitter has complete knowledge of all channel matrices corresponding to its direct link and all the other cross-links in addition to the transmitter power constraints and the receiver noise covariances.

We denote by $\mathbf{G}_k$, the $\mathbb{C}^{M_k \times d_k}$ precoding matrix of the k-th transmitter. Thus $\mathbf{x}_k = \mathbf{G}_k\mathbf{s}_k$, where $\mathbf{s}_k$ is a $d_k \times 1$ vector representing the d_k independent symbol streams for the k-th user pair. We assume $\mathbf{s}_k$ to have a spatio-temporally white Gaussian distribution with zero mean and unit variance, $\mathbf{s}_k \sim \mathcal{N}(0, \mathbf{I}_{d_k})$. The k-th receiver applies $\mathbf{F}_k \in \mathbb{C}^{d_k \times N_k}$ to suppress interference and retrieve its d_k desired streams. The output of such a receive filter is then given by

$$\mathbf{r}_k = \mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k\mathbf{s}_k + \sum_{\substack{l=1 \\ l \neq k}}^{K} \mathbf{F}_k\mathbf{H}_{kl}\mathbf{G}_l\mathbf{s}_l + \mathbf{F}_k\mathbf{n}_k$$

Note that $\mathbf{F}_k$ does not represent the whole receiver but only the reduction from a N_k-dimensional received signal $\mathbf{y}_k$ to a d_k-dimensional signal $\mathbf{r}_k$, to which further (possibly optimal) receive processing is applied.

III. WEIGHTED SUM RATE MAXIMIZATION FOR THE MIMO IFC

The stated objective of our investigation is the maximization of the WSR of MIMO IFC. For a given MIMO IFC, the maximization of the weighted sum rate (WSR) allows to cover all the rate tuples on the rate region boundary. It is for this reason that, in this paper we consider the weighted sum rate maximization problem for a K-user frequency-flat MIMO IFC and propose an iterative algorithm for linear precoder/receiver design. With full CSIT, but only knowledge of $\mathbf{s}_k$ at transmitter k, it is expected that linear processing at the transmitter should be sufficient. On the receive side however, optimal WSR approaches may involve joint detection of the signals from multiple transmitters. In this paper we propose to limit receiver complexity by restricting the modeling of the signals arriving from interfering transmitters as colored noise (which is Gaussian if we consider Gaussian codebooks at the transmitters). As a result, linear receivers are sufficient. For the MIMO IFC, one approach to linear transmit precoder design is the joint design of precoding matrices to be applied at each transmitter based on channel state information (CSI) of all users. Such a *centralized* approach [4] requires (channel) information exchange among transmitters. Nevertheless, studying such systems can provide valuable insights into the limits of perhaps more practical *distributed* algorithms [8] [9] that do not require any information transfer among transmitters.

The WSR maximization problem can be mathematically expressed as follows.

$$\{\mathbf{G}_k^\star, \mathbf{F}_k^\star\} = \arg \min_{\{\mathbf{G}_k, \mathbf{F}_k\}} \mathcal{R} \quad \text{s.t } \text{Tr}(\mathbf{G}_k^H\mathbf{G}_k) = P_k \quad \forall k \tag{2}$$

where

$$\mathcal{R} = \sum_k -w_k R_k.$$

with $w_k \geq 0$ denoting the weight assigned to the k-th user's rate and P_k it's transmit power constraint. We use the notation $\{\mathbf{G}_k, \mathbf{F}_k\}$ to compactly represent the candidate set of transmitters $\mathbf{G}_k$ and receivers $\mathbf{F}_k$ $\forall k \in \{1, \ldots, K\}$ and the corresponding set of optimum transmitters and receivers is represented by $\{\mathbf{G}_k^\star, \mathbf{F}_k^\star\}$. Assuming Gaussian signaling, the k-th user's achievable rate is given by

$$\begin{aligned} R_k &= \log |\mathbf{E}_k|, \\ \mathbf{E}_k &= \mathbf{I}_k + \mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k(\mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k)^H(\mathbf{F}_k\mathbf{R}_{\overline{k}}\mathbf{F}_k^H)^{-1} \end{aligned} \tag{3}$$

where the interference plus noise covariance matrix $\mathbf{R}_{\overline{k}}$ is:

$$\mathbf{R}_{\overline{k}} = \mathbf{R}_{n_k n_k} + \sum_{l \neq k} \mathbf{H}_{kl}\mathbf{G}_l\mathbf{G}_l^H\mathbf{H}_{kl}^H.$$

We use here the standard notation $| \ . \ |$ to denote the determinant of a matrix. The MIMO IFC rate region is known to be non-convex. The presence of multiple local

optima complicates the computation of optimum precoding matrices to be applied at the transmitter in order to maximize the weighted sum rate. What is known however, is that, for a given set of precoders, linear minimum mean squared error (LMMSE) receivers are optimal in terms of interference suppression. In addition we can extend this concept saying that, for a given set of linear beamforming filters applied at the transmitters, the LMMSE interference-suppressing filter applied at the receiver does not lose any information of the desired signal in the process of reducing the N_k dimensional $\mathbf{y}_k$ to a d_k dimensional vector $\mathbf{r}_k$. This is of course under the assumption that all interfering signals can be treated as Gaussian noise. In other words, the linear MMSE interference suppressor filter is information lossless and is thus optimal in terms of maximizing the WSR.

Thanks to this property of the LMMSE Rx filter, we consider a (more tractable) optimization problem where MMSE processing at the receiver is implicitly assumed. The WSR maximization problem in (2) that we consider becomes:

$$\{\mathbf{G}_k^\star\}=\arg \min_{\{\mathbf{G}_k\}} \sum_{k=1}^{K} -w_k \log |\mathbf{E}_k^{-1}| \ \text{s.t} \ \text{Tr}(\mathbf{G}_k^H \mathbf{G}_k) \leq P_k \ \forall k \tag{4}$$

where $\mathbf{E}_k$ is given by

$$\mathbf{E}_k = (\mathbf{I} + \mathbf{G}_k^H \mathbf{H}_{kk}^H \mathbf{R}_{\overline{k}}^{-1} \mathbf{H}_{kk} \mathbf{G}_k)^{-1}. \tag{5}$$

This problem in non convex and hence finding a solution is a complex task. In order to obtain the stationary points for the optimization problem (4), we solve the Lagrangian:

$$L\left(\{\mathbf{G}_k, \lambda_k\}\right) = \sum_{k=1}^{K} -w_k \log |\mathbf{E}_k^{-1}| + \lambda_k (\text{Tr}\{\mathbf{G}_k^H \mathbf{G}_k\} - P_k)$$

Now setting the gradient of the Lagrangian w.r.t. the transmit filter $\mathbf{G}_k$ to zero, we have:

$$\frac{\partial J(\{\mathbf{G}_k, \lambda_k\})}{\partial \mathbf{G}_k^*} = 0$$

$$\sum_{l \neq k} w_l \mathbf{H}_{lk}^H \mathbf{R}_{\overline{l}}^{-1} \mathbf{H}_{ll} \mathbf{G}_l \mathbf{E}_l \mathbf{G}_l^H \mathbf{H}_{ll}^H \mathbf{R}_{\overline{l}}^{-1} \mathbf{H}_{lk} \mathbf{G}_k \\ -w_k \mathbf{H}_{kk}^H \mathbf{R}_{\overline{k}}^{-1} \mathbf{H}_{kk} \mathbf{G}_k \mathbf{E}_k + \lambda_k \mathbf{G}_k = 0 \tag{6}$$

Our approach to the design of the WSR maximizing transmit filters for the MIMO IFC is based on introducing an augmented cost function in which two additional optimization variables appear [10]. The optimization problem that we consider now is

$$\{\mathbf{G}_k^\star, \ \mathbf{F}_k^\star, \ \mathbf{W}_k^\star\} = \\ \arg \max_{\{\mathbf{G}_k, \mathbf{F}_k, \mathbf{W}_k\}} \sum_{k} -w_k (\text{Tr}(\mathbf{W}_k \boldsymbol{\mathcal{E}}_k) - \log |\mathbf{W}_k| - d_k^{max})$$

$$\text{s.t} \sum_{k} \text{Tr}(\mathbf{G}_k \mathbf{G}_k^H) \leq P_k. \tag{7}$$

where $d_k^{max} \leq \min\{N_k, M_k\}$ represents the maximum number of independent data streams that can be transmitted to user k. This cost function is concave or even quadratic in one set of variables, keeping the other two fixed. Hence we shall optimize

it using alternating maximization. Assuming $\mathbb{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}_{d_k}$, the MSE covariance matrix for general Tx and Rx filters is

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_k = \ & \mathbb{E}[(\mathbf{s} - \mathbf{F}_k \mathbf{y}_k)(\mathbf{s} - \mathbf{F}_k \mathbf{y}_k)^H]\} \\
= \ & \mathbf{I} - \mathbf{G}_k^H \mathbf{H}_{kk}^H \mathbf{F}_k^H - \mathbf{F}_k \mathbf{H}_{kk} \mathbf{G}_k \\
& + \mathbf{F}_k \mathbf{H}_{kk} \mathbf{G}_k \mathbf{G}_k^H \mathbf{H}_{kk}^H \mathbf{F}_k^H \\
& + \sum_{l \neq k} \mathbf{F}_k \mathbf{H}_{kl} \mathbf{G}_l \mathbf{G}_l^H \mathbf{H}_{kl}^H \mathbf{F}_k^H + \mathbf{F}_k \mathbf{R}_{n_k n_k} \mathbf{F}_k^H
\end{aligned} \tag{8}$$

The corresponding Lagrangian can be written as:

$$\begin{aligned}
J(\{\mathbf{G}_k, \mathbf{F}_k, \mathbf{W}_k, \lambda_k\}) = & -\lambda_k (\text{Tr}\{\mathbf{G}_k^H \mathbf{G}_k\} - P_k) \\
& - \sum_{k} w_k (\text{Tr}(\mathbf{W}_k \boldsymbol{\mathcal{E}}_k) - \log |\mathbf{W}_k| - d_k^{max})
\end{aligned} \tag{9}$$

This new cost function will be optimized w.r.t. one set of variables, keeping the other two fixed. The first step in our optimization process is the calculation of the optimal Rx filters assuming fixed the matrices $\mathbf{G}_k$ and $\mathbf{W}_k$. It can easily be seen that the optimal Rx filter is an MMSE filter:

$$\mathbf{F}_k^{LMMSE} = \mathbf{G}_k^H \mathbf{H}_{kk}^H (\mathbf{R}_{\overline{k}} + \mathbf{H}_{kk} \mathbf{G}_k \mathbf{G}_k^H \mathbf{H}_{kk}^H)^{-1} \tag{10}$$

The following step in the optimization procedure is the determination of the optimal expression for the matrix $\mathbf{W}_k$ while keeping the other two variable sets fixed.
What we get is:

$$\mathbf{W}_k = \mathbf{E}_k^{-1} \tag{11}$$

The final step is the maximization of the given cost function w.r.t. the BF matrix. To accomplish this task we derive the Lagrangian w.r.t. the matrix $\mathbf{G}_k$ and equate it to zero:

$$\frac{\partial \mathbf{J}(\{\mathbf{G}_k, \lambda_k\})}{\partial \mathbf{G}_k^*} = $$
$$w_k \mathbf{H}_{kk}^H \mathbf{F}_k^H \mathbf{W}_k - \lambda_k \mathbf{G}_k - \sum_{l=1}^{K} w_l \mathbf{H}_{lk}^H \mathbf{F}_l^H \mathbf{W}_l \mathbf{F}_l \mathbf{H}_{lk} \mathbf{G}_k = 0. \tag{12}$$

This leads to the following expression for the optimizing BF:

$$\mathbf{G}_k = \left(\sum_{l=1}^{K} w_l \mathbf{H}_{lk}^H \mathbf{F}_l^H \mathbf{W}_l \mathbf{F}_l \mathbf{H}_{lk} + \lambda_k \mathbf{I} \right)^{-1} \mathbf{H}_{kk}^H \mathbf{F}_k^H \mathbf{W}_k w_k \tag{13}$$

The only variable that still needs to be optimized is the Lagrange multiplier λ_k. First check if $\text{Tr}(\mathbf{G}_k^H \mathbf{G}_k) \leq P_k$ for $\lambda_k = 0$. If yes, than $\lambda_k = 0$. If not, the Tx power equality constraint is active. For this case, [10] for the scenario of a MIMO broadcast channel used the idea developed in [11] for single antenna receivers. Applying the same reasoning to the MIMO IFC we obtain the following optimal expression for λ_k.

$$\begin{aligned}
\lambda_k = \quad & \frac{1}{P_k} \left(\sum_{l \neq k} w_l \text{Tr}\{\mathbf{W}_l \mathbf{F}_l \mathbf{H}_{lk} \mathbf{G}_k (\mathbf{F}_l \mathbf{H}_{lk} \mathbf{G}_k)^H\} \right) \\
& - \frac{1}{P_k} \left(\sum_{l \neq k} w_k \text{Tr}\{\mathbf{W}_k \mathbf{F}_k \mathbf{H}_{kl} \mathbf{G}_l (\mathbf{F}_k \mathbf{H}_{kl} \mathbf{G}_l)^H\} \right) \\
& - \frac{w_k}{P_k} \left(\text{Tr}\{\mathbf{W}_k \mathbf{F}_k \mathbf{R}_{n_k n_k} \mathbf{F}_k^H\} \right).
\end{aligned} \tag{14}$$

With this value of the Lagrange multiplier the final expression for the BF becomes (15). The algorithm proposed in [10] was developed for a MIMO broadcast channel, where only

an overall Tx power constraint is applied on the system and, in addition, maximizing the WSR automatically requires to transmit with full power. On the other hand in the MIMO IFC the WSR maximization may require some links to transmit with a power less than the maximum power available at that links.

At low SNR regime the maximization of the WSR leads to activate only one stream per link, allocating full power on the best singular mode of the direct channel $\mathbf{H}_{kk}$.

For SNR values sufficiently high the maximization of the sum rate converges to an IA solution. IA feasibility may imply zero streams for some links. Here we propose to determine the optimal value of $\lambda_k \geq 0$ using a linear search algorithm.

Grouping together all the optimization steps that describe our maximization procedure we have the following two-steps iterative algorithm to compute the precoders that maximize the weighted sum rate for a given MIMO IFC (c.f Table **Algorithm 1**). Introducing the augmented cost function, for

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Algorithm 1 MWSR Algorithm for MIMO IFC

Fix an arbitrary initial set of precoding matrices $\mathbf{G}_k$, $\forall \in k = \{1, 2 \dots K\}$
set $n = 0$
repeat
    $n = n + 1$
    Given $\mathbf{G}_k^{(n-1)}$, compute $\mathbf{F}_k^n$ and $\mathbf{W}_k^n$ from (10) and (11) respectively $\forall k$
    Given $\mathbf{F}_k^n$ and $\mathbf{W}_k^n$, compute $\mathbf{G}_k^n$ $\forall k$ using (13)
until convergence

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the calculation of the optimal BF matrix that maximize the WSR, we are able to determine an iterative algorithm that can be easily proved to converge to a local optima that corresponds also to an extremum of the original cost function (4).

Each step of our iterative algorithm increases the cost function, which is bounded above (e.g. by cooperative WSR), and hence convergence is guaranteed. In addition the augmented cost function once we substitute $\mathbf{W}_k$ and $\mathbf{F}_k$ with their optimal values, becomes exactly the original WSR cost function (4). Finally using matrix inversion lemma it is possible to rewrite the expression of the MMSE (10) as

$$\mathbf{F}_k = \mathbf{E}_k \mathbf{G}_k^H \mathbf{H}_{kk}^H \mathbf{R}_{\bar{k}}^{-1}.$$

With this representation of the Rx filters it is possible to interpret some quantities in the gradient of the WSR (6) as Rx filters and hence the expression that comes out of this elaboration is the same as the gradient of the augmented cost function w.r.t. the BF matrix (12). This implies that a stationary point of the original cost function is also a stationary point of the augmented cost function.

A final remark can be made about the dimensions imposed on the beamforming matrix. In particular at high SNR we can put $d_k = M_k$ if we want the algorithm to figure out the feasible set of $\{d_k\}$, in this case all IA-feasible solutions represent local optima. Another possible choice is to use d_k

that corresponds to an IA-feasible solution if we want to focus on that particular stream distribution.

At low or medium SNR regime a possible choice is $d_k = \max\{1, d_k^{IA}\}$ where the set $\{d_k^{IA}\}$ form a IA-feasible set.

IV. INTERFERENCE ALIGNMENT FEASIBILITY

The objective in IA is to design aligning matrices to be applied at the transmitters such that, the interference caused by all transmitters at each non-intended RX lies in a common *interference subspace*. Then simple ZF receivers can be applied to suppress the interference and extract the desired signal. Interference alignment can be described by the following conditions:

$$\mathbf{F}_k \mathbf{H}_{kl} \mathbf{G}_l = \mathbf{0} \quad \forall l \neq k \tag{16}$$

$$\mathrm{rank}(\mathbf{F}_k \mathbf{H}_{kk} \mathbf{G}_k) = d_k \quad \forall k \in \{1, 2, \dots, K\} \tag{17}$$

In addition, the traditional single user MIMO constraint $d_k \leq \min(M_k, N_k)$ also needs to be satisfied. To find a set of conditions that needs to be satisfied by a $K-$user MIMO IFC to admit an IA solution we formulate the given IA problem as finding a solution to a system of equations with limited number of variables. Fig. 3 presents a pictorial representation of such a system of equations where the block matrices $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{G}$ on the left hand side (LHS) of the equality represent respectively, the ZF RX, overall channel matrix and beamformers. The block diagonal matrix to the right hand side (RHS) of the equality represents the total constraints in the system that need to be satisfied for an IA solution to exist. The block matrices on the diagonal of $\mathbf{H}$ represent the direct-links and the off diagonal blocks in any corresponding block row k represent the cross channels of the k-th link. The main idea of our
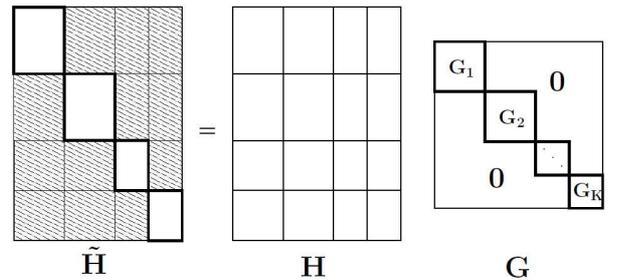
Fig. 2: Interference alignment at all receivers .

approach [12] is to convert the alignment requirements at each RX into a rank condition of an associated interference matrix

$$\mathbf{H}_I^{[k]} = [\mathbf{H}_{k1}\mathbf{G}_1, \dots \mathbf{H}_{k(k-1)}\mathbf{G}_{(k-1)}, \mathbf{H}_{k(k+1)}\mathbf{G}_{(k+1)}, \dots \mathbf{H}_{kk}\mathbf{G}_K]$$

that spans the interference subspace at the k-th RX (the shaded blocks in each block row in Fig.2).Thus the dimension of the Interference subspace must satisfy

$$\mathrm{rank}(\mathbf{H}_I^{[k]}) = r_I^{[k]} \leq N_k - d_k.$$

The equation above prescribes an upperbound for $r_I^{[k]}$ but the nature of the channel matrix (full rank) and the rank requirement of the BF specifies the following lower bound

$$r_I^{[k]} \geq \max_{l \neq k}(d_l - [M_l - N_k]_+).$$

$$\mathbf{G}_k = \left(\sum_{l=1}^{K} \mathbf{H}_{lk}^H \mathbf{F}_l^H \mathbf{W}_l \mathbf{F}_l \mathbf{H}_{lk} - \frac{1}{P_k} \left(\left(\sum_{l \neq k} \mathrm{Tr}\{\mathbf{W}_l \mathbf{J}_l^{(k)}\} - \mathrm{Tr}\{\mathbf{W}_k \mathbf{J}_k^{(l)}\} \right) - \mathrm{Tr}\{\mathbf{W}_k \mathbf{N}_k\} \right) \mathbf{I} \right)^{-1} \mathbf{H}_{kk}^H \mathbf{F}_k^H \mathbf{W}_k \qquad (15)$$

$$\mathbf{J}_l^{(k)} = \mathbf{F}_l \mathbf{H}_{lk} \mathbf{G}_k \mathbf{G}_k^H \mathbf{H}_{lk}^H \mathbf{F}_l^H; \quad \mathbf{J}_k^{(l)} = \mathbf{F}_k \mathbf{H}_{kl} \mathbf{G}_l \mathbf{G}_l^H \mathbf{H}_{kl}^H \mathbf{F}_k^H; \quad \mathbf{N}_k = \mathbf{F}_k \mathbf{R}_{n_k n_k} \mathbf{F}_k^H$$

Imposing a rank $r_I^{[k]}$ on $\mathbf{H}_I^{[k]}$ implies imposing a number of constraints at RX k equal to

$$(N_k - r_I^{[k]})(\sum_{\substack{l=1 \\ l \neq k}}^{K} d_l - r_I^{[k]}).$$

Enforcing the minimum number of constraints on the system implies to have maximum rank: $r_I^{[k]} \leq \min(d_{tot}, N_k) - d_k$ From this consideration it is possible to derive a recursive procedure to evaluate IA feasibility for a general MIMO IFC [12].

If the single user MIMO constraint is satisfied for all links, the first step of the procedure is to ensure that the range for each r_i is non-empty. This amounts to checking if:

$$(\min(d, N_k) - d_k) - \max_{j \in \mathcal{K} - \{k\}} (d_j - [M_j - N_k]_+) \geq 0 \quad \forall k \in \mathcal{K}$$

where $\mathcal{K} = \{1, 2, ...K\}$. Indeed, an IA solution is immediately ruled out if the above relation is not true.

Now, starting from a system with K Rx, verify if the system defined by adding successively one Tx at a time satisfies the following relation: for $k = 1, \ldots, K$,

$$\sum_{i=1}^{k} d_i (M_i - d_i) \geq$$
$$\sum_{i=1}^{k} (N_i - \min(d - d_i, (N_i - d_i)))(d - d_i - \min(d - d_i, (N_i - d_i)))$$
$$+ \sum_{i=k+1}^{K} (N_i - \min(d, (N_i - d_i)))(d - \min(d, (N_i - d_i)))$$

(18)

Finally, interchange the Tx and Rx sides and verify again the previous conditions.

V. WSR MAXIMIZATION AT HIGH SNR

In the first part of the paper we have introduce an iterative algorithm that maximizes the WSR for all possible values of the SNR. In the following we will focus our attention only to the high SNR regime. In particular we study how it is possible to optimize the WSR only in that particular region.

In high SNR regime the behaviour of the rate can be described using two quantities [13] : the *multiplexing gain* or *pre-log* or also *degrees of freedom* (DoF) and the *high SNR rate offset*. The former describes the slope of the asymptote of the rate curve in the high SNR, the latter can be interpreted as the axis intercept of the high SNR asymptote on the rate axis. The approximation can be mathematically represented as:

$$\mathcal{R}_k = \sum_{k=1}^{K} r_k \log(\rho) + \alpha_k + O(\rho)$$

where α_k and r_k represent respectively the rate offset and the pre-log factor for the rate of user k. With ρ we denote the SNR. Using the approximation given before the WSR can be rewritten as:

$$\mathcal{R} = \sum_{k=1}^{K} w_k \mathcal{R}_k = r \log(\rho) + \alpha + O(\rho). \qquad (19)$$

$r = \sum_{k=1}^{K} w_k r_k$ denotes the *weighted sum prelog factor* and $\alpha = \sum_{k=1}^{K} w_k \alpha_k$ is the *weighted sum rate offset*.

In high SNR regime also the expression of the Rx and Tx filter changes. In particular the linear receiver becomes a ZF receiver: $\mathbf{F}_k = \mathbf{F}_k^{IA} + O(\rho)$. Note that with this assumption only the row space of the Rx filter influences the rate so we can assume the Rx filter to be unitary. The interference plus noise covariance matrix $\mathbf{R}_{\bar{k}}^{-1}$ in high SNR becomes: $\mathbf{R}_{\bar{k}}^{-1} = \rho \mathbf{P}_{R_k^I}$, where $\mathbf{P}_{R_k^I}$ is the projection matrix onto orthogonal complement of the column space of the interference matrix $\mathbf{R}_k^I$ at user k.

We assume that the interference subspace at the k−th receiver has dimension $\mathrm{rank}(\mathbf{R}_k^I) = i_k \leq N_k$

With this interpretation of the interference plus noise covariance matrix in high SNR the dominating term in the rate expression becomes:

$$\mathcal{R}_k = \min(d_k, N_k - i_k) \log(\rho) \qquad (20)$$

hence to maximize the rate the Tx filters need to minimize the interference subspace dimension by interference alignment so that $i_k \leq N_k - d_k$, hence d_k should be IA-feasible. If this is the case the rate pre-log factor becomes $r_k = d_k$.

A. Maximization of the pre-log factors

From equation (19) the WSR maximization becomes in first instance the maximization of the weighted sum pre-log factor r:

$$\max_{\{d_k\}} \sum_{k=1}^{K} w_k d_k \qquad (21)$$

this factor is the dominant term between the two quantities in (19) as SNR goes to infinity. The solution of this optimization problem will give the set of pre-log factors $\{d_k^\star\}$ that corresponds to the DoF allocation of the maximum WSR. Because each value of the pre-log factor can vary in a finite set: $d_k \in \{0, 1, \ldots, \min\{M_k, N_k\}\}$ a possible way of solving the optimization problem is using an exhaustive search among all the possible feasible DoF allocations that maximize (21).

A first important remark here is that for a given set of weights $\{w_k\}$ several optimal DoF allocation can be possible. This corresponds to the possibility of the WSR to have several local maxima. Using the proposed approach to determine the

optimal DoF allocation can help to maximize the WSR using the iterative algorithm proposed in the first part of this paper. In particular imposing one of the possible optimal pre-log distribution in our iterative algorithm we can determine which DoF allocation effectively maximize the WSR among all the optimal distribution of streams.

A second remark arise from the observation that the determined optimal prelog-factor distribution is strictly related to the given set of weights $\{w_k\}$. If we change the weights the DoF allocation can change. This means that using the maximization procedure described above it is possible to explore the complete pre-log region varying the set of weights. We recommend that given the set of weights $\{w_k\}$, one determines an optimal choice for the prelogs $\{d_k\}$ with which one then runs the MWSR algorithm.

In the optimal stream allocation it is possible to have that one or more d_k are set to zero. In this case it corresponds to switch off the corresponding users. If we assume that the SNR for all user in the K-MIMO interference channel is expressed as P, with normalized noise variance equal to one, the TX power for user k is: $p_k = \gamma_k P^{\delta_k} \leq \beta_k P$. If now the multiplexing gain for user k is zero the corresponding value of δ_k is zero. This does not correspond to switch off the user but it will not contribute to increase the total number of DoF it will influence only the rate offset causing a negligible level of interference.

B. Maximization of the high SNR rate offsets

Once the optimal multiplexing gain distribution is determined we need to optimize the weighted sum rate offset α. As described in [13] the high SNR rate offset is given by:

$$\alpha_k = \log |\mathbf{G}_k^H \mathbf{H}_{kk}^H \mathbf{P}_{R_k^I}^{-1} \mathbf{H}_{kk} \mathbf{G}_k| \qquad (22)$$

The beamformer can be parametrized as $\mathbf{G}_k = \overline{\mathbf{G}}_k \mathbf{U}_k \boldsymbol{\Delta}_k$, where $\overline{\mathbf{G}}_k$ is determined using IA and satisfies the property: $\overline{\mathbf{G}}_k^H \overline{\mathbf{G}}_k = \mathbf{I}_{d_k}$. The two matrices $\mathbf{U}_k$ and $\boldsymbol{\Delta}_k$ have dimensions $d_k \times d_k$. The former is a unitary matrix and the latter is a diagonal matrix.
Taking the eigendecomposition of the matrix $\overline{\mathbf{H}}_{kk}^H \overline{\mathbf{H}}_{kk} = \overline{\mathbf{G}}_k^H \mathbf{H}_{kk}^H \mathbf{P}_{R_k^I}^{-1} \mathbf{H}_{kk} \overline{\mathbf{G}}_k = \mathbf{V}_k \boldsymbol{\Lambda}_k \mathbf{V}_k^H$, we can choose the unitary matrix $\mathbf{U}_k = \mathbf{V}_k$. With this parametrization the maximization problem of the the rate offset becomes:

$$\alpha_k^\star = \max_{\boldsymbol{\Delta}_k} \log |\boldsymbol{\Delta}_k^2 \boldsymbol{\Lambda}_k| \qquad (23)$$
$$\text{s.t.} \quad \text{Tr}\{\boldsymbol{\Delta}_k^2\} = P_k.$$

But $\log |\boldsymbol{\Delta}_k^2 \boldsymbol{\Lambda}_k| = \log |\boldsymbol{\Delta}_k^2| + \log |\boldsymbol{\Lambda}_k|$. Hence the optimum is reached for uniform power allocation $\boldsymbol{\Delta}_k^2 = \frac{P_k}{d_k}\mathbf{I}$. From this we can see that the expression for the BF at high SNR is:

$$\mathbf{G}_k = \sqrt{\frac{P_k}{d_k}} \overline{\mathbf{G}}_k \qquad (24)$$

Finally we can conclude that the high SNR rate expression is:

$$\mathcal{R}_k = d_k \log(\rho) + d_k \log(\frac{P_k}{d_k}) + \log |\overline{\mathbf{G}}_k^H \mathbf{H}_{kk}^H \mathbf{P}_{R_k^I}^\perp \mathbf{H}_{kk} \overline{\mathbf{G}}_k| \qquad (25)$$

As we said in the previous section IV a necessary condition for the existence of a IA solution is related to the number of variables that we have in the MIMO IFC and the number of constraints that define the problem. Now we want to discuss how the variation of the rate offset can be related to this two quantities.
In particular if we assume that for the given MIMO IFC an IA solution exist we can have the following two cases:

- The number of variables is greater than the number of IA constraints. In this case an excess of variables implies continuously varying α_k (with w_k)
  Consider for example the system $K = 2, M_k = 2, N_k = 2, d = (1,1)$, we can choose the two 2×1 Tx filters arbitrarily, and then the two 1×2 Rx filters are determined by IA.
  It is possible that subsets of equations have no excess of parameters, then the filters involved are not continuously varying
- The number of variables equals the number of IA constraints. Here no excess parameters exist but we may still get a discrete set of solutions $\{\alpha_k\}$ IA is described by a set of polynomial equations hence there are a finite number of solutions. For example in the case $K = 3$, $M_k = N_k = 2N$, 6 filters have N^2 Dof, and $6N^2$ ZF conditions. In this case an IA BF can be determined using the procedure described in [14]. In particular the first BF is determined taking the N eigenvector of a $2N \times 2N$ matrix $\mathbf{H}_{31}^{-1}\mathbf{H}_{32}\mathbf{H}_{12}^{-1}\mathbf{H}_{13}\mathbf{H}_{23}^{-1}\mathbf{H}_{21}$, all the remaining BF can be found from $\mathbf{G}_1$. Using this way to determine the BF we have a different solution for a different choice of the N eigenvectors out of the possible $2N$.

C. Avoiding WSR Local Maxima via Deterministic Annealing

So the above analysis shows that there are potentially many local optima. However, as we observed earlier, at low SNR, MWSR leads (for non-zero weights) to the following global optimum: 1 stream per link, transmitting at full power on the best singular mode of the $\mathbf{H}_{kk}$. Hence this suggests the following *Deterministic Annealing* procedure: gradually increase SNR from low to the desired value, and for each higher SNR use the solution from the lower SNR as initialization. If the SNR step is small enough, the lower SNR solution will remain in the region of attraction of the global optimum at the next higher SNR. The algorithmic specification so far is not enough for the general case of multiple streams per user: at each SNR increase, one needs to test increasing d_k by 1 for each of the users. If the SNR increment is small enough, only at most one stream will turn on at a time. The initizalization of the filters related to the extra stream is still an issue though. Note: some users may also get switched off but that is automatic by the MWSR algorithm.

VI. Zero Forcing Analysis at High SNR

After having determined the optimal stream allocation using the technique described in the previous section it is possible to ascertain a lower bound of the WSR designing the transmitter
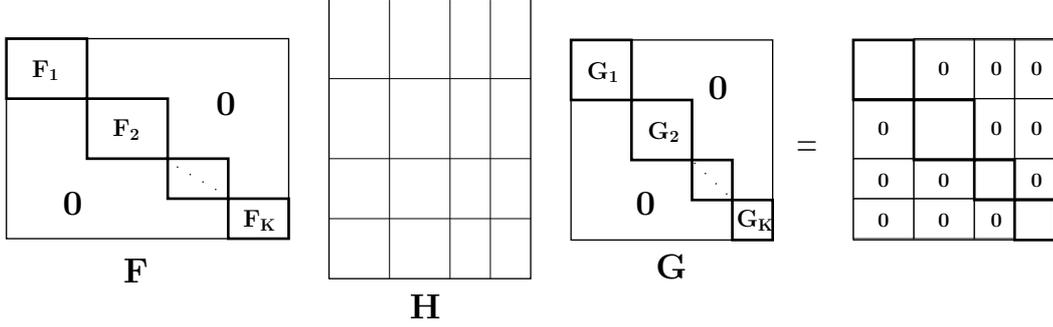
Fig. 3: Block matrix representation of the interference alignment problem.

(Tx) and receiver (Rx) filter using IA. Interference alignment can be thought as joint Tx Rx zero forcing (ZF). From a pictorial point a view it can be represented as in Fig. 3 where the block matrices $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{G}$ on the left hand side (LHS) of the equality represent respectively, the ZF RX, overall channel matrix and beamformers. The block diagonal matrix to the right hand side (RHS) of the equality represents the total constraints in the system that need to be satisfied for an IA solution to exist. The block matrices on the diagonal of $\mathbf{H}$ represent the direct-links and the off diagonal blocks in any corresponding block row k represent the cross channels of the k-th link. We assume that each block has i.i.d. entries with variance $\rho_{k,i}$ and and each block is independent of the other blocks. The interference aligning beamformer matrix $\mathbf{G}_k$ (the diagonal blocks in $\mathbf{G}$) aligns the transmit signal of the k-th user to the interference subspace at all $l \neq k$ users while ensuring the rank of the equivalent channel matrix $\mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k$ is at least d_k. The interference matrix at Rx k is described by:

$$\mathbf{H}_I^{[k]}=[\mathbf{H}_{k1}\mathbf{G}_1, ...\mathbf{H}_{k(k-1)}\mathbf{G}_{(k-1)}, \mathbf{H}_{k(k+1)}\mathbf{G}_{(k+1)}, ...\mathbf{H}_{kk}\mathbf{G}_K],$$

it spans the interference subspace, of dimension $r_k = rank(\mathbf{H}_I^{[k]})$, at that particular receiver. Under the assumption that the entries of the channel matrix $\mathbf{H}_{ki}$ are complex iid the directions of its eigenvectors can be assumed to be uniformly distributed in the signal space. For this reason we can assume that also the entries of the BFs are iid with direction isotropically distributed. Under this assumptions the row of the interference matrix at Rx k are iid (the element within the row are not iid they can have different variance) and hence also the entries of the Rx filter are iid with isotropic directions. Now consider the dual system where the role of the Tx and Rx are interchanged. the interference matrix at receiver k is

$$\mathbf{H}_I^{[k]} = \begin{bmatrix} \mathbf{F}_1\mathbf{H}_{1k} \\ \vdots \\ \mathbf{F}_{k-1}\mathbf{H}_{k-1k} \\ \mathbf{F}_{k+1}\mathbf{H}_{k+1k} \\ \vdots \\ \mathbf{F}_K\mathbf{H}_{Kk} \end{bmatrix}$$

From the previous step we have isotropic matrix $\mathbf{F}_i$ and hence the columns of the interference matrix are iid also in the dual problem. This implies that also the matrix $\mathbf{G}_k$ is isotropic as

we have assumed as initial assumption. After this analysis we can say that the equivalent channel matrix $\tilde{\mathbf{H}}_{kk} = \mathbf{F}_k\mathbf{H}_{kk}\mathbf{F}_k$ has iid entries and the only effect of the Tx and Rx matrices is to shrink the channel dimension to $(N_k - r_k) \times d_k$. After IA the K-user IFC can be interpreted as K parallel MIMO links with reduced channel dimension. The WSR optimization now can be done as for a single user MIMO system, due to the suppression of the interference done using the ZF receiver, using as channel matrix $\tilde{\mathbf{H}}_{kk}$. Because the entries of the equivalent channel matrix are iid the analysis of this optimization problem are the same as for the standard SU-MIMO.

VII. SIMULATION RESULTS

We provide here some simulation results to compare the performance of the proposed max-WSR algorithm. i.i.d Gaussian channels (direct and cross links) are generated for each user. For a fixed channel realization transmit and receiver filters are computed based on IA algorithm and max-WSR algorithm over multiple SNR points. The non convexity of the problem may lead the algorithm to converge to a stationary point that represents a local optimum instead of the global one which we are interested in. To increase the probability of reaching the optimum a common strategy in non convex problem is to to choose multiple random initial beamforming matrices and adopting the solution of the algorithm that determines the best WSR. Using these filters individual rates are computed. The resulting rate-sum is averaged over several hundred Monte-Carlo runs. The average rate-sum plots are used to compare the performance of the proposed algorithm.

In Fig. 4, we plot the results for a 3-user MIMO IFC. The antenna distribution at the receive and transmit side is $M_k = N_k = 2 \ \forall k$. The max-WSR algorithm results in a DoF allocation of $d_1 = 1 \ d_2 = 1 \ d_3 = 1$ with $w_k = 1 \ \forall k$ In Fig. 5, we plot the results for a 3-user MIMO IFC with $M_k = N_k = 3 \ \forall k$. The resulting DoF allocation is $d_1 = 2 \ d_2 = 1 \ d_3 = 1$ with $w_k = 1 \ \forall k$ Finally, Fig. 6 shows the convergence behavior of our algorithm for the same 3-user MIMO IFC with $M_k = N_k = 4 \ \forall k$ in a given SNR point, SNR=5dB

VIII. CONCLUSIONS

In the first part of the paper we addressed maximization of the weighted sum rate for the MIMO IFC. In the second
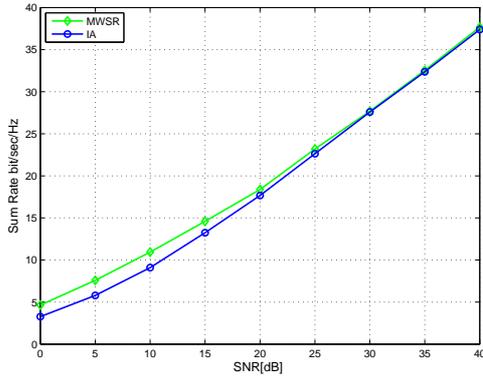
Fig. 4: 3-user MIMO IFC with $M_k = N_k = 2 \; \forall k$.
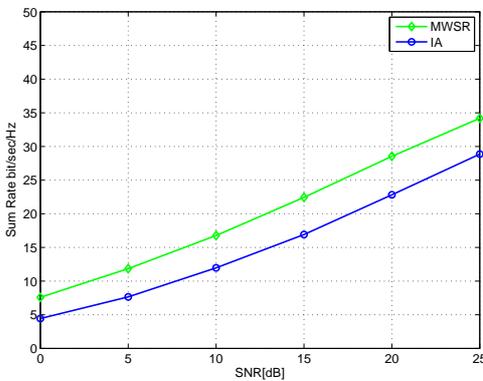
Fig. 5: 3-user MIMO IFC with $M_k = N_k = 3 \; \forall k$

part we analyzed the high SNR characteristic of the WSR. We introduced an iterative algorithm to solve this optimization problem. In the high-SNR regime, this algorithm leads to an optimized Interference Alignment (IA) solution In the finite SNR regime the performance of this algorithm is superior to that of IA and all known algorithms since it maximizes the WSR as opposed to previous attempts that maximize the sum rate. Convergence to a local optimum was also shown experimentally. Convergence to local optima is known and is related to the non-convexity of the MIMO IFC rate region.
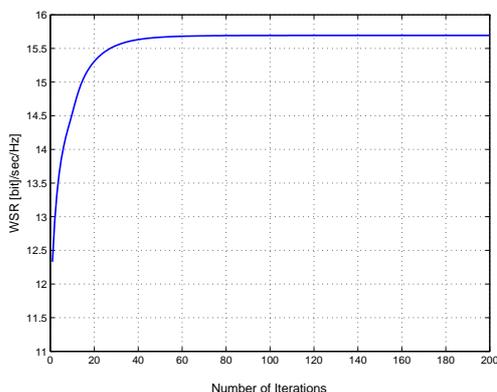
Fig. 6: Convergence behavior for a 3-user MIMO IFC with $M_k = N_k = 4 \; \forall k$ at SNR=5dB

We propose an alternative way to optimize the WSR in high SNR regime based on the maximization of the pre-log factor corresponding to the particular weights distribution of the WSR and, in a second stage, the maximization of the high SNR rate offset. Once the optimal DoF allocation is determined a lower bound on the maximum WSR in high SNR is given. It comes out from the optimization of the equivalent K parallel SU-MIMO links obtained using IA. Finally a new procedure to determine the feasibility of IA is proposed.

IX. ACKNOWLEDGMENT

REFERENCES

[1] R. H. Etkin, D. N. C. Tse, and Hua Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. on Inform. Theory*, vol. 54, no. 12, pp. 5534–5562, 2008.

[2] V.R. Cadambe and S.A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. on Inform. Theory*, vol. 54, no. 8, pp. 3425 –3441, Aug. 2008.

[3] K. Gomadam, V.R. Cadambe, and S.A. Jafar, "Approaching the capacity of wireless networks through distributed interference alignment," in *Proc. IEEE Global Telecommunications Conf. (GLOBECOM)*, Dec 2008.

[4] Sigen Ye and R.S. Blum, "Optimized signaling for MIMO interference systems with feedback," *IEEE Trans. on Signal Processing*, vol. 51, no. 11, pp. 2839–2848, Nov 2003.

[5] S. Serbetli and A. Yener, "Transceiver optimization for multiuser MIMO systems," *IEEE Trans. on Signal Processing*, vol. 52, no. 1, pp. 214 – 226, Jan. 2004.

[6] David A. Schmidt, Shi Changxin, Randall A. Berry, Michael L. Honig, and Wolfgang Utschick, "Minimum mean squared error interference alignment," in *Proc. 43rd IEEE Annual Asilomar Conference on Signals, Systems & Computers*, Pacific Grove,California, USA, Nov 2009.

[7] Steven W. Peters and Robert W. Heath Jr., "Cooperative algorithms for MIMO interference channels," *[Online]*, Feb 2010.

[8] Scutari Gesualdo, Daniel P Palomar, and Sergio Barbarossa, "The MIMO iterative waterfilling algorithm," *IEEE Trans. on Signal Processing*, vol. 57, no. 5, pp. 1917–1935, May 2009.

[9] Shi Changxin, David A. Schmidt, Randall A. Berry, Michael L. Honig, and Wolfgang Utschick, "Distributed interference pricing for the MIMO interference channel," in *Proc. IEEE Int'l Communications Conf. (ICC)*, Dresden, Germany, 2009.

[10] S.S. Christensen, R. Agarwal, E. Carvalho, and J. Cioffi, "Weighted sum-rate maximization using weighted MMSE for MIMO-BC beamforming design," *IEEE Trans. on Wireless Communications*, vol. 7, no. 12, pp. 4792–4799, December 2008.

[11] M. Joham, and K. Kusume, and M. H. Gzara, and W. Utschick, and J. A. Nossek,, "Transmit Wiener Filter," Tech. Rep., Munich University of Technology, January 2002.

[12] F. Negro, S. P. Shenoy, I. Ghauri, and D.T.M. Slock, "Interference alignment feasibility in constant coefficients MIMO interference channel," in *Proc. 11th IEEE International Workshop on Signal Processing Advances in Wireless Communications (SPAWC 2010)*, June 2010.

[13] C. Hellings, D. A. Schmidt, and W. Utschick, "Optimized beamforming for the two stream MIMO interference channel at high SNR," *in Proc. International ITG Workshop on Smart Antennas - WSA 2009*, February 2009.

[14] Hakjea Sung, Seok-Hwan Park, Kyoung-Jae Lee, and Inkyu Lee, "Linear precoder designs for k-user interference channels," *Wireless Communications, IEEE Transactions on*, vol. 9, no. 1, pp. 291 –301, jan. 2010.