INTERFERENCE ALIGNMENT FEASIBILITY IN CONSTANT COEFFICIENT MIMO INTERFERENCE CHANNELS

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ABSTRACT

We consider the K-link constant coefficient multiple-input-multiple-output interference channel (MIMO IFC) where inter-link interference is treated as Gaussian noise (Noisy MIMO IFC). Starting from Interference Alignment (IA) constraints, analytical conditions that need to be satisfied in order to admit an IA solution for such a MIMO IFC are derived. For a given *degrees of freedom* allocation, these conditions, along with a recursive algorithm to check its validity in a given K-link MIMO IFC, allow an analytical evaluation of the existence of IA solutions (or lack thereof). While such an attempt has been made recently for several interesting cases in published literature [1] [2], to the best of our knowledge, this is the first time that such conditions have been derived in full generality (including the multiple stream case). Moreover, for the cases addressed in [2] our approach evaluates the existence of IA solution with much lower complexity.

1. INTRODUCTION

Interference is being increasingly accepted as the major bottleneck limiting the throughput in wireless communication networks. Recent research [3] has however shown that at least in the high signal to noise ratio (SNR) regime interference does not fundamentally limit the channel capacity; that at high SNR the per-user capacity of an interference channel (IFC) with arbitrary number of users scales at half the rate of each user's interference-free capacity. Such a scaling was obtained in [3] using the concept of interference alignment (IA). The key idea behind IA is to process the transmit signal (data streams) at each TX so as to align all the undesired signals at each receiver (RX) in a subspace of suitable dimension. This alignment allows each RX to suppress more interfering streams than it could otherwise cancel. In fact, in the high SNR regime, simple zero-forcing (ZF) receivers suffice to separate the desired signal from the interferers. In a constant coefficient MIMO IFC (channel coefficients are constant over the transmit duration), the total number of streams contributing to the input signal at each RX are typically greater than the number of antennas available at the RX. Aligning the streams at the TX allows each RX to cancel more steams than the number of "spare antennas" at its disposal. Thus underscoring the importance of IA in the high-SNR regime since IA maximizes the sum-capacity pre-log factor, the so called total *degrees of freedom* (dof) for a given antenna distribution in the K-link Noisy IFC (inter-link interference is treated as Gaussian noise) when the processing at the TX and RX is constrained to be linear.

The capacity of an IFC in general has been an open problem for long. Till date, the best result is [4] for the 2-user Gaussian IFC. For $K > 2$, the problem is even more complicated. This has lead to an alternative line of attack; that of characterizing the capacity region in terms of the total dof in the high SNR regime. Such a characterization, assuming time-varying channels was provided in [5] with linear precoders and in [6] where non-linear precoders were considered for the constant coefficient channel. However, the dof characterization for the K-user constant coefficient MIMO IFC with linear processing is still an open problem.

In a K-link MIMO IFC where the k-th link is characterized by a TX with M_k antennas, a RX with N_k antennas and a requirement of d_k independent streams to be communicated over the k-th link, the existence of an IA solution is not known. Numerical solutions in [7] [8] can be used to evaluate their existence through simulations. The feasibility of IA solutions for a constant coefficient MIMO IFC was studied in [1] [2]. In [2], when $d_k = 1\ \forall k$, a MIMO IFC with a given distribution of TX/RX antennas is classified as *proper* or *improper*. All proper systems are almost surely (a.s) feasible. For a system to be proper, it is required that, for *every subset* of equations that arise due to the IA constraints, the number of variables be at least equal to the number of equations in that subset. This condition (that the system be proper) is sufficient but may not be necessary. Moreover, such a classification can be computationally expensive even for systems with relatively small number of transmit and receive antennas. Furthermore, for an arbitrary dof allocation amongst users (d_k not constrained to be 1), additional outerbounds need to be satisfied for a system to be feasible. It turns out however, that for multi-stream transmission, conformance with the outerbounds do not necessarily provide insight into the feasibility of an IA solution. In other words, an IA solution is not guaranteed if the outerbounds are satisfied. An example follows: For a $K = 3$ user MIMO IFC where $d_k = 2\ \forall k$, $M_1 = N_1 = 4$, $M_2 = 5$, $N_2 = 3$, and $M_3 = 6$, $N_3 = 2$, the outerbounds (cf. (21) in [2]) are satisfied. However, the system does not admit an IA solution.

In this paper we build on earlier results in [1] and propose a systematic method to check the feasibility of IA solutions for a given K-link Noisy MIMO IFC and an arbitrary dof allocation. Throughout this paper, when we refer to a K-link MIMO IFC, we mean the K-link constant coefficient Noisy MIMO IFC.

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2. SYSTEM MODEL

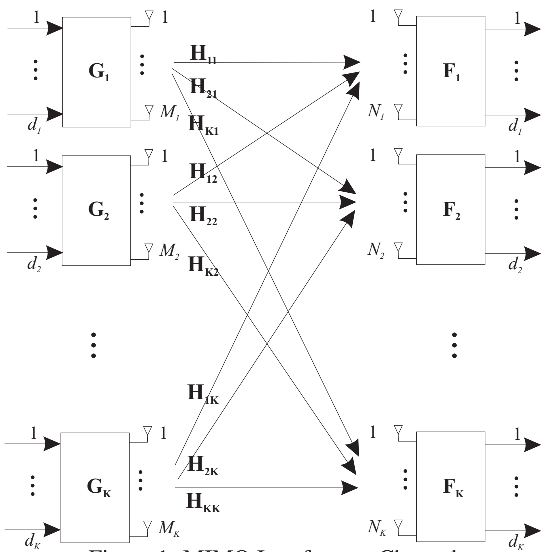

Figure 1: MIMO Interference Channel

Fig. 1 depicts a typical K-link MIMO IFC with K TX-RX pairs. The k-th TX and the k-th TX are equipped with M_k and N_k antennas respectively. The k-th TX generates interference at all $l \neq k$ receivers. Assuming a constant coefficient channel, the $\mathbb{C}^{N_k \times 1}$ received signal $\mathbf{y}_k$ at the k-th RX can be represented as

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{x}_k + \sum_{\substack{l=1 \\ l \neq k}}^{K} \mathbf{H}_{kl}\mathbf{x}_l + \mathbf{n}_k,$$

where $\mathbf{H}_{kl} \in \mathbb{C}^{N_k \times M_l}$ represents the channel matrix between the l-th TX and k-th RX, $\mathbf{x}_k$ the $\mathbb{C}^{M_k \times 1}$ transmit signal vector corresponding to the k-th TX and the $\mathbb{C}^{N_k \times 1}$ vector $\mathbf{n}_k$ represents the additive white Gaussian noise with zero mean and covariance matrix $\mathbf{R}_{n_k}$. Each entry of the channel matrix is a complex random variable drawn from a continuous distribution without any deterministic relation between channel coefficients. We assume centralized processing with complete knowledge of all direct-link and cross-link channel matrices on the transmit side.

Let $\mathbf{G}_k$ denote the $\mathbb{C}^{M_k \times d_k}$ beamforming matrix of the k-th TX. Then $\mathbf{x}_k = \mathbf{G}_k\mathbf{s}_k$, where the $d_k \times 1$ vector $\mathbf{s}_k$ represents the transmitted symbols and d_k the number of independent streams transmitted to its RX. We assume $\mathbf{s}_k$ to have a Gaussian distribution with $\mathcal{N}(0, \mathbf{I}_{d_k})$. At the k-th RX, $\mathbf{F}_k \in \mathbb{C}^{d_k \times N_k}$ is applied to suppress interference and retrieve the d_k desired streams. Applying the interference suppressing filter $\mathbf{F}_k$ to $\mathbf{y}_k$, we obtain the following $d_k \times 1$ vector $\mathbf{r}_k$

$$\mathbf{r}_k = \mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k\mathbf{s}_k + \sum_{\substack{l=1 \\ l \neq k}}^{K} \mathbf{F}_k\mathbf{H}_{kl}\mathbf{G}_l\mathbf{s}_l + \mathbf{F}_k\mathbf{n}_k.$$

3. INTERFERENCE ALIGNMENT FEASIBILITY

The objective in IA is to design aligning matrices to be applied at the transmitters such that, the interference caused by all transmitters at each non-intended RX lies in a common *interference subspace*. Moreover, the interference subspace and the *desired signal subspace* of each RX should be non-overlapping (linearly independent). If alignment is complete, simple ZF can be applied to suppress the interference and extract the desired signal in the high-SNR regime. Since IA is a condition for joint transmit-receive linear ZF, we need to satisfy the following conditions:

$$\mathbf{F}_k\mathbf{H}_{kl}\mathbf{G}_l = \mathbf{0} \quad \forall l \neq k \tag{1}$$

$$\text{rank}(\mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k) = d_k \quad \forall k \in \{1, 2, \ldots, K\} \tag{2}$$

In addition, the traditional single user MIMO constraint $d_k \leq \min(M_k, N_k)$ also needs to be satisfied for d_k streams to be able to pass over the k-th link. The first step towards analytical evaluation of the existence of an IA solution for a given dof allocation in a K-link MIMO IFC is the translation of the above equations into a set of conditions that need to be satisfied to admit an IA solution. To this end, the approach we adopt in this paper is of formulating the given IA problem as finding a solution to a system of equations with limited number of variables dictated by the dimensions of the overall system (the M_ks, N_ks and d_ks of the MIMO IFC). Fig. 2 presents a pictorial representation of such a system of equations where the block matrices $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{G}$ on the left hand side (LHS) of the equality represent respectively, the ZF RX, overall channel matrix and beamformers. The block diagonal matrix to the right hand side (RHS) of the equality represents the total constraints in the system that need to be satisfied for an IA solution to exist. The block matrices on the diagonal of $\mathbf{H}$ represent the direct-links and the off diagonal blocks in any corresponding block row k represent the cross channels of the k-th link. The interference aligning beamformer matrix $\mathbf{G}_k$ (the diagonal blocks in $\mathbf{G}$) aligns the transmit signal of the k-th user to the interference subspace at all $l \neq k$ users while ensuring the rank of the equivalent channel matrix $\mathbf{F}_k\mathbf{H}_{kk}\mathbf{G}_k$ is d_k. In other words, in Fig. 3, the $\mathbf{G}_k$ matrices are designed such

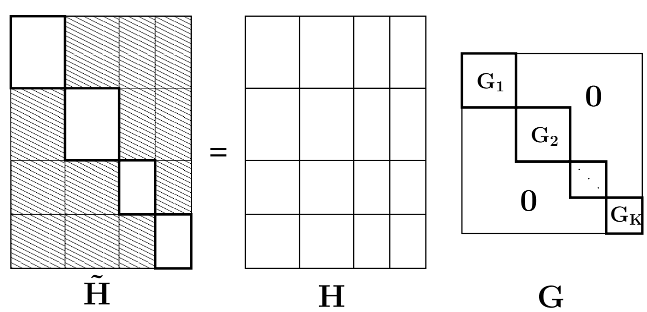

Figure 3: Interference alignment at all receivers .

that premultiplication of the overall beamformer matrix $\mathbf{G}$ with the overall channel matrix $\mathbf{H}$ results in a block matrix $\widetilde{\mathbf{H}}$ in which, all the off-diagonal blocks in any block row k (the shaded blocks of each block row) share a common column space whose dimension is at most $(N_k - d_k)$. With this accomplished, $\mathbf{F}_k$ simply projects the received signal into a subspace orthogonal to the interference subspace to retrieve the desired signal at the k-th RX resulting in a $(d_k \times d_k)$ matrix (the rank d_k equivalent channel) for its desired streams and $(K-1)$ block-zero matrices in the k-th block row of the matrix to the right.

The only requirement on the $(d_k \times d_k)$ matrix that mixes up the desired streams is that it be of full rank. The beamforming matrix therefore, is determined up to an arbitrary $(d_k \times d_k)$ square matrix. Thus, of the total number of $(M_k \times d_k)$ variables available for the design of $\mathbf{G}_k$ matrix, transmission of d_k independent streams results in an immediate loss of d_k^2 variables thus reducing the total number of variables available for the design of an interference aligning beamformer at each TX to $d_k(M_k - d_k)$. The reason for evaluating

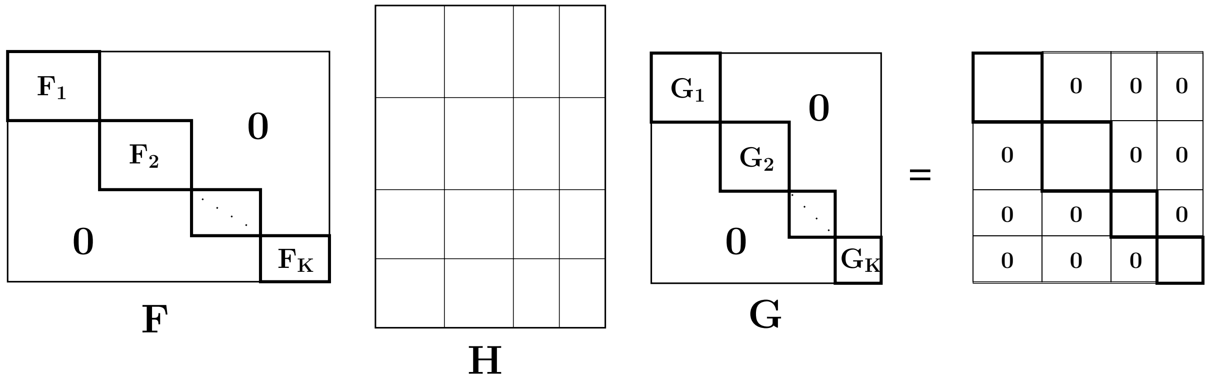

Figure 2: Block matrix representation of the interference alignment problem.

the number of variables available at the TX is the nature of the IA problem. The IA scheme essentially requires that all alignment is done at the TX. Therefore every TX imposes a set of constraints on the entire system (as a consequence of alignment conditions at each non intended RX) whenever it transmits a stream to its RX. Thus, an IA solution will be feasible only if the total number of variables available in the system is greater than or equal to the total number of constraints to be satisfied. Moreover, the variables should be distributed appropriately at each of the TX. In the sequel, we provide a systematic method of counting the number of variables available for the design of an interference aligning beamformer at each TX and comparing them with the constraints imposed on the system by each TX. This method can be seen as arriving at the K-link MIMO IFC for which the existence of an IA solution is to be analyzed, by successively adding a single TX and computing the total number of variables available for the joint design of the interference aligning beamformers at the transmitters and comparing it against the total number of alignment constraints imposed by the TX (due to its d_k streams) at each step of this build-up.

The main idea of our approach is to convert the alignment requirements at each RX into a rank condition of an associated interference matrix. At RX k, the interference due to all other $(K-1)$ transmitters is grouped into a $(N_k \times \sum_{l=1;l\neq k}^{K} d_l)$ matrix

$$\mathbf{H}_I^{[k]} = [\mathbf{H}_{k1}\mathbf{G}_1, \ldots \mathbf{H}_{k(k-1)}\mathbf{G}_{(k-1)}, \mathbf{H}_{k(k+1)}\mathbf{G}_{(k+1)}, \ldots \mathbf{H}_{kk}\mathbf{G}_K],$$

that spans the interference subspace at the k-th RX. We call this the interference matrix at user k. The total signal-space dimension at RX k is given by the total number of receive antennas N_k, of which d_k interference-free signaling dimensions are to be reserved for the signal from the k-th TX. This is achieved when the interference from all other transmitters lies in an independent subspace whose dimension can be at most $(N_k - d_k)$. Thus the dimension of the subspace spanned by the matrix $\mathbf{H}_I^{[k]}$ must satisfy

$$\operatorname{rank}(\mathbf{H}_I^{[k]}) = r_I^{[k]} \leq N_k - d_k \tag{3}$$

While the above equation prescribes an upperbound for the rank of the interference matrix, the nature of the channel matrix (full rank property) combined with the rank requirement of the beamformer at each TX ($\operatorname{rank}(\mathbf{G}_k) = d_k$) specifies the following lower bound on $r_I^{[k]}$

$$r_I^{[k]} \geq \max_{l\neq k}(d_l - [M_l - N_k]_+) \tag{4}$$

where $[x]_+ = \max(0, x)$ and $[M_l - N_k]_+$ discounts the possibility of the columns of $\mathbf{G}_l$ belonging to the orthogonal complement of $\mathbf{H}_{kl}$. Forcing the rank of $n \times m$ matrix $\mathbf{A}$ to some $r \leq \min(m, n)$ implies imposing $(n-r)(m-r)$ constraints. We explain this briefly as follows. Without loss of generality, (w.l.o.g) assume that the columns of this $n \times m$ matrix are partitioned into $\mathbf{A} = [\mathbf{A}_1; \mathbf{A}_2]$ where $\mathbf{A}_1$ is $n \times r$ and is of full column rank. Then imposing a rank r on $\mathbf{A}$ implies that $\mathbf{A}_2$ shares the same column space as $\mathbf{A}_1$ which in turn implies that $\mathbf{A}_1^{\perp T}\mathbf{A}_2 = \mathbf{0}$. Since $\mathbf{A}_1^{\perp}$ is $n \times (n-r)$, it follows that $(n-r)(m-r)$ constraints need to be satisfied for $\mathbf{A}$ to be of rank r. Thus imposing a rank $r_I^{[k]}$ on $\mathbf{H}_I^{[k]}$ implies imposing

$$(N_k - r_I^{[k]})(\sum_{\substack{l=1\\ l\neq k}}^{K} d_l - r_I^{[k]})$$

constraints at RX k. $r_I^{[k]}$ is maximum when the interference contribution of each interferer spans an independent subspace. Which leads us to the upper bound $r_I^{[k]} \leq \sum_{l=1;l\neq k}^{K} d_l$. However, accounting for the inequality in (3) we have

$$r_I^{[k]} \leq \min(d_{tot}, N_k) - d_k \tag{5}$$

where $d_{tot} = \sum_{k=1}^{K} d_k$, and $\min(.)$ operation appears in the above equation due to the fact that the rank of $\mathbf{H}_I^{[k]}$ cannot exceed its dimensions.

4. RECURSIVE PROCEDURE TO EVALUATE FEASIBILITY

In this section we detail a recursive method of evaluating the feasibility of an IA solution for a MIMO IFC and a corresponding dof distribution. As mentioned earlier, the main idea here is to interpret the interference alignment requirement at each RX as forcing a certain rank on the associated interference channel $\mathbf{H}_I^{[k]}$ which in turn imposes a certain number of constraints on the IA problem. In the earlier section we show that this rank is bounded above and below by the system parameters. The first step therefore is to ensure that the range of each r_i is non-empty. From (3) and (4), this amounts to checking if

$$(\min(d_{tot}, N_k) - d_k) - \max_{j\in\mathcal{K}-\{k\}}(d_j - [M_j - N_k]_+) \geq 0 \quad \forall k \in \mathcal{K} \tag{6}$$

where $\mathcal{K} = \{1, 2, \ldots K\}$. Indeed, an IA solution is immediately ruled out if (6) is not true. This is due to the fact that the full rank nature of the cross channel $\mathbf{H}_{kj}$ will ensure that the minimum rank of $\mathbf{H}_I^{[k]}$ due to $j \neq k$ will be d_j unless it possesses a null space of non zero dimension in which case it can shrink the rank by a maximum of $[M_j - N_k]_+$. (6) can be interpreted as check for the minimum values of M_k and N_k $\forall k$ for a given dof allocation.

Proposition: Let $\mathcal{M}_K = \{\{M_k\}, \{N_k\}, \{d_k\}\}$ *represent a* K*-link MIMO IFC where* $\{M_k\}$ *and* $\{N_k\}$ *represent the ordered set of transmit and receive antennas of each user in the system and* $\{d_k\}$ *is the ordered set of the associated dof desired for each user (ordering is by user index). Denote by* $\mathcal{K}_o$ *the ordered set of users with decreasing* d_k *such that users with equal* d_k*s are ordered according to increasing* M_k*. Similarly, define* $\mathcal{M}^{'}_K$ *to be the MIMO IFC and the associated set* $\mathcal{K}^{'}_o$ *obtained by interchanging* $\{M_k\}$ *and* $\{N_k\}$*. Then an IA solution exists if both of the following conditions are satisfied:*

1. (6) holds true for $\mathcal{M}_K$ and $\mathcal{M}^{'}_K$
2. Starting from a system consisting only of the K receivers, if the complete system $\mathcal{M}_K$ (respectively $\mathcal{M}^{'}_K$) is "built" by successively adding one TX at a time from $\mathcal{K}_o$ (respectively $\mathcal{K}^{'}_o$) and (7) is valid (satisfied) at each step of this "build-up".

The need to satisfy both the above conditions for $\mathcal{M}_K$ *and* $\mathcal{M}^{'}_K$ arises due to the alignment duality. From the IA conditions in (1) (2), it is clear that taking the transpose of these equations results in IA conditions for the *dual* MIMO IFC and the same existence conditions should be satisfied for this dual MIMO IFC as well. At each step k of the recursion, (7) accumulates the total number of variables available for designing an IA solution in an associated sub-problem comprising of a k-link MIMO IFC where only k transmitters are transmitting non-zero streams and aligning their streams into some interference subspace of all non-intended receivers in the LHS of (7). The RHS accumulates the total number of constraints at all receivers that arise due to these transmitters. That the number of variables contributed by the i-th TX is given by $d_i(M_i - d_i)$ is obvious from the discussion in the previous section. We now elaborate on the method of obtaining the constraints on the RHS of (7). Forcing a rank on $\mathbf{H}_I^{[k]}$ amounts to satisfying a number of constraints that is a function of the rank and the dimensions of $\mathbf{H}_I^{[k]}$. While we do not have knowledge of the exact rank of $\mathbf{H}_I^{[k]}$ at each k (since that will be the result of the IA design whose feasibility we are evaluating in the first place) we do know the numerical *range* of $r_I^{[k]}$ for each k. Therefore, instead of using the actual rank it is useful to use its upperbound (denoted by $\overline{r}_I^{[k]}$, as specified in (6). On the RHS of (7) the first summation reflects the total number of constraints to be satisfied for an IA solution to exist in a k-link MIMO IFC with k-links transmitting a total of $\underline{d} = \sum_{i=1}^{k} d_i$ streams. For each user i accounted for in this summation, we have to ensure that at RX-i, $r_I^{[i]} \leq (N_i - d_i)$. The column dimension of $\mathbf{H}_I^{[i]}$ is $(\underline{d} - d_i)$. In order to minimize the total number of constraints that we impose of the system (due to the act of forcing a particular $r_I^{[i]}$ at the i-th RX), we choose the maximum possible rank of $r_I^{[i]}$, which we know to be $\min(\text{column dimensions}, N_i - d_i)$ i.e., $\overline{r}_I^{[i]} = \min(\underline{d} - d_i, N_i - d_i)$. The second summation consists of all "un-paired" receivers in the sub-problem i.e., those receivers whose corresponding transmitters are presently not transmitting any streams but still need $\underline{d}$ streams to be aligned in their interference subspace. Therefore, the maximum allowable rank of the interference matrices for all these receivers is $\overline{r}_I^{[i]} = \min(\underline{d}, N_i - d_i)$. Thus, (7) when true at each step, verifies that the number of variables available for the design of IA beamformers at all k transmitters is greater than the number of constraints that are imposed by an IA solution. In fact, it verifies that its is possible to align all the interference not just in the associated k-link MIMO IFC but also in the interference subspace of all un-intended receivers that are not in the k-link MIMO IFC (the un-paired receivers accounted for in the second summation). Finally, the ordering of the users in terms of increasing d_k in $\mathcal{K}_o$ ($\mathcal{K}^{'}_o$ for $\mathcal{M}^{'}_K$) ensures early identification of in-feasibility of an IA solution since a larger dof requirement typically results in smaller number of variables available at the TX in order to meet the rank constraints.

In the next section we present numerical examples to show that our approach is able to check the feasibility (or in-feasibility) of an IA solution for a given MIMO IFC. For a $\mathcal{M}_K$ which conforms to both the conditions of our approach, we are able cross validate that an IA solution exists using an iterative algorithm proposed in [8]. Indeed, it can be shown that the algorithm in [8] will always converge to an optimum solution when our conditions are met since convergence to an optimum solution implies that the d_k minimum eigenvalues of $\sum_{i \neq k} \mathbf{H}_{ki}\mathbf{G}_i\mathbf{G}_i^H\mathbf{H}_{ki}^H$ are zero which will be true if $\text{rank}(H_I^{[k]}) \leq \min(d_{tot}, N_k) - d_k$ which is a part by our systematic approach.

5. NUMERICAL EXAMPLES

In this section we provide some numerical examples to validate the conditions derived in this paper. In all the examples given in this section, when the MIMO IFC that satisfied the conditions in Sec. 4, the numerical algorithm in [8] was able to find an IA solution whereas it failed to find one when these conditions were not satisfied[1].

Example 1: Consider a 2-link MIMO system with $M = 2, N = 4, d = 2$. This system satisfies the 2 conditions in Sec. 4 and IA solutions do exist for this system.

Example 2: Similarly, the 6 user case where $M_k = 3, N_k = 4, d_k = 1 \forall k$, both conditions in Sec. 4 are satisfied and an IA solution is possible for this case.

Example 3: There exists an IA solution for $\mathcal{M}_3$ where $\{M_k\} = \{3, 1, 10\}, \{N_k\} = \{4, 3, 4\}, \{d_k\} = \{2, 1, 2\}$ and it can be shown that indeed, it satisfies the conditions in the previous section.

Example 4: We now look at another 2-link MIMO system with $M_1 = 4, N_1 = 7, d_1 = 3, M_2 = 10, N_2 = 4, d_2 = 2$. For this system, the rank conditions are not satisfied and indeed, there is no IA solution for this case.

Example 5: In the 4-link case characterized by $M_k = 2, N_k = 3$ $k = 1, 2, 3$ and $M_4 = N_4 = 2$ $d_k = 1 \forall k$. The rank conditions are satisfied but (7) is not satisfied. Therefore we conclude that there cannot be an interference alignment solution for this system.

6. ALTERNATIVE ZERO FORCING APPROACH TO IA

Another possible approach to determine if a K-link MIMO interference channel has an IA solution can be obtained interpreting interference alignment as joint transmit-receive linear zero forcing. The idea is that a stream transmitted from TX k and causes interference to the non intended RX j can be suppressed at either the TX or at the RX. Denoting with t_{kj} the size of the subset of streams d_k, that are received at RX j that the k-th TX suppresses, and with r_{kj} the size of the subset of streams d_k, that are received at RX j, that the j-th RX suppresses, the sum of these two quantities should be: $t_{kj} + r_{kj} \geq d_k$. The total number of streams that TX k can suppress is at most $M_k - d_k$ and the total number of streams that the j-th RX can suppress is not greater than $N_k - d_k$. Therefore, to check the feasibility of an interference alignment solution, the following

[1]In addition to these, we tested our conditions extensively with varied antenna and stream distributions. We do not provide these examples here due to space constraints. In particular, all the examples in [2] we also tested.

$$\sum_{i=1}^{k} d_i(M_i - d_i) \geq \sum_{i=1}^{k} (N_i - \underbrace{\min(\underline{d} - d_i, (N_i - d_i))}_{\overline{r}_I^{[i]}})(\underline{d} - d_i - \min(\underline{d} - d_i, (N_i - d_i))) + \sum_{i=k+1}^{K} (N_i - \underbrace{\min(\underline{d}, (N_i - d_i))}_{\overline{r}_I^{[i]}})(\underline{d} - \min(\underline{d}, (N_i - d_i))) \quad (7)$$

conditions should be satisfied:

$$\begin{array}{ll} \sum_{j \neq k} t_{kj} & \leq M_k - d_k \\ \sum_{k \neq j} r_{kj} & \leq N_j - d_j \end{array} \quad (8)$$

$$\forall t_{kj}, r_{kj} \in \{0, 1 \ldots, d_k\}, \text{ and } t_{kj} + r_{kj} = d_k$$

$$\max_{k \neq j}(d_j - [M_k - N_j]) \leq (N_j - d_j) \forall j \in \{1, \ldots, K\}$$

As before, due to alignment duality, (8) must be true when M_k and N_k values are interchanged (the dual channel case). One possible way to verify if all this inequalities are satisfied or not is to check all the possible $\prod_{k=1}^{K}(d_k + 1)^{K-1}$ combination of t_{kj} and r_{kj}. If there is at least a combination that satisfies the constraints, that one corresponds to the interference alignment solution.Such an alternate approach has some interesting implications.

Example 6: Consider $\boldsymbol{\mathcal{M}}_3 = \{\{M_k\} = \{N_k\} = \{1, 3, 6\}, \{d_k\} = \{1, 2, 3\}\}$. w.l.o.g., order the users in terms of increasing d_k, then, the first user pair is in no position to do anything. However, $\mathbf{G}_2$ can be designed to suppress interference caused at the RX of user-1 and $\mathbf{G}_3$ can be designed to suppress interference caused at the receivers of users 1 and 2. Similarly, $\mathbf{F}_2$ can suppress interference generated by user-1 while $\mathbf{F}_3$ can be designed to suppress interference from transmitters of user-1 and user-2. Thereby enabling reception of d_k interference free streams $\forall k$ user pairs. More interestingly, based on the structure of the above problem, we have the following conjecture that draws attention to the benefits of systems with unequal stream distributions.

Conjecture: There exists a MIMO IFC $\boldsymbol{\mathcal{M}}_K^{(u)}$ with unequal antenna and stream distribution for any given network dof d_{tot}, such that the total number of antennas in $\boldsymbol{\mathcal{M}}_K^{(u)}$, $\mathcal{A}_{tot}^{(u)} = \sum_k (M_k + N_k)$, required to achieve d_{tot} is less than the total number of antennas in $\boldsymbol{\mathcal{M}}_K^{(e)}$ where $M_k = M, N_k = N, d_k = d_{tot}/K\ \forall k$. $\boldsymbol{\mathcal{M}}_K^{(e)}$ is the so-called identical stream and antenna configuration (ISAC) [1] or symmetric [2] system.

The conjecture is motivated by the generalization of *Example 6* to any K-link system. Consider a K-link MIMO IFC with user pairs indexed in the order of increasing d_k. Let the following relationship hold.

$$d_{(k+1)} = d_k + 1,\ k \in 2, \ldots, K.$$

Then it can be shown that an IA solution exists if each user pair has the following antenna distribution:

$$M_k = N_k = \sum_{i=1}^{k} d_i,\ k \in \{1, \ldots, K\}.$$

Let $\mathcal{A}_{tot}^{(e)}$ represent the total number of antennas in an ISAC system $\boldsymbol{\mathcal{M}}_K^{(e)}$. We know from [1] [2] that, for $\boldsymbol{\mathcal{M}}_K^{(e)}$ the minimum number of antennas per-user needs to satisfy

$$M + N \geq (K + 1)\frac{d_{tot}}{K}.$$

It is easily verified that, for $K \geq 2$, $\mathcal{A}_{tot}^{(u)} < \mathcal{A}_{tot}^{(e)}$.
It is also possible to prove this starting from a given $\boldsymbol{\mathcal{M}}_K^{(e)}$ and splitting the d_{tot} into a dof allocation where not all users have the same dof.

7. CONCLUDING REMARKS

We considered the problem of analytically evaluating the feasibility of an interference alignment (IA) solution for a given degrees of freedom (dof) allocation in a general K-link MIMO IFC. We derived a set of conditions and presented a systematic method to check if these conditions are satisfied for a given MIMO IFC. We showed that, when an IA solution exists, these conditions are satisfied at every step of this systematic approach.We also show that an IA solution does not exist when these conditions are not satisfied.

8. REFERENCES

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