Rate-of-decay of probability of isolation in dense sensor networks with bounding constraints

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Abstract—**The work establishes the asymptotic rate of decay for the probability of node isolation in bounded wireless sensor networks, in the high density regime. In this regime, the exposition reveals the role of the most isolated neighborhoods of the bounding region in exponentially increasing the average probability of isolation. The problem is treated for a large family of random spatial distributions of nodes, random shapes of node coverage areas, and random topography of the network's bounding region. Different examples are presented to insightfully describe the detrimental effect of boundedness in network isolation. Finally we address different aspects relating to extremely isolating bounding regions, and densities that vary exponentially in time.**

Index Terms—**Wireless sensor network, node isolation, topography, network connectivity, boundary effects, large deviations.**

I. Introduction

We consider the setting of wireless multi-hop networks, where sets of randomly placed nodes cross-communicate in a decentralized manner. In this setting, *node isolation* describes the event where a node finds it self to be out of the range of communication of all other nodes. This event relates to unfortunate realizations of the nodes' random placement and random channels, and is generally seen as undesirable as it reduces the overall information that the network can communicate. Isolation also generally relates to the event of network disconnectedness where there exist pairs of nodes that cannot communicate, even in the presence of multi-hopping. The extend of isolation generally depends on the spatial distribution of nodes, their transmission range, and the topography of the bounding region, i.e., the region that defines the boundaries within which nodes may be placed.

In addition to the classical sensor network application (c.f. [1]), the setting relates to the general settings of information diffusion and epidemic networks (c.f. [2], [3]), where the network's propagation characteristics are a function of the isolation/connectivity properties, as well as a function of the bounding region representing natural or constructed alterations of the network topology.

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A. Prior Work

The early work of Cheng and Robertazzi [4] investigated the percolation depth in a network of nodes that are randomly distributed, according to a homogeneous Poisson point process, over an infinitely large area. Isolation and connectivity were also studied by Piret in [5], who focused on networks of nodes which are randomly distributed according to a uniform probability distribution, along a one-dimensional line segment. The work of Gupta and Kumar [1] then proceeded to provide a solution to the range assignment problem for RF wireless multi-hop dense networks with uniformly distributed nodes. Specifically, under the assumption of a circular *node coverage area,* and given a set of n nodes randomly deployed in a geographical region of fixed area, all having the same transmission range r, the work in [1] identified, in the regime of infinite n, the minimum value of r that almost-surely allows for network connectivity. Related work by Penrose [6] studied the k-*connectivity* of geometric random graph networks, deployed inside d-dimensional cubes, for $d \geq 2$. The work established that, in the same asymptotic setting of $n \to \infty$, the minimal r for which the graph is almost surely k-connected, is equal to the minimum r that almost-surely allows that each node has at least k neighbors. The work by Bettstetter and Zangl [7] calculates a lower bound on the probability that there exist no isolated nodes in a network, in the regime of $n >> 1$, under the assumption of a circular bounding region of radius R, of identical circular node coverage regions of radius $r << R$, and for nodes randomly distributed according to a homogeneous Poisson point process. The work also presents pairs of (r, n) that almost surely allow for a fully connected network. Similar work in [8] focuses on rectangular/square bounding regions and randomly shaped coverage areas.

B. Results

The current work introduces simple and insightful expressions on the probability of node isolation in bounded networks. Specifically the current work studies the asymptotic rate of decay of the probability of node isolation in statistically homogeneous networks of nodes. This is done in the setting of asymptotically high n, or equivalently, of asymptotically high node density, and is achieved for a large family of node-distribution statistics, random shapes of node coverage areas, and random topography of the network's bounding region. A defining novelty is the emphasis given in establishing the rate with which isolation appears, rather than establishing if non-isolation is achieved with probability 1. The approach allows for insight on how increasing resources such as node density as well as volume and shape of the nodes' covering region, affect the node-isolation performance of the network. It also allows for meaningful interpretations of the detrimental role that the bounding region has in increasing the probability of isolation.

C. Notation

We use $\mathbb{R}$ to denote the real numbers, $\mathbb{R}^d$ to denote the d-dimensional Euclidean space, and $V(C)$ to denote the volume of a set $C \subset \mathbb{R}^d$. We use $\doteq$ to denote *exponential equality*, where

$$f \doteq e^{-\rho B} \iff -\lim_{\rho\to\infty} \frac{\log f}{\rho} = B, \quad (1)$$

with $\dot{\leq}, \dot{\geq}$ being similarly defined. For a given event $\mathcal{E}$, we use $\mathbb{I}(\mathcal{E})$ as an indicator function defined as

$$\mathbb{I}(\mathcal{E}) = \begin{cases} 1 & \text{if } \mathcal{E} \text{ occurs} \\ 0 & \text{else.} \end{cases} \quad (2)$$

II. System Model

We proceed to describe the network's spatial node distribution, bounding region, and node coverage area. Let h denote a random realization of node placement, where h is drawn from a statistical distribution p_h. Define a randomly generated subset of points

$$\mathcal{X}^{'}(h) \subset \mathbb{R}^d,$$

parameterized by real parameters $t \geq 0$ and $\rho \geq 0$. The random distribution p_h generating $\mathcal{X}^{'}(h)$ is an arbitrary homogeneous random point process, also parameterized[1] by t, ρ, guaranteeing that the *expected* number of points of $\mathcal{X}^{'}(h)$ inside any area $A \subset \mathbb{R}^d$, $(\mathbb{E}_h[|\{\mathcal{X}^{'}(h) \cap A\}|])$, is a function of $V(A)$ and not of the shape or position of A. The parameter t will be interpreted as the time, and the parameter ρ as the density of $\mathcal{X}^{'}(h)$, defined as

$$\rho \triangleq \frac{\mathbb{E}_h\{|A \cap \mathcal{X}^{'}(h)|\}}{V(A)}, \quad (3)$$

which holds for any $A \subset \mathbb{R}^d$.

We now consider a possibly irregular region $S \subset \mathbb{R}^d$, of finite volume that is independent of ρ, and which is assigned the role of the bounding region, in that it defines the set of nodes

$$\mathcal{X}(h) \triangleq \mathcal{X}^{'}(h) \cap S, \quad (4)$$

that can be regarded as the set of active points inside a specific bounding region of interest (S). We note the subsequent use of the term *node* to specifically refer to elements of $\mathcal{X}(h)$, and the use of the term *point* to refer to non-node elements of $\mathbb{R}^d$.

For any $s \in \mathbb{R}^d$, we further define the *coverage region*

$$B_s \subset \mathbb{R}^d,$$

to be an arbitrary open set, and we ask that $s \in B_s$ and $V(B_s) < \infty$.

When $s \in \mathcal{X}(h)$, the intersection $B_s \cap S$ is regarded as the *effective coverage region* of the node s. We note that the fact that B_s is an open set ensures that $0 < V(B_s \cap S) < \infty$. Furthermore $V(B_s \cap S)$ is independent of ρ. We refrain from placing other conditions on B_s, keeping in line with the substantial variability of different deterministic or random channel models, the variability of different node capabilities, as well as the variability of bounding regions.

[1]For notational simplicity, this parameterization is not denoted.

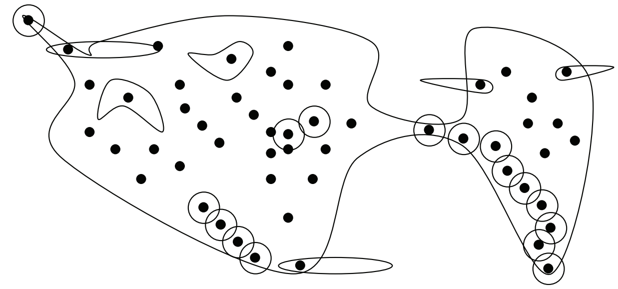

Fig. 1. Network of nodes, bounded by bounding region S. Different nodes are randomly placed at variably isolated neighborhoods of the bounding region, accepting variable degrees of isolation, due to variable intersections between their covering region and the bounding area.

A. Large Deviation Principle

For any $s \in S$, we define the *point degree*

$$k_s \triangleq |\{B_s \cap \mathcal{X}(h)\} \setminus s| \quad (5)$$

to describe the number of active nodes in the coverage region of s. In the high-density limit of $\rho \to \infty$, which we henceforth adopt, the random sequence

$$y_{s,\rho} \triangleq \frac{k_s - \mathbb{E}_h[k_s]}{\rho}, \quad (6)$$

tends towards zero, in the sense that

$$P(|y_{s,\rho}| < \epsilon) \to 1, \epsilon > 0,$$

and the probability $P(k_s = 0)$ that a point $s \in S$ is isolated, vanishes with increasing ρ. Note that the above probability is with respect h, i.e., with respect to the randomness in node placing. Henceforth all probabilities, unless explicitly stated otherwise, will be considered to be with respect to the randomness in node placing.

The aim of this work is to establish the rate of decay of such probabilities for increasing ρ. With this in mind, we ask that the random variable $y_{s,\rho}$ accept the *large deviation principle* (c.f. [9]), such that

$$-\lim_{\rho\to\infty} \frac{\log P(|y_{s,\rho}| > \epsilon)}{\rho} = I_0(\epsilon_s), \quad (7)$$

where $I_0(\epsilon_s) > 0, \quad \forall\ \epsilon_s > 0$. We note that I_0 is a function of p_h, whereas ϵ_s is a function of ϵ and B_s. As a result for any point $s \in S$, there exists a *rate function* I such that

$$-\lim_{\rho\to\infty} \frac{\log P(k_s = 0)}{\rho} = I(f_s(B_s \cap S)), \quad (8)$$

for some function $f_s : B \mapsto \mathbb{R}$, for any $B \subset S$. In accordance with standard properties of rate functions, we ask that I be non-decreasing, and that

$$f_s(B) > 0,\ I(f_s(B)) > 0$$

for all $B \subset S$ such that $V(B) > 0$. Finally we ask that B_s, f_s, I jointly satisfy the basic continuity property

$$\lim_{s_1 \to s_2} I(f_{s_1}(B_{s_1})) = I(f_{s_2}(B_{s_2})). \tag{9}$$

III. Asymptotic rate of decay of probability of node isolation in bounded networks

In the regime of asymptotically increasing density, and under the assumption that $y_{s,\rho}$ accepts the large deviation principle, we proceed to establish the probability of node isolation in the presence boundedness. The proofs follow afterward.

Lemma 1: For $2 \leq K < \infty$ and for any fixed $\epsilon > 0$, there exists an $I^{'} > 0$ such that

$$P(|\mathcal{X}(h)| \geq \epsilon\rho^K) \dot{\leq} e^{-\rho I^{'}}. \tag{10}$$

Theorem 1: Let

$$P_i \triangleq \mathbb{E}_h\{\mathbb{E}_{s\in\mathcal{X}(h)} \mathbb{I}(k_s = 0)\},$$

define the average probability of node isolation. Then the asymptotic rate of decay of P_i is given by

$$-\lim_{\rho\to\infty} \frac{\log P_i}{\rho} = \min_{s\in S} I(f_s(B_s \cap S)). \tag{11}$$

Theorem 1 reveals that in the high-density regime, the rate of decay of the probability of node isolation, averaged over the *nodes*, is dominated by the event of node isolation for a single node, where furthermore, that specific node must be assumed to occupy the most isolated neighborhood of the *bounding region*.

The following further reveals, in the asymptotic setting of interest, the role of boundedness in increasing the probability of node isolation in the network. We first define, for any $s \in S$,

$$k_s^{'} \triangleq |B_s \cap \mathcal{X}'(h) \setminus s|, \tag{12}$$

to be the number of elements in $\mathcal{X}'(h)$ that are inside the coverage region of s. Note that $k_s^{'}$ does not consider S.

Corollary 1a: Let the *boundedness exponent* be defined as

$$\mathcal{C}_S \triangleq \frac{\mathbb{E}_{s\in\mathcal{X}(h)}[\lim\limits_{\rho\to\infty}\frac{1}{\rho}\log P(k_s = 0)]}{\mathbb{E}_{s\in\mathcal{X}(h)}[\lim\limits_{\rho\to\infty}\frac{1}{\rho}\log P(k_s^{'} = 0)]}.$$

Then

$$\mathcal{C}_S = \frac{\min_{s\in S} I(f_s(B_s \cap S))}{\min_{s\in S} I(f_s(B_s))}. \tag{13}$$

The corollary concisely indicates that in the setting of interest, the network's isolation performance is entirely governed by the most isolated regions of the shaping region S. As expected, the unbounded case gives $\mathcal{C}_S = 1$.

We proceed with the proofs of the presented results.

Proof of Lemma 1: For any $2 \leq K < \infty$, (7) gives that

$$-\lim_{\rho\to\infty} \frac{\log P(|\mathcal{X}(h)| \geq \epsilon\rho^K)}{\rho} = -\lim_{\rho\to\infty} \frac{\log P(k_{s_1} \geq \epsilon\rho^K)}{\rho}$$

for any $s_1 \in S$ and $B_{s_1} = S$. From (6) we have that

$$y_{s_1,\rho} \triangleq \frac{k_{s_1} - \rho V(S)}{\rho}, \tag{14}$$

which implies that

$$\begin{aligned} -\lim_{\rho\to\infty} &\frac{\log P(|\mathcal{X}(h)| \geq \epsilon\rho^K)}{\rho} \\ &= -\lim_{\rho\to\infty} \frac{\log P(y_{s_1,\rho} \geq \epsilon\rho^{K-1} - V(s))}{\rho} \\ &\geq -\lim_{\rho\to\infty} \frac{\log P(y_{s_1,\rho} = \epsilon)}{\rho} > 0. \ \square \end{aligned} \tag{15}$$

Proof of Theorem 1: We seek to establish that

$$-\lim_{\rho\to\infty} \frac{\log \mathbb{E}_h\{\sum_{s\in\mathcal{X}(h)} P(s)\mathbb{I}(k_s = 0)\}}{\rho} = \min_{s\in S} I(f_s(B_s \cap S)).$$

We note that

$$\begin{aligned} P_i &= \mathbb{E}_h\{\sum_{s\in\mathcal{X}(h)} P(s)\mathbb{I}(k_s = 0)\} \\ &= \mathbb{E}_h\{\int_{s\in S} P(s)\mathbb{I}(k_s = 0)ds\} \\ &= \int_{s\in S} \mathbb{E}_h\{P(s)\mathbb{I}(k_s = 0)\}ds \\ &\leq \int_{s\in S} \mathbb{E}_h\{\mathbb{I}(k_s = 0)\}ds = \int_{s\in S} P(k_s = 0)ds \\ &\doteq e^{-\rho I(f_{s'}(B_{s'} \cap S))} \end{aligned} \tag{16}$$

where

$$s' := \arg\min_{s\in S} f_s(B_s \cap S) \in S, \tag{17}$$

and where $P(s)$ denotes the probability that, given a specific realization h, a certain node s is chosen among the nodes in $\mathcal{X}(h)$. Note that $P(s) = 0$ for any $s \notin \mathcal{X}(h)$. In the above we also note that the last asymptotic equality comes from Varadhan's lemma [10] and (8), and that s' is independent of h.

Towards meeting the above bound, let

$$s_0(h) := \arg\min_{s\in\mathcal{X}(h)} f_s(B_s \cap S), \tag{18}$$

denote the most isolated *node* in $\mathcal{X}(h)$, and let $\overline{s'} \in S$ denote the sphere of radius δ centered at s', where δ is an arbitrary fixed number independent of ρ. Then homogeneity gives

$$P(s_0(h) \in \overline{s'}) \geq \frac{V(\overline{s'})}{V(S)} \doteq e^{\rho 0} \doteq 1, \tag{19}$$

and the following holds

$$\begin{aligned} P_i &= \mathbb{E}_h\{\sum_{s\in\mathcal{X}(h)} P(s)\mathbb{I}(k_s = 0)\} \\ &\geq \mathbb{E}_h\{P(s_0(h))\mathbb{I}(k_{s_0(h)} = 0)\} \\ &\geq P(|\mathcal{X}(h)| \leq \rho^K) \\ &\quad \mathbb{E}_h\{P(s_0(h))\mathbb{I}(k_{s_0(h)} = 0, |\mathcal{X}(h)| \leq \rho^K)\} \\ &\qquad\qquad \text{for some finite } K, (2 < K < \infty) \\ &= (1 - e^{-\rho I'})\mathbb{E}_h\{\frac{1}{|\mathcal{X}(h)|}\mathbb{I}(k_{s_0(h)} = 0, |\mathcal{X}(h)| \leq \rho^K)\} \\ &\doteq \mathbb{E}_h\{\mathbb{I}(k_{s_0(h)} = 0, |\mathcal{X}(h)| \leq \rho^K)\} \\ &\geq P(s_0(h) \in \overline{s'}) \\ &\quad \mathbb{E}_h\{\mathbb{I}(k_{s_0(h)} = 0, s_0(h) \in \overline{s'}, |\mathcal{X}(h)| \leq \rho^K)\} \end{aligned}$$

$$\begin{aligned} &\doteq \mathbb{E}_h\{\mathbb{I}(k_{s_0(h)}=0, s_0(h)\in\overline{s'}, |\mathcal{X}(h)|\le\rho^K)\} \\ &\doteq e^{-\rho I(f_{s'}(B_{s'}\cap S))}. \end{aligned} \tag{20}$$

In the above, the fourth equality comes from Lemma 1 and the last equality is because, $V(\overline{s'})>0$ can be made arbitrarily small by sufficiently reducing δ, independently of ρ, which in turn gives that

$$I(f_s(B_s\cap S)) = I(f_{s'}(B_{s'}\cap S)),\ \ s\in\overline{s'}, \tag{21}$$

directly from the continuity properties in (9).

As a result

$$P_i \ \ \dot{\ge}\ \ P(k_{s'}=0)\doteq e^{-\rho I(f_{s'}(B_{s'}\cap S))}$$

and, in conjunction with (16) the result is established. □

Proof of Corollary 1a: The proof of the Corollary is direct from Lemma 1 and Theorem 1. □

A. Example cases

In the following we will seek to address specific clarifying example cases.

a) Example case 1 - Uniformly covered network and Poisson point process: We consider the *uniformly covered network* case where all nodes have identically shaped coverage areas, i.e.,

$$B_s = A+s,\quad s\in\mathbb{R}^d, \tag{22}$$

for some arbitrary shaped open set A, centered at the origin of $\mathbb{R}^d$. We then have the following.

Corollary 1b: For a uniformly covered network with average density ρ, and for p_h being a homogeneous Poisson point process, then

$$-\lim_{\rho\to\infty}\frac{\log P_i}{\rho} = \min_{s\in S} V((A+s)\cap S). \tag{23}$$

Proof of Corollary 1b: We recall the well known fact (c.f. [11]) that given a Poisson point process, the number of nodes within any finite subarea B_s, is given by

$$P(|\mathcal{X}(h)\cap B_s| = n) = \frac{(\rho V(B_s))^n}{n!}e^{-\rho V(B_s)}. \tag{24}$$

In the uniformly covered setting of interest where $B_s = A+s$, this implies that the probability of isolation for some $s\in S$, takes the form

$$\begin{aligned} P(k_s=0) &= (\rho V(s+A))e^{-\rho V((A+s)\cap S)} && (25)\\ &= (\rho V(A))e^{-\rho V((A+s)\cap S)} && (26)\end{aligned}$$

which gives

$$-\lim_{\rho\to\infty}\frac{\log P(k_s=0)}{\rho} = V((A+s)\cap S). \tag{27}$$

As a result

$$I(f_s(B_s\cap S)) = V((A+s)\cap S),$$

and

$$-\lim_{\rho\to\infty}\frac{\log P_i}{\rho} = \min_{s\in S} I(f_s(B_s\cap S)) = \min_{s\in S} V((A+s)\cap S). \tag{28}$$

□

b) Example case 2: - Uniform circular coverage, rectangular bounding region, and Poisson point process: The following illustrative example is specific to the setting of the uniformly covered network, the Poisson point process, and specific to the case where, as illustrated in Figure 2, each covering region $A+s$ is a circle of radius r (c.f. [1]), and finally where the bounding region S is a rectangle of length bigger than r. In this setting we can see that

$$s' \triangleq \arg\min_s I(f_s(B_s\cap S))$$

corresponds to the corner points of S, which allow for

$$V(B_{s'}\cap S) = \frac{1}{4}V(A)$$

which in turn shows that the rate of decay of the probability of node isolation, averaged over all the nodes, is given by

$$-\lim_{\rho\to\infty}\frac{1}{\rho}\log P_i = \frac{1}{4}\pi r^2. \tag{29}$$

In the same example, the effect of boundedness is easily seen from (13) to be

$$\begin{aligned} \mathcal{C}_S &= \frac{\min_{s\in S} I(f_s(B_s\cap S))}{\min_{s\in S} I(f_s(B_s))} \\ &= \frac{\min_{s\in S} V((A+s)\cap S)}{V(A)} \\ &= \frac{(\frac{\pi/2}{2\pi})\pi r^2}{\pi r^2} = \frac{\pi/2}{2\pi} = \frac{1}{4} \end{aligned}$$

showing how the probability of isolation is exponentially increased due to the presence of S.

c) Example case 3 - Polygon shaped S: Furthermore, it is easy to see that in the same uniformly covered network setting with Poisson point process, circular A, and a polygon S, then

$$\mathcal{C}_S = \frac{\phi_{\min}}{2\pi},$$

where $\phi_{\min}\in(0,\ \pi)$ is the minimum angle, formed by the vertices of S. Here $\phi_{\min}$ is seen to entirely define the effect of boundedness.

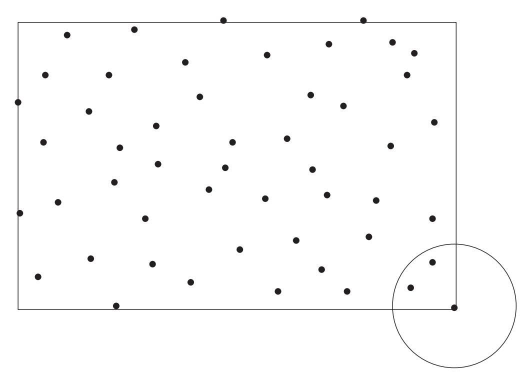

Fig. 2. Randomly distributed nodes, rectangular S and circular $s+A$.

B. Extreme topologies

Focusing on the uniformly covered setting, we here proceed to consider the case where certain neighborhoods of S exhibit extreme variability in shape, in the sense that there exist $s \in S$, such that

$$V((s+A)\cap S) = \rho^{-\alpha} V(A). \tag{30}$$

The following result sheds some light on the asymptotic rate of decay of the probability of isolation in such a setting, averaged over all nodes.

Theorem 2: In a uniformly covered network, under the shaping assumption of (30), the probability of isolation scales with $\rho^{1-\alpha_{\max}}$ (rather than scaling with ρ), in that

$$-\lim_{\rho\to\infty} \frac{\log P_i}{\rho^{1-\alpha_{\max}}} = K,$$

for some constant K, where

$$\alpha_{\max} \triangleq \arg\max_{\alpha\geq 0}\{-\lim_{\rho\to\infty}\frac{1}{\rho}\log P(s{:}\frac{V(s+A\cap S)}{V(A)} \doteq \rho^{-\alpha}) = 0\}.$$

Proof of Theorem 2: Let

$$-\lim_{\rho\to\infty}\frac{1}{\rho}\log P(s:\frac{V((s+A)\cap S)}{V(A)} \doteq \rho^{-\alpha}) = \gamma(\alpha),$$

then $\gamma(\alpha) = 0$ for some $\alpha \geq 0$ since

$$\int_0^\infty P(s:\frac{V((s+A)\cap S)}{V(A)} \doteq \rho^{-\alpha}) d\alpha = 1.$$

The above considers the fact that for $\alpha > 0$ then $\rho^{-\alpha}$ can be made arbitrarily small, whereas $V(S)$ is fixed and finite. Note that the above probability is taken over the different parts of a specific region S, and is not a function of h. Let

$$\mathcal{A} \triangleq \{\alpha \geq 0 : \gamma(\alpha) = 0\}$$

define the typical set of degrees of isolation among the elements of S.

It is then the case that Varadhan's lemma [10] gives

$$\begin{aligned}
&\mathbb{E}_h\big\{\int_0^\infty P(s:\frac{V((s+A)\cap S)}{V(A)} \doteq \rho^{-\alpha})\mathbb{1}(k_s=0,\alpha)d\alpha\big\}\\
&\doteq \mathbb{E}_h\big\{\int_{\alpha\in\mathcal{A}} P(s:\frac{V((s+A)\cap S)}{V(A)} \doteq \rho^{-\alpha})\mathbb{1}(k_s=0,\alpha)d\alpha\big\}\\
&\doteq \mathbb{E}_h\big\{P(s:\frac{V((s+A)\cap S)}{V(A)} \doteq \rho^{-\alpha_{\max}})\mathbb{1}(k_s=0,\alpha_{\max})\big\}
\end{aligned}$$

which concludes the proof. $\square$

C. Time Variations

We here note that the model allows for partial exposition of the case where the average density conditions in the network, vary with time. In practice this variability can relate to environmental changes or node aging. The following gives some insight on the effect of time varying node density on the probability of node isolation. For ease of exposition, we again focus on the uniformly covered setting, symmetric $B_s = s + A$, and polygon S, a volume-related rate function, and for illustrative purposes we focus on the case where the density decays exponentially with time.

Proposition 1: Let

$$I(f_s(\Gamma)) = V(\Gamma),\ \forall\Gamma \in \mathbb{R}^d, \forall s \in S,$$

let $B_s = A + s$, let A be symmetrically shaped, let S be any irregular polygon with smallest angle $\phi_{\min}$ and smallest side bigger than the span of A, and let the average density $\rho(t)$ of the uniformly covered network be such that

$$\lim_{\rho\to\infty}\frac{\rho(t)}{\rho^{1-t\beta}} = D(t) < \infty,$$

for some $\beta > 0$. Then

$$-\lim_{\rho\to\infty}\frac{1}{\rho^{1-t\beta}}\log P_i = \frac{\phi_{\min}}{2\pi}D(t)V(A). \tag{31}$$

The proof follows easily from Theorem 1.

IV. Conclusions

The work introduced simple characterizations of the probability of node isolation in bounded dense networks. The exposition holds for a general setting of node-distribution statistics, and for general shapes of coverage and bounding regions. Emphasis was given in establishing the rate with which isolation appears, rather than establishing if non-isolation is achieved with probability 1.

In addition to the classical sensor network application, the exposition relates to the general settings of information diffusion or epidemic networks, where the network's propagation characteristics are a function of the isolation/connectivity properties of the network, and of the bounding region which represents natural or constructed alterations of the network topology.

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