

# Impact of channel-state information on coded transmission over fading channels with diversity reception

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## Abstract

We study the synergy between coded modulation and antenna-diversity reception on channels affected by slow Rician fading. Specifically, we assess the impact of channel-state information (CSI) on error probability. We show that with a good coding and constant envelope modulations (like PSK) scheme the loss in performance when CSI is not available is moderate (around 1.5 dB). Moreover, as the diversity order grows, the channel tends to become Gaussian.

## 1 Introduction

In a recent paper [2], the authors have studied the synergy between coded modulation and antenna-diversity reception on channels affected by slow Rician fading. System performance was analyzed under the assumption of perfect channel-state information (CSI) at the receiver. In particular, it was shown that antenna diversity makes the channel “more Gaussian.” This has the important consequence of making coding schemes designed for the Gaussian channel to perform well also on a fading channel with diversity.

In this Correspondence we examine the effects of lack of CSI. For uncoded signal sets with equal energy (e.g., PSK) CSI does not affect error probability because the maximum-likelihood decision regions are invariant to a homothety of the signal constellation. However, for coded systems lack of CSI has a negative influence on system performance,

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which we want to assess here. We show that with a good coding scheme the loss in performance with respect to perfect CSI is moderate (around 1.5 dB). Moreover, as the diversity order grows, the channel still tends to become Gaussian.

## 2 Channel model

We consider an  $M$ -branch diversity fading channel with ideal carrier-phase recovery; the input-output relationship is

$$\mathbf{y} = \sqrt{\Gamma/M} \mathbf{a} x + \mathbf{n} \quad (1)$$

where the components of vectors  $\mathbf{a}$  and  $\mathbf{n}$  are defined as follows:  $a_i$  is the absolute value of the  $i$ -th branch channel gain, which we assume to have mean  $\mu_a$  and unit second moment, and  $n_i$  is a circular complex Gaussian random variable with zero mean and variance  $1/2$ , i.e.,  $E|n_i|^2 = 1$ . We write  $n_i \sim \mathcal{N}_c(0, 1/2)$ . All branch gains and noise components are assumed to be independent and identically distributed (i.i.d.). We assume that the transmitted symbols are normalized by  $E|x|^2 = 1$ . The signal-to-noise ratio (SNR) of this channel is

$$\text{SNR} = \frac{\text{Total branch signal power}}{\text{Total branch noise power}} = \frac{\Gamma}{M} \quad (2)$$

Here we consider constant-energy signal sets, antenna diversity with order  $M$ , and two different combining techniques, that is, maximal-ratio combining (which requires exact CSI) and equal-ratio combining (which does not require CSI).

**General diversity combining.** In order to detect the transmitted data, the received signal is first linearly combined as follows:

$$y = \sum_{i=1}^M \beta_i y_i = \sum_{i=1}^M \beta_i (\sqrt{\Gamma/M} a_i x + n_i) \quad (3)$$

Maximum likelihood detection selects the symbol  $x$  that minimizes  $|y - x|^2$ . An unessential rescaling of the received sample yields the received signal sample

$$y = \sqrt{\Gamma} x + z \quad \text{with} \quad z = \sqrt{M} \frac{\sum_{i=1}^M \beta_i n_i}{\sum_{i=1}^M \beta_i a_i} \quad (4)$$

Hence, the SNR of the resulting ‘‘combined channel’’ is  $\Gamma/E[|z|^2]$ .

**Maximal-ratio combining.** Application of maximal-ratio combining (MRC), which corresponds to the choice  $\beta_i = a_i$  in (3), yields the channel output

$$y = \sqrt{\Gamma}x + z \quad \text{with} \quad z = \sqrt{M} \frac{\sum_{i=1}^M a_i n_i}{\sum_{i=1}^M a_i^2} = \frac{n}{\beta} \quad (5)$$

where we set

$$n = \frac{\sum_{i=1}^M a_i n_i}{\sqrt{\sum_{i=1}^M a_i^2}} \quad \text{and} \quad \beta = \sqrt{\frac{1}{M} \sum_{i=1}^M a_i^2} \quad (6)$$

In spite of its apparent complexity,  $n$  is  $\sim \mathcal{N}_c(0, 1/2)$  and is independent of  $\beta$ . To prove this, it is sufficient to observe that, given  $\mathbf{a}$ ,  $n$  is conditionally Gaussian, and its conditional mean and variance are independent of  $\mathbf{a}$ :

$$\mathbf{E}|n|^2 = \mathbf{E}_{\mathbf{a}} \left[ \frac{\sum_{i=1}^M \sum_{j=1}^M a_i a_j \mathbf{E}[n_i n_j^*]}{\sum_{i=1}^M a_i^2} \right] = \mathbf{E}_{\mathbf{a}} \left[ \frac{\sum_{i=1}^M a_i^2}{\sum_{i=1}^M a_i^2} \right] = 1 \quad (7)$$

The SNR of the vector channel is then  $\Gamma/M$ , while that of the combined channel is  $\Gamma/E[\beta^{-2}]$ . Heuristically, since from the weak law of large numbers  $\beta^2$  approaches 1 in probability as  $M \rightarrow \infty$ , we may expect that the SNR approaches  $\Gamma$  as  $M \rightarrow \infty$ . This will be proved rigorously in the following.

Note that the metric obtained by MRC is equivalent to the ML metric. In fact, with the ML metric, we have

$$\arg \min_{\hat{x}} |\mathbf{y} - \sqrt{\Gamma/M} \mathbf{a} \hat{x}|^2 = \arg \min_{\hat{x}} \sum_{i=1}^M |y_i - \sqrt{\Gamma/M} a_i \hat{x}|^2 = \arg \max_{\hat{x}} \sum_{i=1}^M (y_i, a_i \hat{x}) \quad (8)$$

where we denote  $(a, b) = \text{Re}(ab^*)$ . Similarly, with our MRC metric, we have

$$\arg \min_{\hat{x}} |y - \hat{x}|^2 = \arg \min_{\hat{x}} \left| \sum_{i=1}^M a_i y_i - \hat{x} \right|^2 = \arg \max_{\hat{x}} \left( \sum_{i=1}^M a_i y_i, \hat{x} \right) \quad (9)$$

which coincides with (8).

**Equal-gain combining.** Applying equal-gain combining (EGC), i.e., choosing  $\beta_i = 1$  in (3), produces the following equivalent channel output:

$$y = \sqrt{\Gamma}x + z \quad \text{with} \quad z = \frac{\sum_{i=1}^M n_i / \sqrt{M}}{\sum_{i=1}^M a_i / M} = \frac{n}{\beta} \quad (10)$$

Here, we set

$$n = \frac{1}{\sqrt{M}} \sum_{i=1}^M n_i \quad \text{and} \quad \beta = \frac{1}{M} \sum_{i=1}^M a_i \quad (11)$$

It is immediate to see that  $n \sim \mathcal{N}_c(0, 1/2)$ . The SNR of the vector channel is then  $\Gamma/M$ , while that of the combined channel is  $\Gamma/E[\beta^{-2}]$ . Again, the weak law of large numbers

shows that  $\beta$  approaches 1 in probability as  $M \rightarrow \infty$ , so that we expect the SNR to approach  $\Gamma$  as  $M \rightarrow \infty$ , with a consequent asymptotic loss of  $20 \log_{10}(1/\mu_a)$  dB with respect to MRC.

### 3 Convergence to the Gaussian channel

In both cases of diversity combining considered here,  $\beta$  depends on  $M$ , and is expected to converge to 1 in probability as  $M \rightarrow \infty$ . We now show that the distribution of  $z$  converges to  $n \sim \mathcal{N}_c(0, 1/2)$ , and that the rate of convergence is not affected by CSI.

**Channel Gaussianity.** We measure the ‘‘Gaussianity’’ of the channel by computing the divergence (or Kullback-Leibler distance) between the probability density function (pdf) of  $z$ , say  $f_z(z)$ , and the pdf of a Gaussian random variable  $z_G$ . This is [1]:

$$D(z \parallel z_G) = \int_{\mathbb{R}^2} f_z(z) \ln \frac{f_z(z)}{f_{z_G}(z)} dz \quad (12)$$

where the integration is over the two-dimensional space  $\mathbb{R}^2$ . Since the two pdf have circular symmetry, after some algebra, we obtain

$$D(z \parallel z_G) = \int_0^\infty e^{-u} \mathbf{E}_\beta [\beta^2 e^{(1-\beta^2)u}] \ln \mathbf{E}_\beta [\beta^2 e^{(1-\beta^2)u}] du \quad (13)$$

This expression can be evaluated in closed form asymptotically as  $M \rightarrow \infty$ . With MRC we obtain

$$D(z \parallel z_G) = \frac{\mu_4^2}{M^2} + O(M^{-3}) \quad (14)$$

where  $\mu_4 = E[a_i^4] - 1$ . This result corrects an error in [2], where  $D(z \parallel z_G)$  was computed for Rayleigh-distributed channel gains.

In the special case of Rayleigh-distributed branch gains and MRC we have the following exact result

$$D(z \parallel z_G) = \frac{1}{M(M-1)} \quad (15)$$

With EGC,

$$D(z \parallel z_G) = \frac{25}{2} \frac{(1/\mu_a^2 - 1)^2}{M^2} + O(M^{-3}) \quad (16)$$

### 4 Analysis of error probability

In this section we briefly review how to obtain the error probability of the uncorrelated and independent fading diversity channel for a coded-modulation scheme. Resorting to

standard union upper bounds [4], it is sufficient to calculate the pairwise error probability (PEP) between the two  $L$ -symbol sequences  $\mathbf{x}$  and  $\hat{\mathbf{x}}$ :

$$P(\mathbf{x} \rightarrow \hat{\mathbf{x}}) = P\left(\sum_{k=1}^L |y_k - \hat{x}_k|^2 < \sum_{k=1}^L |y_k - x_k|^2\right) = P\left(\Delta = \sum_{k=1}^L 2\text{Re}[y_k(x_k - \hat{x}_k)^*] < 0\right) \quad (17)$$

where

$$y_k = \sum_{i=1}^M \beta_{ki} y_{ki} = \sum_{i=1}^M \beta_{ki} [a_{ki} x_k + n_{ki}] \quad (18)$$

Here,  $\beta_{ki} = a_{ki}$  with MRC and  $\beta_{ki} = 1$  with EGC. Moreover,

$$\begin{aligned} \Delta &= \sum_{k=1}^L \sum_{i=1}^M 2\text{Re}[\beta_{ki} y_{ki} (x_k - \hat{x}_k)^*] \\ &= \sum_{k=1}^L \sum_{i=1}^M \beta_{ki} \sqrt{\Gamma/M} a_{ki} |x_k - \hat{x}_k|^2 + 2\beta_{ki} \text{Re}[n_{ki} (x_k - \hat{x}_k)^*] \end{aligned} \quad (19)$$

Thus,  $\Delta$  is Gaussian distributed conditionally on the fading gain sequence  $(\mathbf{a}_k)_{k=1}^L$ . More precisely,

$$\Delta \sim \begin{cases} \mathcal{N}(\sqrt{\Gamma/M} \sum_{k=1}^L d_k^2 \sum_{i=1}^M a_{ki}^2, 2 \sum_{k=1}^L d_k^2 \sum_{i=1}^M a_{ki}^2) & \text{(MRC)} \\ \mathcal{N}(\sqrt{\Gamma/M} \sum_{k=1}^L d_k^2 \sum_{i=1}^M a_{ki}, 2M \sum_{k=1}^L d_k^2) & \text{(EGC)} \end{cases} \quad (20)$$

where we set  $d_k = |x_k - \hat{x}_k|$ . The pairwise error probability (17) can be obtained by the *inversion formula* [3]

$$P(\Delta < 0) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \Phi_{\Delta}(s) \frac{ds}{s} \quad (21)$$

where  $\Phi_{\Delta}(s) = \mathbf{E}[e^{-s\Delta}]$  and  $c > 0$  is chosen so as to achieve convergence of (21).

Setting  $\gamma = \Gamma/M$  we obtain

$$\Phi_{\Delta}(\sqrt{\gamma}s) = \prod_{k=1}^L \left[ \frac{K+1}{K+1 + \gamma d_k^2 s(1-s)} \exp\left(-\frac{K\gamma d_k^2 s(1-s)}{(K+1)(K+1 + \gamma d_k^2 s(1-s))}\right) \right]^M \quad (22)$$

in the case of MRC [2] and

$$\Phi_{\Delta}(\sqrt{\gamma}s) = \prod_{k=1}^L \left[ \mathbf{E}[\exp(\gamma d_k^2 s(s-R))] \right]^M \quad (23)$$

in the case of EGC with  $R$  representing a Rician distributed random variable with unit second moment and Rician factor  $K$ . Only for Rayleigh fading is a closed-form expression available:

$$\Phi_{\Delta}(\sqrt{\gamma}s) = \prod_{k=1}^L \left[ \exp(\gamma d_k^2 s^2) \left( 1 - \frac{\sqrt{\pi}}{2} \gamma d_k^2 s \exp(\gamma^2 d_k^4 s^2/4) \text{erfc}(\gamma d_k^2 s/2) \right) \right]^M \quad (24)$$

**Chernoff bounds.** Here we restrict ourselves to Rayleigh fading. It is well known [3] that, minimizing  $\Phi_{\Delta}(s)$  with respect to (real)  $s$ , we obtain an upper bound (Chernoff bound) to the PEP. With MRC, from (22) we obtain

$$\min_s \Phi_{\Delta}(s) = \Phi_{\Delta}(1/2) = \prod_{k=1}^L (1 + \gamma d_k^2/4)^{-M} \approx \exp \left[ ML \left( \log(4/\gamma) - \frac{2}{L} \sum_{k=1}^L \log(d_k) \right) \right] \quad (25)$$

With EGC, we were unable to find the minimum in closed form.

By choosing  $s_0 = [(1/L) \sum_{k=1}^L d_k^2]^{-1/2}$  (which minimizes  $\Phi_{\Delta}(s)$  as  $\gamma \rightarrow \infty$ ) we obtain

$$\min_s \Phi_{\Delta}(s) \approx \Phi_{\Delta}(s_0) = \exp \left[ ML \left( \log(2e/\gamma) - \frac{4}{L} \sum_{k=1}^L \log(d_k) + \log \left( \frac{1}{L} \sum_{k=1}^L d_k^2 \right) \right) \right] \quad (26)$$

This corresponds to the following asymptotic gain of MRC with respect to EGC:

$$\begin{aligned} 10 \log_{10}(\gamma_{\text{EGC}}/\gamma_{\text{MRC}}) &\approx 10 \log_{10}(e/2) + 10 \log_{10} \left[ \frac{1}{L} \sum_{k=1}^L d_k^2 \right] - 10 \frac{1}{L} \sum_{k=1}^L \log_{10}(d_k^2) \\ &\geq 10 \log_{10}(e/2) = 1.33 \text{ dB} \end{aligned} \quad (27)$$

where Jensen's inequality has been applied to the log function.

Computing the equivalent SNR's with MRC and EGC (see, e.g., [5, sec. 5.6.2]), a difference of 1.05 dB is found. However, we think that our 1.33 dB is more significant when digital transmission (either coded or uncoded) is concerned as shown by the results of Fig. 1.

## 5 Numerical results

Numerical results not reported here for the sake of brevity show an excellent agreement between the asymptotic approximation and the true value of  $D(z || z_G)$  even for small diversity order  $M$ .

Figure 1 reports the bit error rate (BER) for uncoded QPSK ( $L = 1$ ,  $d_1 = 2$ ) over a Rayleigh fading channel with MRC/EGC diversity and  $M = 8, 16$ , and  $32$  branches. The figure shows that the asymptotic gain of 1.33 dB is attained for sufficiently high diversity order and  $E_b/N_0$ .

Figure 2 reports the union bound to the BER obtained with the Ungerboeck 8-state rate-2/3 8-PSK TCM [4] over a Rayleigh fading channel. The curves correspond to  $M = 1, 2, 4$  diversity branches with MRC and EGC. The asymptotic gain is obtained by considering the dominant error event of the code, for which  $L = 2$ ,  $d_1^2 = 2$ , and  $d_2^2 = 4$ , yielding 1.59 dB.

## **6 Conclusions**

We have examined the effects of lack of CSI in coded systems operating over a flat fading channel with antenna diversity. We show that the loss in performance with respect to perfect CSI may be as low as 1.6 dB. Moreover, as the diversity order grows, the channel tends to become Gaussian.

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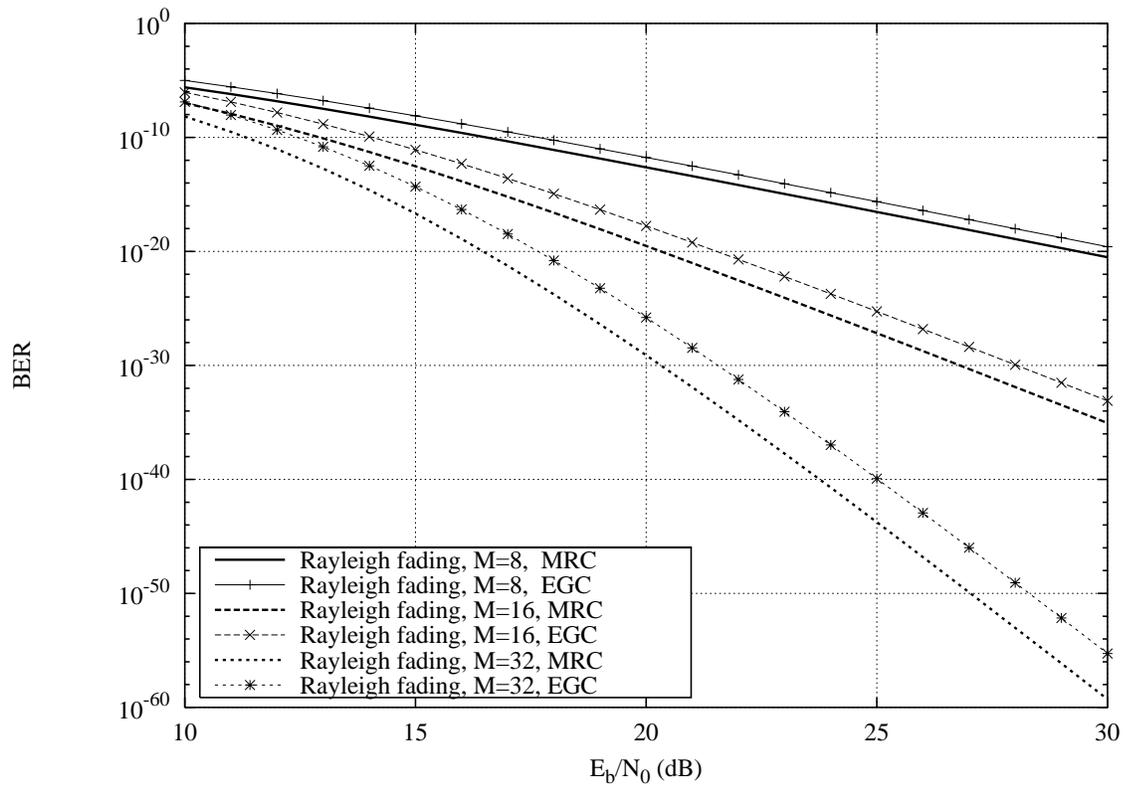


Figure 1: BER vs.  $E_b/N_0$  for uncoded QPSK and Rayleigh fading. Curves correspond to  $M = 8, 16, 32$  diversity branches with MRC and EGC diversity.

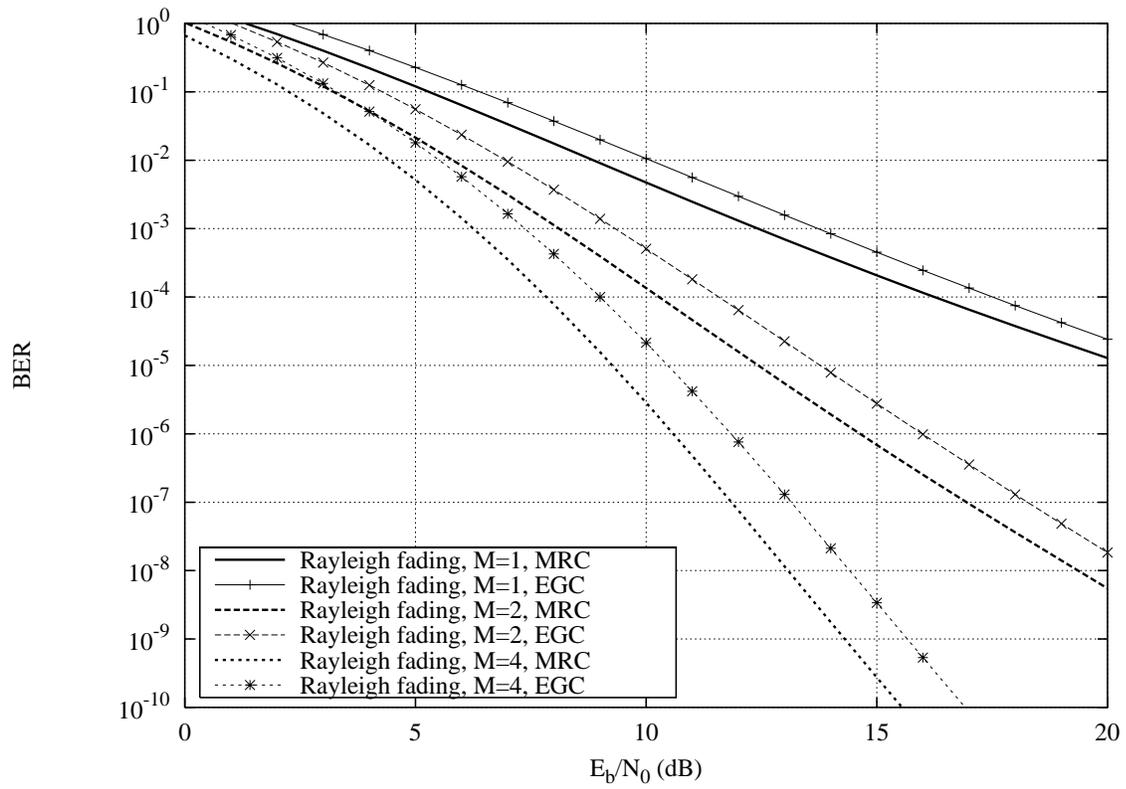


Figure 2: BER vs.  $E_b/N_0$  for Ungerboeck 8-state rate-2/3 8-PSK TCM. Curves correspond to  $M = 1, 2, 4$  diversity branches with MRC and EGC diversity.