Adaptive Linear Receivers for DS/CDMA: Steady-State Performance Analysis

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Abstract— **We present closed-form simple expressions for the signal-to-interference plus noise ratio at the output of well-known adaptive implementations of the linear minimum mean-square error receiver for CDMA signals. A very accurate Gaussian approximation of the resulting steady-state bit-error probability at the receiver output is derived for 4PSK modulation with Gray mapping. Our results apply to both data-aided and non-data aided algorithms of LMS and RLS type. In agreement with previous works, we show that non-data aided algorithms may suffer from poor steady-state performance.**

Keywords— **CDMA, adaptive algorithms, linear receivers.**

I. Introduction

Multiuser detection has been a fruitful and rapidly growing research field for the last decade. Broadly speaking, this is motivated by the fact that the techniques developed for single-user communications, mostly devoted to combat Gaussian white noise, fail to give near-optimal performance if used in the presence of multiple-access interference (MAI). Under the common name of *multiuser detection* we find a broad range of receivers, which differs in complexity and performance (see [1] for a complete survey and a comprehensive list of references). We can distinguish between *centralized* and *decentralized* receivers. Centralized receivers make use of side information about all interfering users (spreading sequences, timing and propagation channels). These receivers are suited for base-station processing (uplink), where all this side information is either available or can be estimated consistently. Among centralized receivers we can mention the *optimal multiuser receiver* and receivers based on decision feedback or on parallel interference cancellation [1]. On the contrary, decentralized receivers exploit the knowledge of the spreading sequence, of the timing and of the propagation channel of the user of interest only. Remarkably, this is the same information necessary for a conventional matched filter that ignores the presence of MAI. Decentralized receivers are suited for mobile-station processing (downlink), where information relative to the other users is either difficult to obtain and/or forbidden, because of privacy reasons.

Because of the lack of side information about MAI, decentralized receivers treat the superposition of MAI and background Gaussian noise as a random process, the statistics of which must be learned from the received signal, via some adaptive algorithm. In this paper, we are concerned with decentralized *adaptive linear receivers*, i.e., receiver formed by the concatenation of an adaptive linear filter with a suitable (non-linear) detection operation acting on the filter output. Algorithms that make use of a known data sequence (training sequence) for adaptation are referred to as data-aided (DA). Algorithms that do not require a training sequence are referred to in the literature under different names (e.g., "blind" [3] or "code-aided" [4]). We shall refer to these algorithms as non-data aided (NDA). In Section III we present closed form formulas for the steady-state output signal-to-interference plus noise ratio (SINR) of the adaptive algorithms presented in Section II. Also, we derive a Gaussian approximation for the steady-state bit-error rate (BER) of these adaptive receivers, in the case of 4PSK with Gray mapping. This offers a rapid and accurate tool to predict the performance of adaptive receivers. DA algorithm suffer from a SINR degradation of at most 3 dB with respect to optimum. On the contrary, NDA algorithms might be very far from the optimum SINR.

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II. Background

A. Discrete-time finite-memory signal model

Consider a system with K users. The k-th user signal is given by

$$u_k(t) = \sum_m b_k[m] s_k(t - mT) \qquad (1)$$

where $s_k(t)$ and $b_k[m]$ are the signature waveform and the m-th information symbol of user k, respectively. Users transmit individually and mutually uncorrelated sequences of unit-variance zero-mean complex symbols. In DS/CDMA, the signature waveforms are given by $s_k(t) = \sum_{\ell=0}^{L-1} s_{k,\ell}\psi(t - \ell T_c)$, where $\mathbf{s}_k = (s_{k,0}, \ldots, s_{k,L-1})^T$ is the k-th user spreading sequence, $T_c = T/L$ is the chip interval, L is the processing gain and $\psi(t)$ is the chip pulse, assumed to be bandlimited in $[-W/2, W/2]$ and common to all users. We assume normalized energy $\int |s_k(t)|^2 dt = 1$.

User k transmits with delay τ_k through a channel with baseband equivalent impulse response $c_k(\tau)$. Without loss of generality, we consider delays $\tau_k = q_k/W + \gamma_k$ where q_k is an integer and $0 \leq \gamma_k < 1/W$. The *fractional part* γ_k of the delay can be modeled as an effect of the propagation channel, while the *integer part* q_k/W can be thought as an effect of asynchronous transmission. Then, user k signal contribution at the receiver is obtained by convolving $u_k(t - q_k/W)$ with $c_k(\tau - \gamma_k)$. From the bandlimited assumption, by using the sampling theorem we obtain

$$v_k(i/W) = \sum_i \left[\sum_j c_k[j] u_k((i - j - q_k)/W) \right] \qquad (2)$$

where the coefficients of the discrete-time channel impulse response are given by $c_k[j] = \int c_k(j/W - \gamma_k - \tau)\mathrm{sinc}(W\tau)d\tau$. The overall received signal, given by the superposition of all users' signals plus background noise, is given by $y(t) = \sum_{k=1}^{K} v_k(t) + \nu(t)$, where $\nu(t)$ is a white circularly-symmetric complex Gaussian process with power spectral density N_0.

The baseband receiver front-end is formed by an ideal lowpass filter with bandwidth $[-W/2, W/2]$ and gain $1/\sqrt{W}$ followed by sampling at rate $W = N_c/T_c$ with arbitrary sampling epoch. In order to obtain an approximated finite-memory signal model, we assume that the discrete-time impulse responses $c_k[j]$ and the sampled chip pulse $\psi(j/W)$ are negligible for $j \notin [0, P]$ and for $j \notin [-Q, Q]$, respectively, where P and Q are suitable integers. In particular, the k-th discrete-time channel impulse response is represented by the vector $\mathbf{c}_k = (c_k[0], \ldots, c_k[P])^T$. Moreover, we assume that the receiver has a finite-length *processing window*, i.e., for each symbol time n it processes a window of samples with indexes $i \in [nLN_c - M_1, nLN_c + M_2]$. The *processing window* size $\widetilde{L} = M_1 + M_2 + 1$ is left as a design parameter (notice that if $M_1 > 0$ and $M_2 > LN_c - 1$ the receiver processing window is larger than one symbol interval T). Accordingly, we define the n-th channel output vector $\mathbf{y}[n]$ as the content of the receiver processing window at symbol time n, i.e., $\mathbf{y}[n] = (y[nLN_c + M_2], y[nLN_c + M_2 - 1], \ldots, y[nLN_c - M_1])^T$ and we let $\boldsymbol{\nu}[n]$ be the corresponding vector of noise samples.

After a little algebra, we obtain

$$\mathbf{y}[n] = \sum_{k=1}^{K} \sum_{m=-B_1}^{B_2} \mathbf{S}_k[m]\mathbf{c}_k b_k[n-m] + \boldsymbol{\nu}[n] \tag{3}$$

where the matrices $\{\mathbf{S}_k[m] : m = -B_1, \ldots, B_2\}$, of size $(M_1+M_2+1)\times P$, are uniquely defined by q_k and $s_k(t)$, and can be easily calculated, and where the summation limits B_1 and B_2 are explcitly found as functions of N_c, P, Q and L. After suitable renumbering of the user symbols so that $(k, m) \mapsto u$ and $b_k[n-m] \mapsto b_u[n]$, and by defining the modified normalized spreading sequences $\mathbf{p}_u = \mathbf{S}_k[m]\mathbf{c}_k/\sqrt{\mathcal{E}_u}$ where $\mathcal{E}_u = |\mathbf{S}_k[m]\mathbf{c}_k|^2$ is the average energy contribution of symbol $b_u[n]$, we can rewrite (3) as

$$\mathbf{y}[n] = \sum_{u=1}^{U} \sqrt{\mathcal{E}_u}\mathbf{p}_u b_u[n] + \boldsymbol{\nu}[n] \tag{4}$$

where $U = (B_1 + B_2 + 1)K$ is the number of equivalent users in the asynchronous CDMA channel and $b_1[n]$ is the desired symbol of the desired user.

B. Linear decentralized receivers

We constrain the receiver to be *linear*, i.e., formed by a linear FIR filtering operation $z[n] = \mathbf{h}^H\mathbf{y}[n]$ followed by a suitable (non-linear) processing with input sequence $\{z[n]\}$. We can rewrite (4) by putting in evidence the useful signal component, as

$$\mathbf{y}[n] = \sqrt{\mathcal{E}_1}\mathbf{p}_1 b_1[n] + \mathbf{w}[n] \tag{5}$$

where $\mathbf{w}[n]$ collects noise+ISI+MAI. A relevant measure of performance for the filter $\mathbf{h}$ is its output SINR. In our setting, this is defined by

$$\mathrm{SINR} \triangleq \frac{E[|\overline{z}(b_1[n])|^2]}{E[\sigma_z^2(b_1[n])]} \tag{6}$$

where $\overline{z}(b_1[n]) = E[z[n]|b_1[n]]$ and $\sigma_z^2(b_1[n]) = E[|z[n]|^2|b_1[n]] - |\overline{z}(b_1[n])|^2$ are the conditional mean and the conditional variance of the filter output $z[n]$ given the useful symbol $b_1[n]$, respectively.

Single-user matched filter (SUMF). The baseline linear receiver filter is the SUMF $\mathbf{h} = \mathbf{p}_1$, matched to the useful signal component as if $\mathbf{w}[n]$ was a white noise vector. The SUMF requires the knowledge of user 1 signature waveform $s_1(t)$, coarse timing q_1 and channel vector $\mathbf{c}_1$, so that $\mathbf{p}_1 = \mathbf{S}_1[0]\mathbf{c}_1$ can be calculated. The output SINR achieved by the SUMF is given by

$$\mathrm{SINR}_{\mathrm{sumf}} = \frac{\mathcal{E}_1}{\mathbf{p}_1^H\mathbf{R}_w\mathbf{p}_1} \tag{7}$$

where $\mathbf{R}_w = E[\mathbf{w}[n]\mathbf{w}[n]^H]$. In the absence of ISI and MAI, $\mathbf{R}_w = N_0\mathbf{I}$ so that $\mathrm{SINR}_{\mathrm{sumf}} = \mathrm{SNR}_1 \triangleq \mathcal{E}_1/N_0$.

Linear minimum MSE receiver (LMMSER). A classical criterion for the design of the filter $\mathbf{h}$ is the minimization of the MSE [2] $J = E[|b_1[n] - \mathbf{h}^H\mathbf{y}[n]|^2]$. The minimum MSE (MMSE) filter vector is the Wiener filter $\mathbf{h}_{\mathrm{opt}} = \sqrt{\mathcal{E}_1}\mathbf{R}_y^{-1}\mathbf{p}_1$, where we let $\mathbf{R}_y = E[\mathbf{y}[n]\mathbf{y}[n]^H]$. The resulting output SINR is given by

$$\mathrm{SINR}_{\mathrm{opt}} = \mathcal{E}_1\mathbf{p}_1^H\mathbf{R}_w^{-1}\mathbf{p}_1 \tag{8}$$

and it is the maximum SINR over all possible filters $\mathbf{h}$ [2] (this motivate the subscript "opt"). Notice that any two filter vectors $\mathbf{h}'$ and $\mathbf{h}''$ which differ by a scalar (non-zero) multiplicative term provide the same SINR (we shall write $\mathbf{h}' \propto \mathbf{h}''$). Then, any filter $\mathbf{h} \propto \mathbf{h}_{\mathrm{opt}}$ is also optimal in terms of output SINR. Adaptive implementations of the LMMSER are obtained from standard DA LMS and RLS algorithms [7].

Constrained minimum MOE receiver (CMMOER). In [3], the receiver filter $\mathbf{h}$ is designed in order to minimize the mean output energy (MOE) $\xi \triangleq E[|z[n]|^2] = \mathbf{h}^H\mathbf{R}_y\mathbf{h}$ subject to the constraint $\mathbf{h}^H\mathbf{p}_1 = 1$. The solution of this constrained minimization problem is readily obtained as $\mathbf{h}_{\mathrm{moe}} = \xi_{\min}\mathbf{R}_y^{-1}\mathbf{p}_1$ where $\xi_{\min} = \mathcal{E}_1\left(1 + \frac{1}{\mathrm{SINR}_{\mathrm{opt}}}\right)$ is the constrained minimum MOE. Since $\mathbf{h}_{\mathrm{moe}} \propto \mathbf{h}_{\mathrm{opt}}$, also the CMMOER attains $\mathrm{SINR}_{\mathrm{opt}}$. Adaptive implementations of the CMMOER are obtained by using the NDA LMS and RLS algorithms described in [3], [4].

Generalized constrained minimum MOE receiver (GCMOER). An elegant generalization of the CMMOER which avoids explicit knowledge of $\mathbf{p}_1$ has been proposed in [8], [9]. This receiver, referred here as the GCMOER, is the result of the min-max problem: choose $(\mathbf{h}, \mathbf{g})$ such that the MOE is minimized with respect to $\mathbf{h}$ subject to the constraint $\mathbf{S}_1[0]^H\mathbf{h} = \mathbf{g}$ and maximized with respect to $\mathbf{g}$ subject to the constraint $|\mathbf{g}|^2 = 1$. The resulting filter vector is given by [8] $\mathbf{h}_{\rm gmoe} = \xi_1 \mathbf{R}_y^{-1}\mathbf{S}_1[0]\mathbf{u}_1$ where $1/\xi_1$ is the minimum eigenvalue of the matrix $\mathbf{S}_1[0]^H\mathbf{R}_y^{-1}\mathbf{S}_1[0]$ and $\mathbf{u}_1$ is a corresponding unit-norm eigenvector. It is easy to show that the resulting MOE is given by $\xi = \xi_1$. The GCMOER is well-defined if the minimum eigenvalue of $\mathbf{S}_1[0]^H\mathbf{R}_y^{-1}\mathbf{S}_1[0]$ has multiplicity 1. This is verified under the *unique identifiability condition* of [9, Prop. 1]. In the following we assume that this is always the case. The GCMOER needs only the prior knowledge of q_1 and $s_1(t)$ in order to calculate $\mathbf{S}_1[0]$. Therefore, it requires even less side information than the SUMF.

The output SINR achieved by the GCMOER is easily obtained as

$$\mathrm{SINR}_{\rm gmoe} = \frac{1}{\xi_1/(\mathcal{E}_1|\mathbf{u}_1^H\mathbf{c}_1|^2) - 1} \tag{9}$$

In this paper we consider the following straightforward NDA adaptive implementation of the GCMOER based on recursive estimation of $\mathbf{R}_y^{-1}$ [10]: [1]

$$\begin{aligned}
\mathbf{k}[n] &= \left(\alpha + \mathbf{y}[n]^H\mathbf{M}[n-1]\mathbf{y}[n]\right)^{-1}\mathbf{M}[n-1]\mathbf{y}[n] \\
\mathbf{M}[n] &= \frac{1}{\alpha}(\mathbf{I} - \mathbf{k}[n]\mathbf{y}[n]^H)\mathbf{M}[n-1] \\
\mathbf{u}_1[n] &= \arg\min_{\mathbf{g}} \frac{\mathbf{g}^H\mathbf{S}_1[0]^H\mathbf{M}[n]\mathbf{S}_1[0]\mathbf{g}}{\mathbf{g}^H\mathbf{g}} \\
\xi_1[n] &= \left(\mathbf{u}_1[n]^H\mathbf{S}_1[0]^H\mathbf{M}[n]\mathbf{S}_1[0]\mathbf{u}_1[n]\right)^{-1} \\
\mathbf{h}[n] &= \xi_1[n]\mathbf{M}[n]\mathbf{S}_1[0]\mathbf{u}_1[n]
\end{aligned} \tag{10}$$

where α is the exponential forgetting factor and $\mathbf{M}[0] = \gamma\mathbf{I}$. Because of the similarity with the NDA-RLS, the above algorithm will be referred to as "generalized" NDA-RLS (GNDA-RLS).

III. Steady-state performance analysis

For a given adaptive algorithm, $\mathbf{h}[n]$ is a random vector. We assume the following convergence conditions:

1. Convergence of the mean filter vector: $\lim_{n\to\infty} E[\mathbf{h}[n]] = \overline{\mathbf{h}}$, where $\overline{\mathbf{h}}$ is a given constant vector.
2. Convergence of the MSE: $\lim_{n\to\infty} J[n] = J$, where we define $J[n] = E[b_1[n] - \mathbf{h}[n-1]^H\mathbf{y}[n]|^2]$ and J is a given finite constant.

The constants $\overline{\mathbf{h}}$ and J depend on the specific algorithm and on the channel parameters (user channels, spreading sequences etc...). We are allowed to write $\mathbf{h}[n] = \overline{\mathbf{h}} + \boldsymbol{\epsilon}[n]$ where $\boldsymbol{\epsilon}[n]$ is an asymptotically zero-mean error vector. It can be shown that, for a large class of adaptive algorithms, $\boldsymbol{\epsilon}[n]$ is also asymptotically WSS [12]. We make this assumption here. The steady-state MSE can be written as [7] $J = J_0 + J_{\rm ex}$ where $J_0 = E[|b_1[n] - \overline{\mathbf{h}}^H\mathbf{y}[n]|^2]$ is the MSE achieved by a non-adaptive receiver with deterministic filter vector $\overline{\mathbf{h}}$ and $J_{\rm ex}$ is the steady-state excess MSE. Under the *independence assumption* [7], $\boldsymbol{\epsilon}[n-1]$ is treated as statistically independent of $b_1[n]$ and of $\mathbf{y}[n]$. This allows us to write $J_{\rm ex} = \mathrm{trace}(\mathbf{R}_y\mathbf{R}_\epsilon)$, where $\mathbf{R}_\epsilon = \lim_{n\to\infty} E[\boldsymbol{\epsilon}[n]\boldsymbol{\epsilon}[n]^H]$. Closed-form expressions for $J_{\rm ex}$ are known for several adaptive algorithms [7], [13], [3], [4].

A. SINR analysis

We derive a general expression of the steady-state output SINR in terms of $\overline{\mathbf{h}}$, J_0 and $J_{\rm ex}$. Then, we evaluate it for the DA-LMS, DA-RLS, NDA-LMS, NDA-RLS and GNDA-RLS algorithms presented in SectionII. For the sake of notation simplicity, we drop the time index. Then, all the following expressions should be interpreted as limits for $n \to \infty$. Let $\mathbf{h} = \overline{\mathbf{h}} + \boldsymbol{\epsilon}$, where $\overline{\mathbf{h}}$ is deterministic and $\boldsymbol{\epsilon}$ is random with mean zero and independent of b_1 and $\mathbf{y}$. From (4) we obtain immediately $\overline{z}(b_1) = \sqrt{\mathcal{E}_1}(\overline{\mathbf{h}}^H\mathbf{p}_1)b_1$ and $E[\sigma_z^2(b_1)] = J - |1 - \sqrt{\mathcal{E}_1}(\overline{\mathbf{h}}^H\mathbf{p}_1)|^2$. Then, by using the above expressions in (6) and by letting $J = J_0 + J_{\rm ex}$, we obtain the steady-state SINR of a general adaptive filter as

$$\mathrm{SINR} = \frac{\mathcal{E}_1|\overline{\mathbf{h}}^H\mathbf{p}_1|^2}{J_0 + J_{\rm ex} - |1 - \sqrt{\mathcal{E}_1}(\overline{\mathbf{h}}^H\mathbf{p}_1)|^2} \tag{11}$$

DA algorithms. Both the DA-LMS and the DA-RLS have the property that $\overline{\mathbf{h}} = \mathbf{h}_{\rm opt}$ [7]. We have

$$\begin{aligned}
\mathbf{h}_{\rm opt}^H\mathbf{p}_1 &= \frac{1}{\sqrt{\mathcal{E}_1}}\frac{\mathrm{SINR}_{\rm opt}}{1+\mathrm{SINR}_{\rm opt}} \\
J_0 &= \frac{1}{1+\mathrm{SINR}_{\rm opt}}
\end{aligned} \tag{12}$$

By substituting this into (11) and by using the fact that for both DA-LMS and DA-RLS we have that $J_{\rm ex} = \eta J_0$, where η is referred to as *MSE misadjustment* and it is explicitly given (see [7], [13]), we obtain

$$\mathrm{SINR}_{\rm DA} = \frac{\mathrm{SINR}_{\rm opt}}{1 + \eta + \eta/\mathrm{SINR}_{\rm opt}} \tag{13}$$

NDA algorithms. Both the NDA-LMS and the NDA-RLS have the property that $\overline{\mathbf{h}} = \mathbf{h}_{\rm moe}$ [3], [4]. Then, we have

$$J_0 = \frac{\mathcal{E}_1}{\mathrm{SINR}_{\rm opt}} + (1 - \sqrt{\mathcal{E}_1})^2 \tag{14}$$

By using this in (11) and by using the fact that $J_{\rm ex} = \eta\xi_{\min}$ (see [3], [4]), where η (the *MOE misadjustment*) is given in [4], [3], we obtain

$$\mathrm{SINR}_{\rm NDA} = \frac{\mathrm{SINR}_{\rm opt}}{1 + \eta + \eta\mathrm{SINR}_{\rm opt}} \tag{15}$$

Now, we consider the GNDA-RLS algorithm. In [10] it is shown that, as long as the GCMOER is well-defined, the GNDA-RLS has the property that $\overline{\mathbf{h}} = \mathbf{h}_{\rm gmoe}$. Then, we obtain

$$\begin{aligned}
\mathbf{h}_{\rm gmoe}^H\mathbf{p}_1 &= \mathbf{u}_1^H\mathbf{c}_1 \\
J_0 &= 1 - \sqrt{\mathcal{E}_1}\mathrm{Re}\{\mathbf{u}_1^H\mathbf{c}_1\} + \xi_1
\end{aligned} \tag{16}$$

[1] See [10], [11] for computationally-efficient stochastic-gradient adaptive GCMOER implementations.

The evaluation of $J_{\rm ex}$ for the GNDA-RLS algorithm is complicated by the presence of the eigenvector computation step in the recursion (10). Then, we approximate $J_{\rm ex}$ by the asymptotic excess MSE of the following modified recursion:

$$\begin{aligned} \mathbf{k}[n] &= \left(\alpha + \mathbf{y}[n]^H \mathbf{M}[n-1]\mathbf{y}[n]\right)^{-1} \mathbf{M}[n-1]\mathbf{y}[n] \\ \mathbf{M}[n] &= \frac{1}{\alpha}(\mathbf{I} - \mathbf{k}[n]\mathbf{y}[n]^H)\mathbf{M}[n-1] \\ \widetilde{\xi}_1[n] &= \left(\widetilde{\mathbf{p}}_1^H \mathbf{M}[n]\widetilde{\mathbf{p}}_1\right)^{-1} \\ \mathbf{h}[n] &= \widetilde{\xi}_1[n]\mathbf{M}[n]\widetilde{\mathbf{p}}_1 \end{aligned} \tag{17}$$

where we let $\widetilde{\mathbf{p}}_1 = \mathbf{S}_1[0]\mathbf{u}_1$. This is motivated by the fact that, for large n, the inverse covariance matrix $\mathbf{M}[n]$ behaves like a *quasi-deterministic* quantity when $\widetilde{L}(1-\alpha) \ll 1$ (see [4] and references therein). Therefore, $\lim_{n\to\infty} \mathbf{M}[n] \simeq E[\mathbf{M}[n]] = (1-\alpha)\mathbf{R}_y^{-1}$, which implies that, for large n, $\mathbf{u}_1[n] \simeq \mathbf{u}_1$.

Recursion (17) is formally equivalent to the NDA-RLS algorithm. Then, it is rather straightforward to duplicate the derivation of [4] with the change $\mathbf{p}_1 \to \widetilde{\mathbf{p}}_1$ and obtain $J_{\rm ex} \simeq \eta\xi_1$ with the same MOE misadjustment as for the NDA-RLS. We obtain:

$$\mathrm{SINR}_{\rm GNDA} = \frac{\mathrm{SINR}_{\rm gmoe}}{1 + \eta + \eta \mathrm{SINR}_{\rm gmoe}} \tag{18}$$

Remark 1. For $\mathrm{SINR}_{\rm opt} \gg 1$, we have that $\mathrm{SINR}_{\rm DA} \simeq \mathrm{SINR}_{\rm opt}/(1+\eta)$. In normal working conditions it is reasonable to expect that the excess MSE due to adaptation does not exceed the MMSE, therefore $0 < \eta < 1$ and DA algorithms at the steady-state are suboptimal by at most 3 dB. On the contrary, NDA have $\mathrm{SINR}_{\rm NDA} \leq 1/\eta$, which might be much less than $\mathrm{SINR}_{\rm opt}$. By comparing (15) and (18), since $\mathrm{SINR}_{\rm gmoe} \leq \mathrm{SINR}_{\rm opt}$, we notice that the price for not knowing the channel vector $\mathbf{c}_1$ is a decreased steady-state SINR. However, in most cases this penalty is small [9].

B. Symbol-by-symbol error probability

It is customary to compare the receiver efficiency also in terms of symbol-by-symbol error probability. In order to compare different receivers in terms of their error probability, we should discuss the issue of phase ambiguity of the receiver filter vector $\mathbf{h}$. While multiplying $\mathbf{h}$ by a complex non-zero scalar has no impact on the output SINR, it might have a major impact on the error probability, depending on the decision rule. In particular, it is well-known that blind equalization schemes based on second-order statistics are able to equalize the channel up to a phase rotation [14]. This is actually the case for the CMMOER and GCMOER. Suppose that user 1 carrier phase is not known a priori by the receiver. Then, the CMMOER makes use of a "rotated" version $e^{j\phi}\mathbf{p}_1$ of the nominal useful signal vector in order to compute its filter vector, where ϕ is an arbitrary phase offset between transmitter and receiver. Similarly, the minimal eigenvector $\mathbf{u}_1$ defining the GCMOER filter vector can be determined up to an arbitrary phase factor $e^{j\phi}$. The NDA adaptive implementations of CMMOER and GCMOER are affected by analogous phase ambiguity. In practice, this can be resolved in several standard ways, as for example by non-coherent or differential (block) detection or by explicit phase estimation. Phase acquisition and tracking can be performed *after filtering* and are facilitated by the fact that they operate at the filter output SINR, i.e., after interference rejection. By using (5) and $\mathbf{h} = \overline{\mathbf{h}} + \boldsymbol{\epsilon}$, we can the filter output $z = \mathbf{h}^H\mathbf{y}$ in the form

$$z = \sqrt{\mathcal{E}_1}(\overline{\mathbf{h}}^H \mathbf{p}_1)b_1 + \zeta \tag{19}$$

where ζ is the residual interference plus noise at the filter output, and takes into account also the effect of the random filter error $\boldsymbol{\epsilon}$. For the sake of space limitations, we cannot investigate here the details of phase recovery. Then, in order to make fair comparisons between different receivers, we assume that the phase of the deterministic useful signal component $(\overline{\mathbf{h}}^H\mathbf{p}_1)$ is perfectly known to the receiver.

In this paper, we assume that the symbols b_u belong to a 4PSK signal set with Gray mapping, i.e.,

$$b_u = (d_u^I + jd_u^Q)/\sqrt{2} \tag{20}$$

where d_u^I and d_u^Q are i.i.d. antipodal random variables taking on values in $\{\pm 1\}$ with equal probability (the superscripts I and Q denote the in-phase and the quadrature rails), and we consider the following simple suboptimal symbol-by-symbol threshold detection rule: $\widehat{d_1^I} = \mathrm{sign}(\mathrm{Re}\{\widetilde{z}\})$ and $\widehat{d_1^Q} = \mathrm{sign}(\mathrm{Im}\{\widetilde{z}\})$, where $\widetilde{z} = \rho^{-1}(\overline{\mathbf{h}}^H\mathbf{p}_1)^*\mathbf{h}^H\mathbf{y}$ is the phase-compensated filter output and where we let $\rho \triangleq |\overline{\mathbf{h}}^H\mathbf{p}_1|$. Because of the symmetry, we can assume $d_1^I = 1$. The detector input can be written as

$$\mathrm{Re}\{\widetilde{z}\} = \rho\sqrt{\frac{\mathcal{E}_1}{2}}\left(1 + \beta_0 + \sum_{i=1}^{2U-1}\beta_i d_i + \widetilde{\nu}\right) \tag{21}$$

where the d_i's are i.i.d., uniformly distributed over $\{\pm 1\}$, $\widetilde{\nu} \sim \mathcal{N}(0, N_0|\mathbf{h}|^2/(\mathcal{E}_1\rho^2))$, $\beta_0 = \mathrm{Re}\left\{\boldsymbol{\epsilon}^H\mathbf{p}_1\mathbf{p}_1^H\overline{\mathbf{h}}\right\}/\rho^2$, $\beta_1 = \mathrm{Im}\left\{\boldsymbol{\epsilon}^H\mathbf{p}_1\mathbf{p}_1^H\overline{\mathbf{h}}\right\}/\rho^2$ and

$$\beta_i = \begin{cases} \sqrt{\mathcal{E}_u/\mathcal{E}_1}\mathrm{Re}\left\{\mathbf{h}^H\mathbf{p}_u\mathbf{p}_1^H\overline{\mathbf{h}}\right\}/\rho^2 & i = 2u-2 \\ \sqrt{\mathcal{E}_u/\mathcal{E}_1}\mathrm{Im}\left\{\mathbf{h}^H\mathbf{p}_u\mathbf{p}_1^H\overline{\mathbf{h}}\right\}/\rho^2 & i = 2u-1 \end{cases} \tag{22}$$

for $u = 2, \ldots, U$. The β_i's are random variables, since they are functions of $\boldsymbol{\epsilon}$. The BER conditioned on $\boldsymbol{\epsilon}$ is immediately obtained as [15]

$$P(e|\boldsymbol{\epsilon}) = \frac{1}{2^{2U-1}}\sum_{d_1,\ldots,d_{2U-1}} Q\left(\sqrt{\frac{\mathcal{E}_1\rho^2}{N_0|\mathbf{h}|^2}}\left(1 + \beta_0 + \sum_{i=1}^{2U-1}\beta_i d_i\right)\right) \tag{23}$$

where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$ is the Gaussian distribution tail function.

Methods for the evaluation of (23) have been extensively studied in the framework of ISI channels, for which the channel output samples is formally analogous to (21) for fixed (i.e., non-random) coefficients β_i. Here we use the very efficient and numerically accurate method based on Discrete-Cosine Transform (DCT) is provided in [16]. In order to compute the steady-state $P(e)$ we should average (23) with respect to the steady-state distribution of the filter error vector $\boldsymbol{\epsilon}$. This appears to be prohibitively complex, since this distribution is not known exactly. The desired steady-state error probability can be evaluated by the semi-analytic Monte Carlo (MC) method, that is expensive in terms of computation. An efficient alternative to the MC method is the steady-state Gaussian approximation (SSGA), which consists of modeling the residual noise variable ζ in (19) as a Gaussian zero-mean random variable. It is easy to check that, $\rho^2 \mathcal{E}_1/E[|\zeta|^2]$ is equal to the steady-state SINR of the adaptive filter. Therefore we can write $P(e) \approx Q(\sqrt{\text{SINR}})$ where SINR is provided by Propositions 1,2 and 3, for the algorithms considered here.

IV. Results

In order to validate the analysis carried out in the previous section, we considered a system with $K = 10$ users and processing gain $L = 31$. Each user is given a distinct sequence from a Gold set. Without loss of generality, we let $q_1 = 0$ and we generated independently the delays q_k for $k = 2, \dots, K$, uniformly distributed over the integers in $[-L/2, L/2)$, and γ_k for $k = 1, \dots, K$, uniformly distributed over $[0, T_c)$. We considered continuous-time Rayleigh channel impulse responses in the form $c_k(\tau) = \sum_{p=0}^{P'} g_p \delta(\tau - \tau_p)$, with $g_p \sim \mathcal{CN}(0, \sigma_p^2)$. The set of pairs (σ_p^2, τ_p) defines the *delay-intensity profile* (see Table I). The assignment of the delays q_k, of the channel vectors $\mathbf{c}_k$ and of the spreading sequences $\mathbf{s}_k$ is fixed throughout the simulations. Therefore, *we are not averaging over these parameters*. The receiver processing window is chosen to span two symbol intervals. with the useful symbol falling approximately in the middle of the processing window. We considered two SNR assignments: (a) all users have the same SNR= 13 dB (corresponding to $E_b/N_0 = 10$ for uncoded 4PSK); (b) users $k = 1, \dots, 5$ have SNR= 13 dB and users $k = 6, \dots, 10$ have SNR= 28 dB. These situations are representative of perfect power-control and of uncompensated near-far effect.

Fig. 1 and 2 show BER vs. the number of symbol intervals (i.e., algorithm iterations) for DA, NDA and GNDA RLS algorithms in cases (a) and (b), respectively. The curves are obtained by the semi-analytic MC method averaged over $N = 50$ independent simulation runs. At each iteration step, (23) is evaluated via the DCT method of [16], for the current value of the filter error vector. The horizontal lines indicate the steady-state BER obtained via the SSGA. The BER of the ideal (non-adaptive) SUMF, LMMSER and GCMOER are shown for comparison. These BER values are again computed via the DCT method and coincide with their exact value up to four significant digits. Fig. 3 and 4 show analogous results for DA and NDA LMS algorithms.

V. Concluding remarks

The steady-state SINR analysis provided by Propositions 1, 2 and 3 is very accurate for all algorithms considered. The results of Section IV show also that the SSGA for the steady-state BER is very good. Actually, in all our experiments we never found cases where the Gaussian approximation was accurate for the (non-adaptive) LMMSER and convergence conditions were satisfied but the SSGA was not very close to the steady-state BER of adaptive algorithms. Eventually, our analysis provides a useful tool for performance evaluation of adaptive linear receivers, avoiding heavy computer simulations.

It is intuitive to see that, for a reasonable choice of the user spreading waveforms, the eigenvalue spread [7] of $\mathbf{R}_y$ increases with the power unbalance between the users. Then, in heavy near-far situations like case (b), the convergence of LMS-type algorithms is very slow. On the contrary, RLS-type algorithms do not suffer from the near-far effect, since, as it is well-known, their convergence rate is almost independent of the eigenvalue spread of $\mathbf{R}_y$. Hence, we observe that even if an ideal (non-adaptive) receiver is intrinsically near-far resistant [1], its adaptive implementation might converge so slowly to prevent any practical utility. Driven by this consideration, we suggest an informal definition of *near-far resistant adaptive receiver*: an adaptive receiver is near-far resistant if, for a given desired steady-state performance, its convergence time is (almost) independent of the interfering users SNR. With respect to this definition, RLS-type algorithms are near-far resistant while LMS-type algorithms are not.

The performance gap between DA and NDA algorithms is evident. In our examples, the BER achieved by NDA algorithms is about one order of magnitude larger than that of their DA counterparts (a slightly larger degradation can be observed for the GNDA-RLS).

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p	σ_p^2	τ_p/T_c
0	1.0	0.0
1	0.5	1.2
2	0.2	3.4
3	0.1	5.6

TABLE I

Delay-intensity profile of the Rayleigh channel used in the simulations.

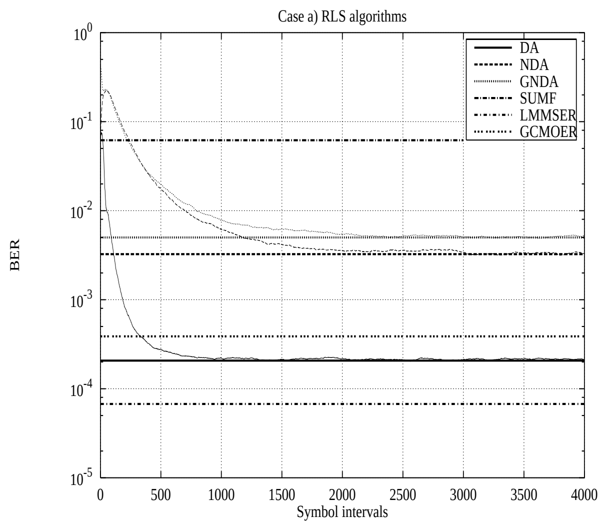

Fig. 1. BER vs. number of symbols for the RLS algorithms in near-far case (a).

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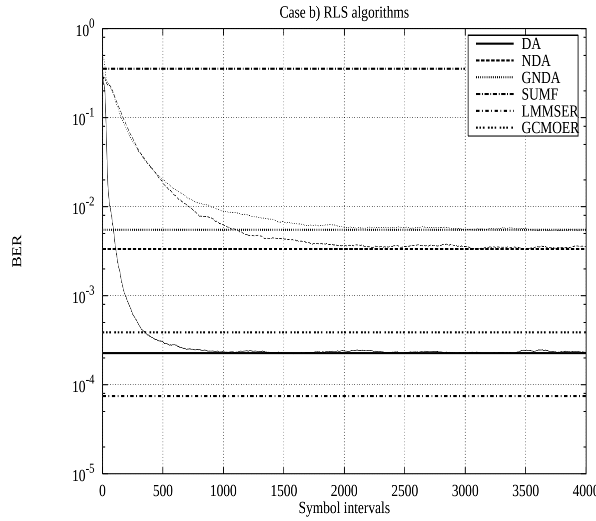

Fig. 2. BER vs. number of symbols for the RLS algorithms in near-far case (b).

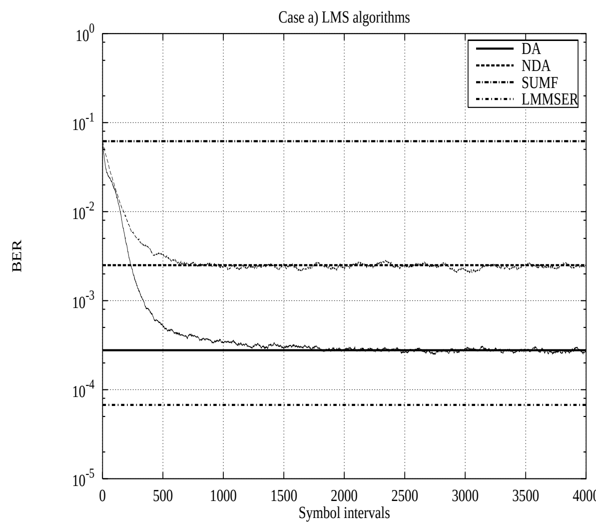

Fig. 3. BER vs. number of symbols for the LMS algorithms in near-far case (a).

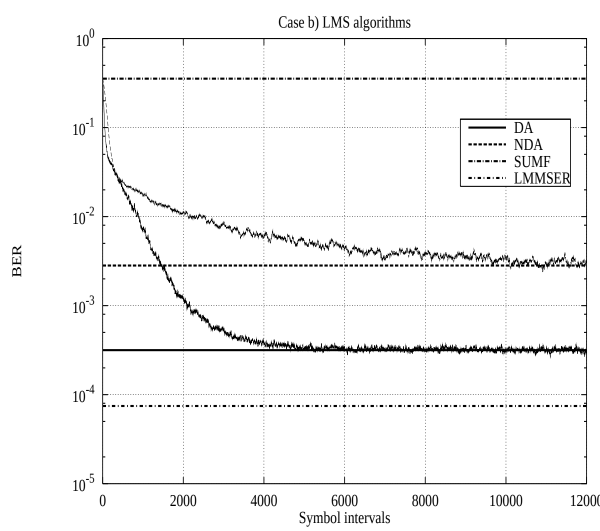

Fig. 4. BER vs. number of symbols for the LMS algorithms in near-far case (b).