D-MG Tradeoff and Optimal Codes for a Class of AF and DF Cooperative Communication Protocols

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***Abstract*—Cooperative relay communication in a fading channel environment under the orthogonal amplify-and-forward (OAF), nonorthogonal and orthogonal selection decode-and-forward (NSDF and OSDF) protocols is considered here. The diversity–multiplexing gain tradeoff (DMT) of the three protocols is determined and DMT-optimal distributed space–time (ST) code constructions are provided. The codes constructed are sphere decodable and in some instances incur minimum possible delay.**

Included in our results is the perhaps surprising finding that the orthogonal and the nonorthogonal amplify-and-forward (NAF) protocols have identical DMT when the time durations of the broadcast and cooperative phases are optimally chosen to suit the respective protocol. Moreover our code construction for the OAF protocol incurs less delay.

Two variants of the NSDF protocol are considered: fixed-NSDF and variable-NSDF protocol. In the variable-NSDF protocol, the fraction of time occupied by the broadcast phase is allowed to vary with multiplexing gain. The variable-NSDF protocol is shown to improve on the DMT of the best previously known static protocol when the number of relays is greater than two. Also included is a DMT optimal code construction for the NAF protocol.

***Index Terms*—Cooperative diversity, cyclic division algebra codes, distributed space–time (ST) code, diversity–multiplexing gain tradeoff (DMT), nonorthogonal amplify-and-forward (NAF), orthogonal amplify-and-forward(OAF), selection decode-and-forward, ST codes.**

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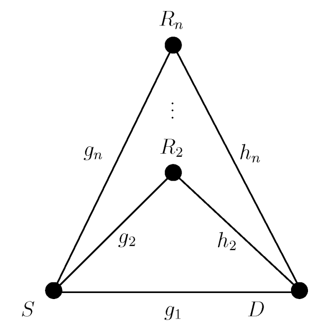

Fig. 1. Cooperative relaying in networks.

I. Introduction

COOPERATIVE relay communication is a promising means of wireless communication in which cooperation is used to create a virtual transmit array between the source and the destination, thereby providing spatial diversity for combating the fading channel. Such cooperative communication is under consideration for example, by the IEEE 802.16j Relay Task Group, see [12], [26].

Consider a two-hop relay network as shown in Fig. 1, in which there are $n+1$ nodes that cooperate in the communication between source node S and destination node D. The remaining $(n-1)$ nodes thus act as relays.

A. Assumptions

We follow the literature in making the assumptions listed below concerning the channel. Our description is in terms of the equivalent, complex-baseband, discrete-time channel.

- All nodes have a single transmit and single receive antenna and are assumed to transmit synchronously.
- The T channel uses over which communication relating to a single message vector takes place is short enough to invoke the quasi-static assumption, i.e., the channel fading coefficients $\{g_i, h_j\}$ are fixed for the duration of the communication, but vary randomly from one block of T channel uses to the next. Here, g_1 denotes the fading coefficient of the channel between S and D, and $\{g_i, h_i, 2 \leq i \leq n\}$ represent fading coefficients of the channels between S and the relay nodes $\{R_i\}$, and channels between the $\{R_i\}$ and D, respectively (see Fig. 1).
- All channels are assumed to be Rayleigh fading i.e., all fade coefficients are independent and identically distributed (i.i.d.), circularly symmetric complex Gaussian $\mathbb{C}\mathcal{N}(0,1)$ random variables.

- We assume half-duplex operation at each node, i.e., at any given instant a node can either transmit or receive, but not do both.
- The additive-noise variables at the receivers are assumed to be i.i.d., circularly symmetric, complex Gaussian, $\mathbb{C}\mathcal{N}(0,\sigma^2)$ distributed.
- The destination is assumed to know all fading coefficients in the case of amplify-and-forward (AF) protocols but needs to know only the source–destination and relay–destination channel-fading coefficients in the case of decode-and-forward (DF) protocols. We assume that a relay knows only the corresponding source–relay channel fading coefficient, i.e., relay R_j will know only g_j in the case of DF protocols. However, in the case of AF protocols, it is not necessary for the relays to know any of the channel-fading coefficients.

B. *Protocols*

Several cooperative communication protocols are studied in this paper. All protocols considered here involve two-phase communication, where during the first phase lasting for p channel uses, the source S broadcasts to the relays and the destination. In the second cooperation phase lasting for q channel uses, the relays and (or) the source communicate with the destination. In all the protocols, one channel use corresponds to one use of the two-hop network. A protocol is said to be either nonorthogonal or orthogonal depending upon whether the source continues to transmit (to the destination) in the cooperation phase or otherwise. It is said to be a DF or AF protocol depending upon whether the relays decode or amplify the received signal, prior to forwarding it.

Within the class of DF protocols, one distinguishes between fixed decode-and-forward (FDF) and selection decode-and-forward (SDF) protocols. Under an SDF protocol, a relay participates in the cooperation phase only if its measurements of the corresponding source–relay channel fading coefficient g_i reveal the particular source–channel link to lie outside the outage region. In FDF, a relay always decodes. Both the FDF and SDF protocols are static protocols, where the time for which relay participates in the cooperation does not depend on the source–relay channel strength. There is also the class of dynamic protocols which includes the dynamic decode-and-forward protocol (DDF) as well as the enhanced dynamic decode-and-forward protocol (E-DDF) or the partial dynamic decode-and-forward protocol (PDDF). In a dynamic protocol, the relay listens to the source as long as it is necessary to ensure that it is not in outage. Under the DDF protocol, the relay is required to decode the complete source message before it takes part in the cooperation phase, whereas under the E-DDF (PDDF) protocol the relay is required to decode only a part of the source message. We will use OAF, OSDF to refer to orthogonal AF and SDF protocols while NAF and NSDF are the labels applied to nonorthogonal versions of the corresponding protocols. Other protocols such as compress-and-forward [22], incremental AF [5], and quantize-and-map [13] are not considered here.

C. *Channel Model*

The combination of physical wireless channel depicted in Fig. 1 and a cooperative-communication protocol results in an equivalent point-to-point multiple-input multiple-output (MIMO) channel which we shall refer to as the *induced* MIMO channel. The number of transmit and receive antennas in this induced MIMO model is a function of the parameters of the particular protocol. As we shall see, the induced channel model is of the form

$$\underline{y} = H\underline{x} + \underline{w} \tag{1}$$

where $\underline{y}$ corresponds to the signal at the destination, H is the induced channel matrix, $\underline{x}$ is the vector transmitted by the source in the case of an AF protocol, and is the compound vector formed by concatenating the transmissions of the source and the participating relays in the case of DF protocols and where $\underline{w}$ is the noise vector observed at the destination. In a two-phase protocol, the induced channel model represents the relation between the signal transmitted by the source and the relays, and the signal received at the destination at the end of the two phases (broadcast phase of duration p channel uses and cooperation phase of duration q channel uses) of the protocol. As a result, the vector $\underline{y}$ is of size $m \times 1$, where $m = p + q$. As we shall see later, the entries of the induced channel matrix H may not be Rayleigh. In the case of DF protocols, even the structure of the induced channel matrix depends on the number of relays participating in the cooperation phase.

D. *Diversity–Multiplexing Gain Tradeoff (DMT)*

The diversity–multiplexing gain tradeoff (DMT) was introduced by Zheng and Tse in [23] as a means of assessing the capabilities of point-to-point MIMO channels. Given a two-hop network operating under a particular protocol, one can apply this theory to the associated induced MIMO-channel model given in (1). By this means, the DMT can be used to evaluate and compare various proposed protocols for the two-hop network. We shall now describe the DMT of the induced channel associated with a particular protocol for the two-hop network.

Let ρ be the average signal-to-noise ratio (SNR) of any link in the network. Let $\{C(\rho,\wp)\}$ be a family of distributed codes indexed by ρ, each one of block length T, under a particular protocol $\wp$. Let $R(\rho,\wp)$ be the average rate of the code $C(\rho,\wp)$, in bits per channel use (network use). The rate of the code is allowed to vary with ρ as

$$R(\rho,\wp) \approx r\log\rho \tag{2}$$

where r is called the multiplexing gain of the coding scheme $\{C(\rho,\wp)\}$. The network is said to be in outage for a fixed SNR ρ, under a particular protocol, if the induced channel H of that protocol is unable to support communication at a rate $R(\rho,\wp)$ bits per channel use.

a) Outage of the network under a particular protocol: As mentioned earlier, the combination of two-hop network and protocol give rise to an induced channel model. Since we fix the network and consider various protocols here, we will associate the induced channel with the corresponding protocol. We will denote a protocol as $\wp$. The probability of outage of the protocol $\wp$ is then defined as

$$P_{\text{out},\wp}(r\log\rho) = \inf_{\substack{\Sigma_x \succeq 0 \\ \text{Tr}(\Sigma_x)\leq p\mathcal{E}}} \Pr\left[I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H}=H) < mr\log\rho\right] \tag{3}$$

where $I(.;.)$ denotes the mutual information. We have imposed an average energy constraint on the transmitted signal $\underline{x}$ by upper-bounding the trace of its covariance matrix Σ_x. The notation $\Sigma_x \succeq 0$ means Σ_x is a positive-semidefinite Hermitian matrix. In general, if A and B are matrices, then $A \succeq B$ ($A \preceq B$) indicates that $A - B$ ($B - A$) is a positive-semidefinite Hermitian matrix.

The outage exponent $d_{\text{out},\wp}(r)$ of the two-hop network, operating under protocol $\wp$ is defined as

$$d_{\text{out},\wp}(r) = -\lim_{\rho\to\infty}\frac{\log P_{\text{out},\wp}(r\log\rho)}{\log\rho}. \tag{4}$$

The outage exponent of the two-hop network is then defined as the supremum of the outage exponents taken over all possible protocols.

b) DMT of the network under a particular protocol: A coding scheme $\{C(\rho,\wp)\}$ for the induced channel in (1) is said to achieve multiplexing gain r and diversity gain $d_\wp(r)$ if

$$\lim_{\rho\to\infty}\frac{R(\rho,\wp)}{\log\rho} = r \tag{5}$$

$$\lim_{\rho\to\infty}\frac{\log P_e(\rho,\wp)}{\log\rho} = -\,d_\wp(r) \tag{6}$$

where $P_e(\rho,\wp)$ is the average error probability of the code $C(\rho,\wp)$ under maximum-likelihood decoding and under the protocol $\wp$. In a slight abuse of terminology, we will simply write *code* in place of coding scheme. We shall also refer to $d_\wp(r)$ as the SNR exponent of the code $C(\rho,\wp)$. The optimal DMT of the two-hop network operating under the protocol $\wp$, denoted as $d^*_\wp(r)$, is defined as the supremum of $d_\wp(r)$ taken over all possible codes i.e.,

$$d^*_\wp(r) = \sup_{C(\rho,\wp)} d_\wp(r). \tag{7}$$

It can be shown from an application of Fano's inequality (see [23]) that for any code

$$d_\wp(r) \leq d_{\text{out},\wp}(r). \tag{8}$$

For all the protocols proposed in this paper, we identify explicit codes which satisfy the above bound with equality, i.e., codes whose SNR exponent equals the outage exponent of the two-hop network under the particular protocol $\wp$. As a result, the outage exponent $d_{\text{out},\wp}(r)$ will coincide with the DMT $d^*_\wp(r)$ of the two-hop network operating under the protocol $\wp$. We shall also drop the subscript $\wp$ henceforth and refer to the DMT of a protocol as $d^*(r)$ and the SNR exponent of a code as $d(r)$ since the protocol under consideration will be clear from the context. The code which achieves the bound in (8) is known as the DMT optimal code for that particular protocol.

c) Upper-bound on the DMT of the network: We first state two definitions before discussing the upper bound on the DMT of the network.

The DMT of the network $d_{\text{nw}}(r)$ is defined as the supremum of the outage exponents taken over all possible protocols and codes, i.e.,

$$\begin{aligned} d_{\text{nw}}(r) &= \sup_\wp\, d^*_\wp(r) \\ &= \sup_\wp\, \sup_{C(\rho,\wp)} d_\wp(r). \end{aligned} \tag{9}$$

The outage exponent of a matrix H of size $m\times n$ is defined as the outage exponent of the associated channel $\underline{y} = H\underline{x} + \underline{w}$, where $\underline{y}$ is an $m\times 1$ vector, $\underline{x}$ is an $n\times 1$ vector, and $\underline{w}$ is an $m\times 1$ vector distributed as $\mathbb{C}\mathcal{N}(0,I)$.

The cut-set bound developed for general multiterminal networks [24] can be used to obtain an upper bound on the DMT of any network (see [22]). The nodes in a network can be partitioned into two sets (Ω,Ω^c), such that the source belongs to Ω and the destination belongs to Ω^c. Such a partition is termed as a cut. We shall denote a cut (Ω,Ω^c) by Ω and let Λ be the set of all such cuts.

Let X_Ω and X_{Ω^c} be the signals transmitted by the nodes in Ω and Ω^c, respectively, and Y_{Ω^c} be the signals received by the nodes in Ω^c. Let H_{nw} be the channel matrix consisting of all the fading coefficients of the network. The event that the cut Ω is in outage is defined as

$$\mathcal{O}_\Omega = \left\{H_{\text{nw}} : I(X_\Omega; Y_{\Omega^c}|X_{\Omega^c}, H_{\text{nw}}) < r\log\rho\right\}, \tag{10}$$

and the probability of outage of the cut Ω is given by

$$P_{\text{out},\Omega}(r\log\rho) = \inf_{p(X_{\Omega,\Omega^c})} \Pr(\mathcal{O}_\Omega) \tag{11}$$

where $p(X_{\Omega,\Omega^c})$ is the probability distribution of the transmitted signals. The outage exponent $d_{\text{out},\Omega}(r)$ is then defined as

$$d_{\text{out},\Omega}(r) = -\lim_{\rho\to\infty}\frac{\log P_{\text{out},\Omega}(r\log\rho)}{\log\rho}. \tag{12}$$

Then, for any coding scheme $C(\rho,\wp)$, under any particular protocol $\wp$, it can be shown using Fano's inequality and the cut-set bound for the general multiterminal networks that

$$d_\wp(r) \leq d_{\text{out},\Omega}(r). \tag{13}$$

Since the above bound is true for every cut Ω, we get

$$d_\wp(r) \leq \min_{\Omega\in\Lambda} d_{\text{out},\Omega}(r). \tag{14}$$

Each cut has an associated channel matrix H_Ω, which is the channel matrix between the nodes on the source and the destination sides of the cut. It can be shown that $d_{\text{out},\Omega}(r)$ is the outage exponent of the channel matrix H_Ω. Now from the definition of DMT of the network and (14) we get

$$d_{\text{nw}}(r) \leq \min_{\Omega\in\Lambda} d_{\text{out},\Omega}(r) \tag{15}$$

$$:= d_{\text{cut-set}}(r) \tag{16}$$

where $d_{\text{cut-set}}(r)$ is known as the cut-set bound.

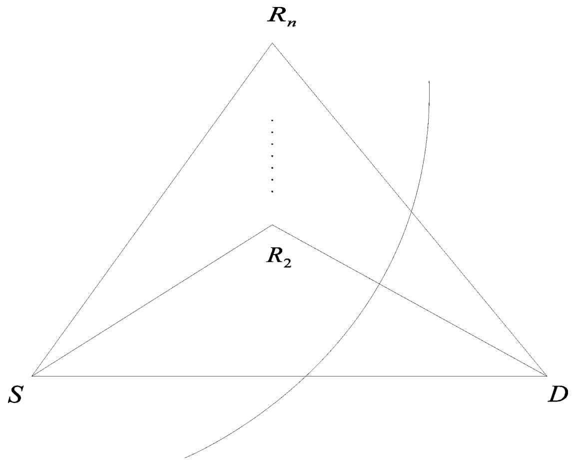

Fig. 2. Dominating cut in the two-hop network.

In the case of the two-hop network considered here, the dominating cut turns out to be the one in which the source and all the relays are on one side of the cut and the destination alone is on the other side of the cut (see Fig. 2). Thus, for the two-hop network the cut-set bound is given by

$$d_{\text{cut-set}}(r) = n(1-r) \tag{17}$$

where $n-1$ is the total number of relays. The cut-set bound of the two-hop network is also referred to in the literature as the transmit diversity bound. Construction of protocols whose DMT equals $d_{\text{cut-set}}(r)$ remains to be an open problem when the number of relays is greater than one.

We use the symbol $\doteq$ to denote exponential equality, i.e., the expression

$$\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho} = b \tag{18}$$

is denoted as $f(\rho) \doteq \rho^b$ and $\dot{\geq}$, $\dot{\leq}$ are similarly defined.

E. Prior Work

The idea of obtaining spatial diversity through the cooperation of mobile users was first proposed in [16], [17] while the quasi-static setting adopted in this paper is drawn from [5]. The SDF and incremental AF protocols are introduced in [5], where the focus is on the case of a single relay. The DMT of the OAF, FDF, SDF, and incremental AF protocols is determined there for the single-relay case when the two phases of the protocols are of equal duration.

In [10], the authors expand their attention to the network as a whole. They consider a wireless network with n cooperating terminals $\mathcal{M} = \{1, 2, \ldots, n\}$, with each terminal $s \in \mathcal{M}$ having an interest in communicating to a corresponding destination terminal $d(s) \notin \mathcal{M}$. The communication between source s and destination $d(s)$, with the remaining terminals in $\mathcal{M}\backslash\{s\}$ acting as potential relays, is envisaged to take place over a collection of orthogonal channels (for, e.g., orthogonal in frequency), one channel per source terminal in $\mathcal{M}$. The focus in [10] is on OSDF protocols. A terminal in $\mathcal{M} \setminus s$ will decode the source message only if the corresponding source–terminal channel is not in outage and will then proceed to relay the signal to the destination. Two variations of the OSDF protocol are considered, labeled as repetition and space–time (ST) coded OSDF protocols, respectively. Under the repetition-code-based OSDF protocol, each relay node is assigned a distinct time slot in which it repeats the decoded message to the destination. In the ST coded version, all relays are permitted to transmit simultaneously and are permitted to re-encode the message using independent codebooks. The DMT of the repetition-based OSDF protocol is determined as are bounds on the DMT of the ST-coded version of the protocol.

In [1], Azarian *et al.* analyze the class of NAF protocols, introduced earlier by Nabar *et al.* [11]. The authors show that these NAF protocols have improved performance in comparison with either the OAF protocol presented in [5] or the class of OSDF protocols considered in [10]. The authors also introduce the DDF protocol, where the relay listens to the source as long as it is necessary to ensure that it is not in outage. They compute the DMT of DDF protocol and show that the DMT achieves the cut-set bound, $d_{\text{cut-set}}(r)$, for multiplexing gains $r \leq \frac{1}{n}$, where $n-1$ is the total number of relays. The authors also consider protocols for the cooperative broadcast and cooperative multiple-access channels and determine the corresponding DMT.

Prasad and Varanasi [14], consider improved versions of the DF protocol, namely, enhanced static decode-and-forward protocol (E-SDF) and enhanced dynamic decode-and-forward protocol (E-DDF). The DMT analysis here is carried out for the case of a single relay. In both the protocols, the source message is split into two parts and the time durations of the first and the second phases are fixed. The source broadcasts the first and the second part of the message, in the first and the second phases, respectively. In the E-SDF protocol, at the end of the first phase, if the relay decodes the first part of the message, it will re-encode the message and transmit it, otherwise it remains silent. In the E-DDF protocol, the relay waits until it is able to decode the first part of the message and then participates in the cooperation. The E-DDF protocol achieves the transmit diversity bound for $r \leq 0.5$ and performs better than DDF for $r > 0.5$.

Avestimehr and Tse [3] also consider an improved version of the DDF protocol, namely, the PDDF. The PDDF and the E-DDF protocol described above are similar in nature and share the same DMT.

Jing and Hassibi [9] consider an OAF protocol in which relay nodes are permitted to apply a linear transformation to the received signal. The two-hop relay network considered here does not have a direct link between the source and the destination. The authors restrict attention to the case where both the source and the relays transmit for equal-time durations and the linear transformations applied by the relays are unitary. Performance is measured in terms of pairwise error probability. Subsequent work on this protocol, including the construction of distributed ST codes with lesser decoding complexity, can be found in [15]. DMT-optimal codes for the NAF protocol are constructed in [20]. In [21], the same authors consider a new class of NAF protocols called slotted amplify-and-forward (SAF) protocols, and show that these improve upon the performance of the NAF protocol of [1] for the case of two relays. The authors also provide an upper bound to the DMT of the SAF protocol when the

TABLE I
DMT OF VARIOUS AMPLIFY-AND-FORWARD PROTOCOLS

Protocol	Authors	No of relays	Duration of two phases	$d^*(r)$	Delay of optimal code
OAF	LTW [5]	1	$p = q$ (chosen)	$2(1-2r)$	$*$
	EVAK (present paper)	$n-1$	$p = n$, $q = n-1$ (optimal)	$(1-r)^+ + (n-1)(1-2r)^+$	$2n-1$
NAF	AGS [1]	$n-1$	$p = q$ (optimal)	$(1-r)^+ + (n-1)(1-2r)^+$	$4(n-1)$ [20]
SAF	YB [21]	2	$\frac{p}{q} = \frac{1}{2}$ (chosen)	$(1-r)^+ + \left(1-\frac{3}{2}r\right)^+$ 3-slot	9 [21]
		$n-1$	$\frac{p}{q} = \frac{1}{n-1}$ (chosen)	$\leq (1-r)^+ + \left(1-\frac{M}{M-1}r\right)^+$ M-slot	M^2 (under relay isolation)

*There is no known construction of explicit DMT optimal code for E-SDF protocol.

number of slots is finite. This upper bound tends towards the cut-set bound as the number of slots increases. Under the assumption of relay isolation, the naive SAF scheme proposed in [21] achieves the SAF-protocol upper bound for any number of slots. A general means of constructing DMT-optimal codes can be found here, although the codes so constructed are not necessarily of minimum delay. A key distinction between the SAF and NAF protocols is that whereas the NAF protocol permits relays to only forward signals received by them during the period when no other relay is transmitting, the SAF protocol does not place such a restriction.

Yuksel and Erkip [22] have considered the DMT of DF and compress-and-forward (CF) protocols for both full-duplex and half-duplex relays. For the case of single full-duplex relay, when all the terminals (source, relay, and destination) have one antenna each, the DF protocol is shown to achieve $d_{\text{cut-set}}(r)$. However, when all the terminals have multiple antennas, the DF protocol fails to achieve $d_{\text{cut-set}}(r)$ whereas the CF protocol is shown to achieve it. The CF protocol makes the added assumption that all the fading coefficients are known at the relay. They also show that for a single half-duplex relay, with all terminals having multiple antennas, the CF protocol achieves the cut-set bound. The multiple-access relay channel (MARC), where there are multiple sources, one destination, and one relay, is studied in this paper. For the single-antenna half-duplex MARC, the authors compute the DMT of the CF protocol, and show that it is optimal for a range of multiplexing gains.

Recently, Pawar *et al.* [13] propose a protocol which uses the quantize-and-map scheme of [2]. Under this protocol, the source continuously transmits to the destination. In the case of a single relay, the relay listens to the source for half the channel uses, then quantizes the received signal at the noise level, re-encodes the quantized signal, and transmits it to the destination for the remaining channel uses. It is shown that this protocol achieves $d_{\text{cut-set}}(r)$ for the case of a single relay. The authors also generalize the protocol to the case of arbitrary number of relays and show that the generalized protocol achieves $d_{\text{cut-set}}(r)$ under the assumption of relay isolation.

F. Results

Our results relate to cooperative relay communication under the orthogonal OAF, OSDF, and NSDF protocols. Our protocols differ from those considered by other researchers in that, we permit the time durations p, q of the respective broadcast and cooperation phases to be chosen optimally. Moreover, we permit the parameters p and q to vary with the multiplexing gain r to improve the DMT performance.

We determine the DMT of the OAF, NSDF, and OSDF protocols. For all three protocols, we construct codes that are DMT optimal. Code construction draws from elementary number theory as well as from the theory of cyclic division algebras, see [4], [6], [7], [18].

We have listed various AF and DF cooperative communication protocols proposed in the literature along with the corresponding DMT in Tables I and II. For the purpose of comparison, we have also listed the DMT of the protocols proposed in this paper.

1) Orthogonal Amplify and Forward: As mentioned earlier, this protocol was introduced by Laneman *et al.* [5], who analyze this protocol for the case of a single relay. As the name suggests, under this two-phase protocol, the source broadcasts a signal to relay and destination, in the first phase. This is then followed by a relaying phase, wherein the relay amplifies and forwards the signals received by it to the destination.

Our version of the protocol is slightly more general than that in [5] since we permit the relays to operate on the received signal

TABLE II
DMT OF VARIOUS DECODE-AND-FORWARD PROTOCOLS WITH $n-1$ RELAYS

Protocol	Authors	Duration of two phases	$d^*(r)$	Delay of optimal code
DDF	AGS [1]	$\frac{p}{q}$ varies with$\{g_i\}$	$\begin{array}{ll} n(1-r), & 0 \le r \le \frac{1}{n} \\ 1+\frac{(n-1)(1-2r)}{1-r}, & \frac{1}{n} \le r \le \frac{1}{2} \\ \frac{1-r}{r}, & \frac{1}{2} \le r \le 1 \end{array}$	delay $\to \infty$
Variable NSDF	EVAK (present paper)	$p=\kappa q$ κ varies with r (optimal)	$\begin{array}{ll} n\left(1-\frac{(n-1)(\kappa_n+1)}{n}r\right), & 0 \le r \le \frac{1}{\kappa_n+1} \\ \frac{(n-r)(1-r)}{(n-2)r+1}, & \frac{1}{\kappa_n+1} \le r \le 1 \end{array}$, where $\kappa_n = \frac{1+\sqrt{1+4(n-1)^2}}{2(n-1)}$	For fixed (p,q) $(p+q)(p+nq)$
Variable OSDF	EVAK (present paper)	$p=\kappa q$ κ varies with r (optimal)	$\begin{array}{ll} n\left(1-\frac{2n-1}{n}r\right), & 0 \le r \le \frac{n-1}{2n-1} \\ \frac{n(1-r)}{(n-1)r+1}, & \frac{n-1}{2n-1} \le r \le 1 \end{array}$,	For fixed (p,q) $(p+q)(p+(n-1)q)$
E-SDF	PV [14]	$p=\kappa q$ $\kappa=2$	$\begin{array}{ll} 2-3r, & 0 \le r \le \frac{1}{3} \\ \frac{3(1-r)}{2}, & \frac{1}{3} \le r \le 1 \end{array}$ (one relay)	*
E-DDF PDDF	PV [14] AT [3]	$\frac{p}{q}$ varies with $\{g_i\}$	$\begin{array}{ll} 2(1-r), & 0 \le r \le \frac{1}{2} \\ \hat{c}(r), & \frac{1}{2} \le r \le 1 \end{array}$ (one relay) where $\hat{c}(r)$ is the unique root of the polynomial $c^3+(2r-3)c^2+(2-r)(1-r)c+(1-r)^2=0$ s.t $\hat{c}(r) \in \left(\frac{1-r}{r}, 2(1-r)\right)$	delay $\to \infty$

*However, transmitting uncoded QAM can achieve the DMT of the protocol which results in a delay of two channel uses.

using a linear transformation[1] and we allow the source and the relays to transmit for unequal time slots. For this more general version of the protocol, we are able to determine the best possible DMT as well as construct, in a simple way, DMT optimal codes that incur minimum delay.

Included in our results, is the perhaps surprising finding that the DMT of the OAF protocol is identical to the DMT of the NAF protocol. If p and q are the time durations of the broadcast and cooperative phases, respectively, then the best OAF performance results when the ratio $\frac{p}{q}$ is chosen to equal $\frac{n}{(n-1)}$, where $(n-1)$ is the number of relays. We also show how one can construct an ST code that is DMT optimal for the OAF protocol and incurs a total delay of $2n-1$ channel uses as compared to the delay of $4(n-1)$ incurred by the DMT optimal code for the NAF protocol having smallest known delay [20].

2) Selection Decode and Forward: This class of protocols was introduced by Laneman and Wornell [10]. This is also a two-phase protocol, with both phases being of equal time duration. In the first phase, the source broadcasts a message to the destination and the relays. In the second phase, all the relays that are not in outage, independently decode the source message, re-encode it, and then transmit it to the destination, while the source remains silent. In the present paper, we permit the time durations of broadcast and cooperative phases denoted by p and q, respectively, to be unequal.

We consider two versions of this protocol in the present paper:

- *Nonorthogonal Selection Decode and Forward (NSDF):* In this version of the protocol, the source continues to transmit during the second phase. At this point, it is convenient to distinguish between two variants of this protocol.
  - — *Variable-NSDF:* In this variant, in order to obtain the best DMT possible, we allow p and q to vary with the multiplexing gain r. Note that since, p and q are not a function of the channel fading coefficients, this protocol falls within the category of static protocols.
    - The variable-NSDF protocol is shown to improve on the DMT of the best previously known static protocol when the number of relays is greater than two. In the case of single relay, the variable NSDF protocol has better DMT compared to the E-SDF protocol [14], for $r \le 0.5$.

[1]In this sense, a more appropriate term for this protocol we consider here might be "linearly transform and forward." However, following Jing and Hassibi [9] we shall regard this protocol as falling under the category of AF protocols.

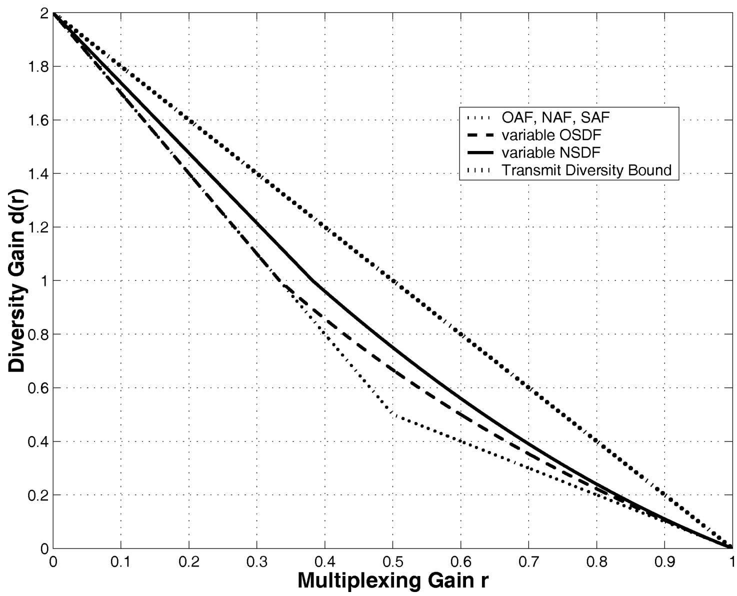

Fig. 3. Optimal DMT for single-relay cooperative communication protocols.

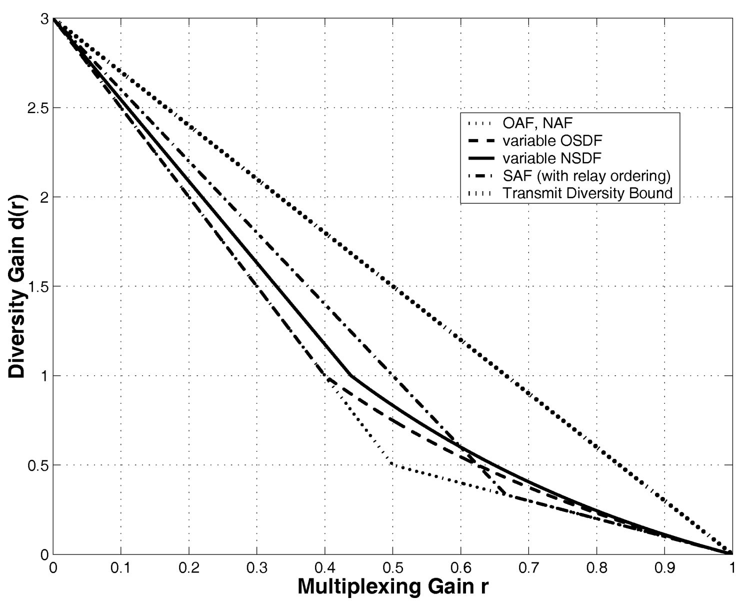

Fig. 4. Optimal DMT for two-relay cooperative communication protocols.

- The DMT of the variable-NSDF protocol for the case of two relays is better than the tradeoff of the SAF protocol [21] for $r > 0.6$ (see Fig. 4).

— *Fixed-NSDF:* Here p and q are fixed and independent of r. The DMT for this variant of the NSDF protocol is determined here for every pair (p, q) with $p \geq q$.

- For values of the ratio $\kappa = \frac{p}{q}$ in the range $1 < \kappa < \frac{n}{n-1}$, the fixed-NSDF protocol dominates the NAF protocol for multiplexing gains r, lying in the range $0 \leq r \leq \frac{p}{p+q}$, beyond which they both have the same DMT.

- *Orthogonal Selection Decode and Forward (OSDF):* In this protocol, the source remains silent in the second phase. As in the case of the NSDF protocol, there are two variants of this protocol, fixed and variable. Under the variable-OSDF protocol, we allow p and q to vary with r in order to compute the best DMT. We determine the DMT for both the fixed and variable-OSDF protocols.

  The DMT of the variable-OSDF protocol, for the case of two relays, improves on the tradeoff of the SAF protocol [21] for $r > \frac{5}{8}$. The DMT of the variable-NSDF protocol is, however, better than the DMT of the variable-OSDF protocol for all r and any number of relays.

For both NSDF and OSDF protocols we construct simply describable codes based on cyclic division algebras.

All the OAF, NSDF, and OSDF protocols considered in this paper are static protocols. In Figs. 3 and 4, we show the optimal DMT of the OAF, NSDF, and OSDF protocols for the case of one and two relays, respectively. Fig. 5 shows the optimal DMT of all the DF protocols, for the case of single relay. In the figures,

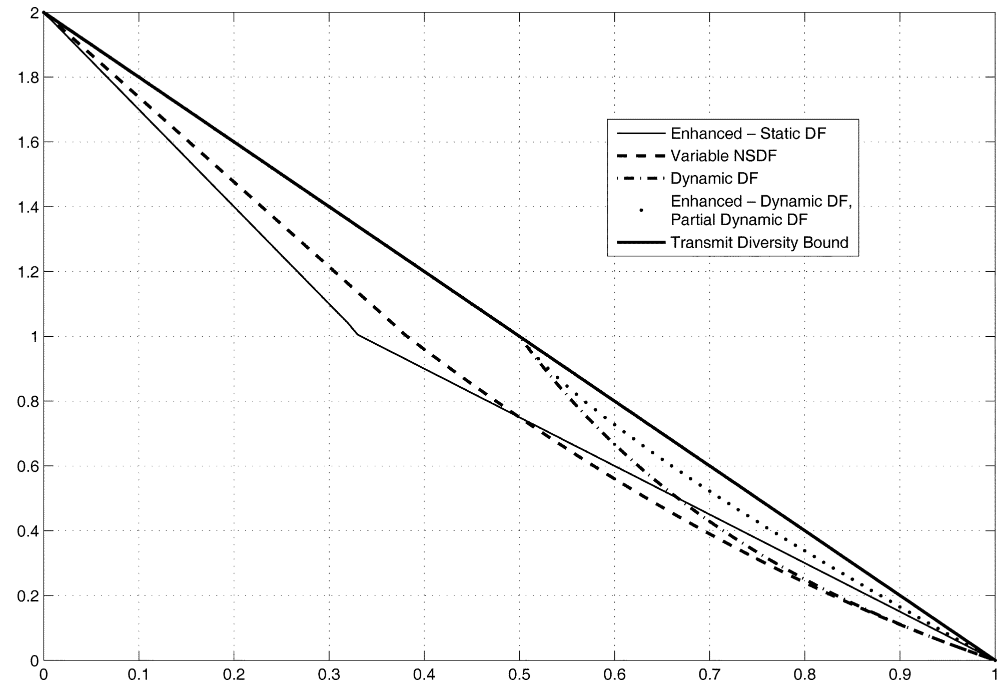

Fig. 5. Optimal DMT for single relay DF protocols.

we also show the DMT of the NAF and SAF protocols for the purpose of comparison. Note that the NAF and SAF protocols have the same DMT in the case of a single relay.

As our final result, we present a code that achieves the DMT of the NAF protocol considered by Azarian *et al.* [1]. This construction was first presented in [8].

G. Organization of the Paper

In Section II, we discuss the class of OAF protocols and compute their DMT. Section III contains the description and DMT analysis of the NSDF protocols. We present the DMT of the OSDF protocols in Section IV. We also provide explicit constructions of DMT optimal codes for the OAF, NSDF, and OSDF protocols, based on elementary number theory and cyclic division algebras, in the corresponding sections. Also, a code construction which is DMT optimal for the NAF protocol is given in Section V. In Section VI, we present the numerical results on the outage probabilities of the proposed protocols and also the performance of the explicit code proposed for the OAF protocol. Finally, we give concluding remarks in Section VII.

Notation: The norm of a vector and the Frobenius norm of a matrix are denoted by $||.||$ and $||.||_F$, respectively. The symbol $|.|$ denotes the determinant of a matrix as well as the modulus of a complex scalar.

II. The Orthogonal Amplify and Forward Protocol

Under this protocol, the source S transmits a signal to the relays $\{R_j\}$ and to the destination D for p channel uses. Over the next q channel uses, each relay R_j applies a linear transformation A_j to the received signal and simultaneously transmits it to the destination, while the source remains silent. Tha main aim of this section is to prove the following theorem.

Theorem 1: The DMT of the OAF protocol when the number of relays is $n-1$ is given by

$$d^*(r) = \begin{cases} n\left(1 - \frac{(2n-1)r}{n}\right), & 0 \leq r \leq \frac{1}{2} \\ (1-r), & \frac{1}{2} < r \leq 1. \end{cases} \quad (19)$$

Remark: The DMT of the OAF protocol is obtained by choosing $\frac{p}{p+q} = \frac{n}{2n-1}$ when $r \leq 1/2$ and $\frac{p}{p+q} = 1$ when $r > 1/2$. The case $\frac{p}{p+q} = 1$ corresponds to noncooperation, i.e., the source continuously transmits to the destination while the relays remain silent. While this choice of parameters (p,q) is the outcome of an optimization procedure we now provide a heuristic explanation as to why noncooperation wins over cooperation for $r > \frac{1}{2}$.

Consider the noncooperation case when the rate $r \geq \frac{1}{2}$ and the relays transmit to the destination for a fraction $\frac{q}{m}$ of channel uses where $m = p+q$. If $\frac{p}{m} < \frac{1}{2}$, the source–destination link will be in outage with probability 1. If $\frac{p}{m} > \frac{1}{2}$, i.e., $\frac{q}{m} < \frac{1}{2}$, then it can be shown that the multiple-input single-output (MISO) channel linking the $(n-1)$ relays to the sink will be in outage for any $\frac{q}{m} < \frac{1}{2}$. On the other hand, any channel uses allocated to the relay–destination link represent channel uses taken away from the source–destination link. It is not surprising therefore that noncooperation wins over cooperation in the regime $\frac{1}{2} \leq r \leq 1$.

Outline of Proof of Theorem 1: The proof will proceed as follows.

- The theorem considers a class of OAF protocols where each protocol corresponds to a distinct choice of the parameters (p,q) and relay matrices $\{A_j\}$. For every such protocol we identify its induced channel matrix.
- We then in Lemma 3 obtain an upper bound to the outage exponent $d_{\text{out}}(r)$ of each protocol that belongs to this class.

The upper bound depends only on the parameters (p, q) and not on the relay matrices $\{A_j\}$. We show that the choice $\frac{p}{p+q} = \frac{n}{2n-1}$ for $r \le \frac{1}{2}$ and $\frac{p}{p+q} = 1$ for $r > \frac{1}{2}$, achieves the largest upper bound among the class of all OAF protocols. This largest upper bound equals $d^*(r)$ given by Theorem 1.

- An interesting feature of the outage analysis presented here is that it results in a suggested specific protocol that achieves the largest upper bound on the outage exponent for $r \le \frac{1}{2}$. This specific protocol is then identified and its outage exponent is computed in Lemma 4. The outage exponent of this specific protocol along with noncooperation for $r > \frac{1}{2}$, is referred to as the outage exponent $d_{\text{out}}(r)$ of the OAF protocol. This is summarized in Lemma 5.
- The proof is completed by identifying a code whose SNR exponent $d(r)$ equals the outage exponent $d_{\text{out}}(r)$ of the OAF protocol, thereby establishing the DMT $d^*(r)$ given in the statement of the theorem.

A. OAF Channel Model

Based on the above signaling protocol, we have the following model for the received signal:

$$\underline{y}_1 = g_1 \underline{x} + \underline{w}_1 \tag{20}$$

$$\underline{r}_j = g_j \underline{x} + \underline{z}_j, \quad j = 2, 3, \ldots, n \tag{21}$$

$$\begin{aligned}\underline{y}_2 &= \sum_{j=2}^{n} h_j A_j \underline{r}_j + \underline{w}_2 \\ &= \left[\sum_{j=2}^{n} g_j h_j A_j\right] \underline{x} + \sum_{j=2}^{n} h_j A_j \underline{z}_j + \underline{w}_2\end{aligned} \tag{22}$$

where

- $\underline{x}$ is the $p \times 1$ signal vector;
- $[\underline{y}_1^t \ \underline{y}_2^t] = \underline{y}^t$ is the $m \times 1$ vector received at the destination;
- $\underline{r}_j$ is the signal received by the jth relay;
- $\{A_j\}$ are $(q \times p)$ matrices that represent the linear transformation taking place at the relay nodes;
- $\{g_i\}$ and $\{h_j\}$ are the various channel fading coefficients;
- and the vectors $\{\underline{z}_j\}_{j=2}^{n}$ and $\{\underline{w}_1, \underline{w}_2\}$ represent the additive noise seen by the receivers located at the relay nodes and destination respectively.

1) Induced Channel Model: The received signal can be written as

$$\underline{y} = H\underline{x} + \underline{w}. \tag{23}$$

where

$$H = \begin{bmatrix} g_1 I_p \\ \sum_{j=2}^{n} g_j h_j A_j \end{bmatrix}, \tag{24}$$

and[2]

$$\underline{w} = \begin{bmatrix} \underline{w}_1 \\ \sum_{j=2}^{n} h_j A_j \underline{z}_j + \underline{w}_2 \end{bmatrix}. \tag{25}$$

[2]Strictly speaking, the channel matrix H notation should have indicated a dependence on the protocol parameters $(p, q, \{A_j\})$, but we have suppressed this for notational simplicity.

This represents the induced MIMO channel model for the OAF protocol. The covariance matrices of noise and signal vector are given, respectively, by

$$\begin{aligned}\Sigma_w &:= \mathbb{E}(\underline{w}\underline{w}^\dagger) \\ &= \begin{bmatrix} \sigma_w^2 I_p & 0 \\ 0 & \sigma_w^2 I_q + \sum_{j=2}^{n} |h_j|^2 \sigma_{z_j}^2 A_j A_j^\dagger \end{bmatrix}\end{aligned} \tag{26}$$

and

$$\Sigma_x := \mathbb{E}(\underline{x}\underline{x}^\dagger), \tag{27}$$

where $\sigma_{z_j}^2, \sigma_w^2$ denote the variances of the corresponding noise vectors variables. We have assumed that the variance of the noise added at the destination in the first and the second phase of the protocol to be the same, i.e., $\sigma_{w_1}^2 = \sigma_{w_2}^2 = \sigma_w^2$. We impose the energy constraint

$$\text{Tr}(\Sigma_x) \le p\mathcal{E} \tag{28}$$

where $\mathcal{E}$ denotes the average energy available for transmission of a source symbol. We define

$$\rho = \frac{\mathcal{E}}{\sigma_w^2} \tag{29}$$

as the average received SNR at the destination. We assume the ratio of the noise variances $\{\sigma_{z_j}^2\}$ and σ_w^2 to be a constant independent of ρ. The average energy of the signal transmitted by the jth relay is bounded above (from (28)) by

$$\begin{aligned}&\mathbb{E}\{\|g_j A_j \underline{x} + A_j \underline{z}_j\|^2\} \\ &= |g_j|^2 \mathbb{E}\{\|A_j \underline{x}\|^2\} + \text{Tr}(A_j \mathbb{E}\{\underline{z}_j \underline{z}_j^\dagger\} A_j^\dagger) \\ &\le |g_j|^2 p\mathcal{E} \text{Tr}\{A_j A_j^\dagger\} + \sigma_w^2 \text{Tr}\{A_j A_j^\dagger\} \\ &= \alpha_j^2 (p\mathcal{E}|g_j|^2 + \sigma_w^2)\end{aligned} \tag{30}$$

where α_j^2 is the squared Frobenius norm of the relay matrices $\{A_j\}$, i.e.,

$$\|A_j\|_F^2 = \text{Tr}(A_j A_j^\dagger) := \alpha_j^2. \tag{31}$$

If we impose the constraint that the average energy transmitted by a relay satisfy

$$\mathbb{E}\{\|g_j A_j \underline{x} + A_j \underline{z}_j\|^2\} \le \mathcal{E} \tag{32}$$

i.e.,

$$\alpha_j^2 \le \frac{\mathcal{E}}{p\mathcal{E}|g_j|^2 + \sigma_w^2}, \tag{33}$$

then this constraint can be met with probability one by choosing α_j^2 to be a suitably large constant.

B. Upper Bounds on the Outage Exponent of OAF Protocols

We first state a useful lemma concerning nonnegative definite matrices.

Lemma 2: Let $\{g_j, h_j, A_j\}$ be as defined above. Let

$$B = \sum_{j=2}^{n} g_j h_j A_j. \tag{34}$$

Then

$$BB^\dagger \preceq (n-1)\sum_{j=2}^{n} \gamma_j A_j A_j^\dagger \tag{35}$$

where $\gamma_j = |g_j h_j|^2$.

Proof: Please see Appendix A. □

Next, we establish upper bounds on the outage exponent of OAF protocols.

Lemma 3 (Upper-bounds on outage exponent of OAF protocols): Consider the collection of OAF protocols described above where each protocol corresponds to a distinct choice of parameters p, $q = m - p$, and relay matrices $\{A_j\}$ as mentioned earlier. Then, for any choice of the transformation matrices $\{A_j\}$, the outage exponent of the corresponding protocol satisfies the upper bounds given below.

If $\frac{p}{m} \le \frac{n}{2n-1}$, then

$$d_{\text{out}}(r) \le \begin{cases} n(1-\frac{mr}{p}), & 0 \le r \le \frac{p}{m} \\ 0, & \frac{p}{m} < r \le 1. \end{cases} \tag{36}$$

If $\frac{p}{m} \ge \frac{n}{2n-1}$

$$d_{\text{out}}(r) \le \begin{cases} n\left(1-\frac{(n-1)mr}{nq}\right), & 0 \le r \le \frac{q}{m} \\ \frac{p}{p-q}\left(1-\frac{mr}{p}\right), & \frac{q}{m} \le r \le \frac{p}{m} \\ 0, & \frac{p}{m} < r \le 1. \end{cases} \tag{37}$$

For $0 \le r \le \frac{1}{2}$, the least restrictive upper bound results from choosing $\frac{p}{m} = \frac{n}{2n-1}$. For $\frac{1}{2} \le r \le 1$, the least restrictive upper bound results from choosing $\frac{p}{m} = 1$. These statements taken together lead to the following upper bound on the outage exponent:

$$d_{\text{out}}(r) \le \begin{cases} n\left(1-\frac{(2n-1)r}{n}\right), & 0 \le r \le \frac{1}{2} \\ (1-r), & \frac{1}{2} < r \le 1. \end{cases} \tag{38}$$

Proof: We shall use the induced MIMO channel model (23) to find the upper bound on the outage exponent. The probability of outage, for the channel in (23), is given by

$$P_{\text{out}}(r\log\rho) = \inf_{\substack{\Sigma_x \succeq 0 \\ \text{Tr}(\Sigma_x) \le p\mathcal{E}}} \Pr\left[I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H}=H) < mr\log\rho\right] \tag{39}$$

where $I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H}=H)$ is the mutual information between $\underline{x}$ and $\underline{y}$, conditioned on the knowledge of H at the receiver. The input distribution can be assumed to be circularly symmetric Gaussian with covariance matrix Σ_x, without loss of optimality. Then the mutual information is given by

$$I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H}=H) = \log | I_m + H\Sigma_x H^\dagger \Sigma_w^{-1} |. \tag{40}$$

Substituting the above expression in (39) we get

$$P_{\text{out}}(r\log\rho) = \inf_{\Sigma_x \succeq 0,\ \text{Tr}(\Sigma_x) \le p\mathcal{E}} \Pr\left[\log | I_m + H\Sigma_x H^\dagger \Sigma_w^{-1} | < mr\log\rho\right]. \tag{41}$$

Since our interest is only in the outage exponent

$$d_{\text{out}}(r) = -\lim_{\rho\to\infty} \frac{P_{\text{out}}(r\log\rho)}{\log\rho} \tag{42}$$

by arguing as in [23], we can show that insofar as computation of $d_{\text{out}}(r)$ is concerned we may replace Σ_x by $\mathcal{E}I_m$. As a result we have

$$P_{\text{out}}(r\log\rho) \doteq \Pr\left[\log | I_m + \mathcal{E}HH^\dagger\Sigma_w^{-1} | < mr\log\rho\right]. \tag{43}$$

Whenever such situations are encountered in the sequel, we shall simply say that in the scale of interest we may replace Σ_x by $\mathcal{E}I_m$. Let

$$\mathcal{J} := I_m + \mathcal{E}HH^\dagger\Sigma_w^{-1} \tag{44}$$

so that we have

$$P_{\text{out}}(r\log\rho) \doteq \Pr(\log | \mathcal{J} | < mr\log\rho). \tag{45}$$

Since $d_{\text{out}}(r)$ is the negative ρ exponent of $P_{\text{out}}(r\log\rho)$, to obtain an upper bound on $d_{\text{out}}(r)$ we upper-bound the determinant of $\mathcal{J}$. Substituting for H (see (24)) and Σ_w (see (26)) in (44) we get

$$\mathcal{J} = I_m + \frac{\mathcal{E}}{\sigma_w^2}\begin{bmatrix} |g_1|^2 & g_1 B^\dagger \\ g_1^* B & BB^\dagger \end{bmatrix}\begin{bmatrix} I_p & 0 \\ 0 & C^{-1} \end{bmatrix} \tag{46}$$

where $C = I_q + \sum_{j=2}^{n} |h_j|^2 \frac{\sigma_{z_j}^2}{\sigma_w^2} A_j A_j^\dagger$, $\gamma_1 = |g_1|^2$, and B is as defined in Lemma 2. Using the relation $\rho = \frac{\mathcal{E}}{\sigma_w^2}$ we get

$$\mathcal{J} = \begin{bmatrix} I_p(1+\rho\gamma_1) & \rho g_1 B^\dagger C^{-1} \\ \rho g_1^* B & I_q + \rho BB^\dagger C^{-1} \end{bmatrix}. \tag{47}$$

Upon row reduction of $\mathcal{J}$, we obtain

$$\begin{aligned} | \mathcal{J} | &= | (1+\rho\gamma_1)I_p | \cdot | I_q + \frac{\rho}{1+\rho\gamma_1} BB^\dagger C^{-1} | \\ &= (1+\rho\gamma_1)^p | C^{-1} | | C + \frac{\rho}{1+\rho\gamma_1} BB^\dagger |. \end{aligned} \tag{48}$$

In order to upper-bound the determinant of $\mathcal{J}$ we upper-bound each of the determinants in the above expression. The matrix C is positive–definite, with all the eigenvalues greater than or equal to 1 due to the fact that

$$\sum_{j=2}^{n} |h_j|^2 \frac{\sigma_{z_j}^2}{\sigma_w^2} A_j A_j^\dagger = \sum_{j=2}^{n} \left(\frac{\sigma_{z_j}}{\sigma_w} h_j A_j\right)\left(\frac{\sigma_{z_j}}{\sigma_w} h_j A_j\right)^\dagger \tag{49}$$

is nonnegative–definite. Hence, we get

$$| C | \ge | I_q | \tag{50}$$

or

$$| C^{-1} | \le 1. \tag{51}$$

Also

$$C + \frac{\rho}{1+\rho\gamma_1} BB^\dagger = I_q + \sum_{j=2}^{n} |h_j|^2 \frac{\sigma_{z_j}^2}{\sigma_w^2} A_j A_j^\dagger + \frac{\rho}{1+\rho\gamma_1} BB^\dagger. \tag{52}$$

By applying Lemma 2, we have

$$C + \frac{\rho}{1+\rho\gamma_1} BB^\dagger \preceq I_q + \sum_{j=2}^{n} \left(|h_j|^2 \frac{\sigma_{z_j}^2}{\sigma_w^2} + \frac{(n-1)\rho}{1+\rho\gamma_1}\gamma_j\right) A_j A_j^\dagger$$

$$\preceq I_q + \mathrm{Tr}\left(\sum_{j=2}^{n}\left(|h_j|^2\frac{\sigma_{z_j}^2}{\sigma_w^2}+\frac{(n-1)\rho}{1+\rho\gamma_1}\gamma_j\right)A_jA_j^\dagger\right)I_q$$

$$= I_q + \left(\sum_{j=2}^{n}\left(|h_j|^2\frac{\sigma_{z_j}^2}{\sigma_w^2}+\frac{(n-1)\rho}{1+\rho\gamma_1}\gamma_j\right)\|A_j\|_F^2\right)I_q. \quad (53)$$

Substituting (51) and (53) in (48), we get

$$|\mathcal{J}| \leq (1+\rho\gamma_1)^p \cdot \left| I_q + \left(\sum_{j=2}^{n}\left(|h_j|^2\frac{\sigma_{z_j}^2}{\sigma_w^2}+\frac{(n-1)\rho}{1+\rho\gamma_1}\gamma_j\right)\|A_j\|_F^2\right)I_q \right|. \quad (54)$$

Since $\|A_j\|_F^2$, $\sigma_{z_j}^2$, σ_w^2, and n are constants, in the scale of interest we get

$$|\mathcal{J}| \dot{\leq} (1+\rho\gamma_1)^p\left(1+\sum_{j=2}^{n}\left[|h_j|^2+\frac{\rho}{1+\rho\gamma_1}\gamma_j\right]\right)^q. \quad (55)$$

As in [1], we define

$$\gamma_1 \doteq \begin{cases}\rho^{-u}, & u \geq 0\\ 0, & u<0.\end{cases} \quad (56)$$

For $j=2,3,\ldots,n$, let us set

$$\gamma_j \doteq \begin{cases}\rho^{-v_j}, & v_j \geq 0\\ 0, & v_j<0\end{cases} \quad (57)$$

$$|h_j|^2 \doteq \begin{cases}\rho^{-s_j}, & s_j \geq 0\\ 0, & s_j<0\end{cases} \quad (58)$$

and

$$v=\min\{v_j\}_{j=2}^n, \quad s=\min\{s_j\}_{j=2}^n. \quad (59)$$

This gives us

$$|\mathcal{J}| \dot{\leq} \rho^{(p-q)(1-u)^+ + q\max\{-s,(1-u),(1-u-s),(1-v)\}^+}. \quad (60)$$

Substituting the above expression in (45) we have

$$P_{\text{out}}(r\log\rho) \dot{\geq} \Pr\left[(p-q)(1-u)^+ + q\max\{-s,(1-u),(1-u-s),(1-v)\}^+ < mr\right]$$
$$= \int \rho^{-u}\rho^{-\sum_{j=2}^n v_j}\rho^{-\sum_{j=2}^n s_j}\,du\,dv_2\ldots dv_n ds_2\ldots ds_n \quad (61)$$

where the integration is computed over the region

$$(p-q)(1-u)^+ + q\max\{-s,(1-u),(1-u-s),(1-v)\}^+ < mr. \quad (62)$$

By applying Varadhan's lemma [25] we have

$$d_{\text{out}}(r) \leq \inf\ u+(n-1)v+(n-1)s \quad (63)$$

where the infimum is computed over the region

$$(p-q)(1-u)^+ + q\max\{-s,(1-u),(1-u-s),(1-v)\}^+ < mr. \quad (64)$$

It is clear that it is enough to consider $u,v \leq 1$. Hereafter, we will consider u and v to lie in the range $0 \leq u,v \leq 1$ and $s \geq 0$. Therefore, we can set s to be 0 in the solution of the above optimization problem to get

$$d_{\text{out}}(r) \leq \inf_{(p-q)u+q\min\{u,v\}>p-mr} u+(n-1)v. \quad (65)$$

By solving the above optimization problem, we get the statement of the lemma. Please see Appendix B for the details. □

C. Specific Protocol Achieving the Outage Exponent Upper Bound

Within the class of OAF protocols, there are different protocols corresponding to various choices of p, q, and $\{A_j\}_{j=2}^n$. As seen in Section II-C, for a given number $(n-1)$ of relays, the upper bound on $d_{\text{out}}(r)$ is maximized when $\frac{p}{p+q}=\frac{n}{2n-1}$. In deriving an upper bound on the outage exponent of OAF protocols we used the inequality

$$\left(\sum_{j=2}^n g_jh_jA_j\right)\left(\sum_{j=2}^n g_jh_jA_j\right)^\dagger \preceq (n-1)\left(\sum_{j=2}^n |g_jh_j|^2 A_jA_j^\dagger\right). \quad (66)$$

Equality will occur in (66) if

$$A_jA_k^\dagger = 0, \quad \text{for all } j \neq k \quad (67)$$

i.e., if the row spaces of the matrices $\{A_j\}_{j=2}^n$ are pairwise orthogonal. The arguments presented above serve as a motivation for the particular choice of the matrices $\{A_j\}$ outlined next. With this specific choice of $\{A_j\}$, satisfying the constraints in (67), we can achieve the upper bound on the outage exponent given in Lemma 3.

Lemma 4: Consider a specific OAF protocol, as described above, with parameters $p=n$ and $q=n-1$. Choose the $(n-1)\times n$ matrices $\{A_j\}$ as follows:

$$A_j(k,l)=\begin{cases}\alpha_j, & k=j-1, l=j\\ 0, & \text{elsewhere}\end{cases} \quad (68)$$

i.e., the $(j-1,j)$th entry of A_j is equal to α_j and all remaining entries are 0. The outage exponent of this specific protocol is given by

$$d_{\text{out}}(r)=n-(2n-1)r, \quad 0 \leq r \leq \frac{n}{2n-1}. \quad (69)$$

Proof: With the above choice of $\{A_j\}$ we get

$$BB^\dagger = \begin{bmatrix}\alpha_2^2\gamma_2 & & \\ & \ddots & \\ & & \alpha_n^2\gamma_n\end{bmatrix}. \quad (70)$$

By substituting for $BB^\dagger$ in (48), and setting $p=n$ and $q=n-1$, we get

$$|\mathcal{J}| \doteq (1+\rho\gamma_1)^n \cdot \left|\begin{bmatrix}1+\frac{\rho}{1+\rho\gamma_1}\gamma_2 & & \\ & \ddots & \\ & & 1+\frac{\rho}{1+\rho\gamma_1}\gamma_n\end{bmatrix}\right|$$

$$
\begin{aligned}
&= (1+\rho\gamma_1)\prod_{j=2}^{n}(1+\rho\gamma_1+\rho\gamma_j) \\
&\doteq \rho^{(1-u)^+}\prod_{j=2}^{n}\rho^{(1-\min\{u,v_j\})^+}.
\end{aligned} \tag{71}
$$

As in Lemma 3, the outage probability is given by

$$
\begin{aligned}
&P_{\text{out}}(r\log\rho) \\
&\doteq \Pr(\log|\mathcal{J}| < (2n-1)r\log\rho) \\
&\doteq \Pr\left((1-u)^+ + (1-\sum_{j=2}^{n}\min\{u,v_j\})^+ < (2n-1)r\right).
\end{aligned} \tag{72}
$$

Without loss of optimality, we can consider u and v_j to lie in the range $0 \le u, v_j \le 1$. Then we have

$$
d_{\text{out}}(r) = \inf_{u+\sum_{j=2}^{n}\min\{u,v_j\} > n-(2n-1)r} u + \sum_{j=2}^{n} v_j. \tag{73}
$$

By solving the optimization problem, we get

$$
d_{\text{out}}(r) = n-(2n-1)r,\;\; 0 \le r \le \frac{n}{2n-1}. \tag{74}
$$

□

The outage exponent of this protocol is equal to the largest upper bound of the class of OAF protocols for $r \le \frac{1}{2}$. Now when the source transmits continuously to the destination, with no relay participating we obtain an outage exponent of $1-r$, which achieves the largest upper bound of the class of OAF protocols for $r > \frac{1}{2}$ (see Lemma 3, equation (38)). The combination of the specific protocol discussed above along with noncooperation for $r > \frac{1}{2}$, is hereafter referred to as the OAF protocol. Thus, we have the following lemma.

Lemma 5: The outage exponent of the OAF protocol is given by

$$
d_{\text{out}}(r) = \begin{cases} n\left(1-\frac{(2n-1)r}{n}\right), & 0 \le r \le \frac{1}{2} \\ (1-r), & \frac{1}{2} < r \le 1. \end{cases} \tag{75}
$$

Since $(n, 2n-1) = 1$, the smallest value of delay parameter $m = (p+q)$ satisfying the condition $\frac{p}{p+q} = \frac{n}{2n-1}$ corresponds to the choice $p = n,\ q = (n-1)$. Hence, the above protocol has minimum possible delay required to achieve the best outage exponent.

Example 1: Let the number of relays be two. Therefore, $n = 3$. We choose $p = 3, q = 2$, and

$$
A_2 = \begin{bmatrix} 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},\; A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}. \tag{76}
$$

For these parameters, the outage exponent of the OAF protocol is

$$
d_{\text{out}}(r) = \begin{cases} 3-5r, & 0 \le r \le \frac{1}{2} \\ 1-r, & \frac{1}{2} < r \le 1. \end{cases} \tag{77}
$$

and is shown in Fig. 4.

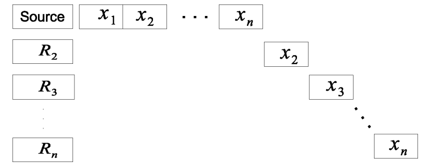

Fig. 6. OAF protocol with $n-1$ relays.

Fig. 6 shows the frame structure of the optimal OAF protocol described above for the case of $n-1$ relays. The structure of the matrix A_j selected gives rise to the following sequence of transmissions. The source transmits a signal for the first n channel uses. In the $(n+j-1)^{\text{th}}$ channel use, relay $R_j, 2 \le j \le n$ transmits the signal it received from the source in the $(j-1)$th channel use.

Proof of Theorem 1: In the previous subsection we computed the outage exponent $d_{\text{out}}(r)$ of the OAF protocol (see Lemma 5). We now provide an explicit construction of a DMT optimal code for the OAF protocol, i.e., a code whose SNR exponent $d(r)$ equals the outage exponent $d_{\text{out}}(r)$ given by Lemma 5.

D. DMT Optimal Code for the OAF Protocol

The code construction is based on elementary number theory. Since any protocol is identified by its induced channel, it is enough to construct a code which achieves the outage exponent of the induced channel. The induced channel model for the OAF protocol is given by

$$
\underline{y} = \theta H \underline{x} + \underline{w} \tag{78}
$$

where

$$
H = \begin{bmatrix} g_1 I_p \\ \sum_{j=2}^{n} g_j h_j A_j \end{bmatrix} \tag{79}
$$

and $\theta\underline{x}$ is the $p \times 1$ codeword vector transmitted by the source, $\underline{y}$ is the received vector at the destination, and $\underline{w}$ is the noise vector which is not white. θ is a scalar chosen to ensure that

$$
\|\theta\underline{x}\|_F^2 \le m\rho. \tag{80}
$$

Remark: Typically, the use of a DMT-optimal code results in time expansion, i.e., the $p \times 1$ vector $\underline{x}$ in

$$
\underline{y} = \theta H \underline{x} + \underline{w} \tag{81}
$$

is replaced by a $p \times T$ code matrix X associated with the equation

$$
Y = \theta H X + W. \tag{82}
$$

However, it turns out that in the case of OAF protocol considered here, no time expansion is necessary, i.e., it is possible to construct a DMT optimal code whose time expansion parameter T equals one.

$$\begin{array}{ll} \mathbb{L} & \mathcal{O}_{\mathbb{L}} = <\beta_1,\ldots,\beta_n> \\ | & | \\ \mathbb{F}=\mathbb{Q}(\imath) & \mathcal{O}_{\mathbb{F}} = \mathbb{Z}[\imath] \end{array}$$

Fig. 7. Structure of field extension.

Now if there are $n-1$ relays, the best OAF protocol is obtained by choosing $p=n, q=n-1$ and the matrices $\{A_j\}$ as given by Lemma 4. The induced channel model then becomes

$$\underline{y} = \theta \begin{bmatrix} & & g_1 I_n & \\ 0 & \alpha_2 g_2 h_2 & & \\ \vdots & & \ddots & \\ 0 & & & \alpha_n g_n h_n \end{bmatrix} \underline{x} + \underline{n} \tag{83}$$

where $\underline{x} = [x_1 \ldots x_n]^t$ is the vector transmitted by the source and $\underline{y}$ is the received vector of length $m = 2n-1$. It can be shown that without loss of optimality, the noise vector $\underline{w}$ can be assumed to be white in the scale of interest. We now drop the symbols $y_2 \ldots y_n$ of the received vector (which can only worsen the performance of the code) and work with the following row deleted induced channel matrix:

$$H = \begin{bmatrix} g_1 & & & \\ & \alpha_2 g_2 h_2 & & \\ & & \ddots & \\ & & & \alpha_n g_n h_n \end{bmatrix}. \tag{84}$$

It can be shown that in the scale of interest the channel H is equivalent to a parallel channel and hence it is sufficient to construct a code which achieves the outage exponent of a parallel channel.

a) Code Construction: Let $\mathbb{L}$ be a degree-n cyclic Galois extension field of $\mathbb{F} = \mathbb{Q}(\imath)$, where $\mathbb{Q}(\imath) = \{a + \imath b \mid a, b, \in \mathbb{Q}\}$. $\mathbb{Q}$ is the set of all rational numbers. Let σ be the generator of the cyclic Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$. Let $\mathcal{O}_{\mathbb{F}}$ and $\mathcal{O}_{\mathbb{L}}$ denote the ring of algebraic integers in $\mathbb{F}$ and $\mathbb{L}$, respectively. It is known that $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}[\imath]$. Let $\{\beta_1, \ldots, \beta_n\}$ be an integral basis for $\mathcal{O}_{\mathbb{L}}/\mathcal{O}_{\mathbb{F}}$ (see Fig. 7).

For M even, let $\mathcal{A}_{\mathrm{QAM}}$ denote the M^2-QAM constellation given by

$$\mathcal{A}_{\mathrm{QAM}} = \{a + \imath b \mid |a|, |b| \le M-1,\ a, b \text{ odd}\} \tag{85}$$

and

$$\mathcal{A}_{\mathrm{QAM}}(\beta_1, \ldots, \beta_n) = \left\{ \sum_i a_i \beta_i \mid a_i \in \mathcal{A}_{\mathrm{QAM}} \right\}. \tag{86}$$

Now, the code vector is given by

$$\theta \underline{x} = \theta \left[\ell_0 \quad \sigma(\ell_0) \quad \ldots \quad \sigma^{n-1}(\ell_0) \right]^t \quad \text{where } \ell_0 \in \mathcal{A}_{\mathrm{QAM}}(\beta_1, \ldots, \beta_n). \tag{87}$$

Interpreting the above code for OAF protocol, the source transmits the signal $\theta \underline{x}$ in the first $p = n$ channel uses. In the next $n-1$ channel uses relay $R_j,\ 2 \le j \le n$, transmits a distinct symbol received from the source in the jth channel use, such that no two relays transmit simultaneously (see Fig. 6).

Lemma 6: The SNR exponent of the above code for the OAF protocol is

$$d(r) = n - (2n-1)r, \ \ 0 \le r \le \frac{n}{2n-1} \tag{88}$$

where r is the multiplexing gain of the code.

Proof: We show that the chosen code is an optimal code for the parallel channel since the induced channel model for the OAF protocol is equivalent to a parallel channel in the scale of interest. Let $R_p = r_p \log(\rho)$ be the target data rate of the induced parallel channel. To support a data rate R_p we need $M^2 = \rho^{\frac{r_p}{n}}$. The energy requirement forces $\theta^2 \doteq \rho^{1-\frac{r_p}{n}}$. Now, the product of the squared norms of the normalized difference code matrices (obtained by scaling the code vectors with $\frac{1}{\sqrt{\rho}}$) is given by

$$\begin{aligned} \frac{1}{\rho^n} |\theta \ell_0|^2 \cdot |\theta \sigma(\ell_0)|^2 \cdots |\theta \sigma^{n-1}(\ell_0)|^2 &= \frac{\theta^{2n}}{\rho^n} \prod_{i=0}^{n-1} |\sigma^i(\ell_0)|^2 \dot{\ge} \frac{\rho^{(1-\frac{r_p}{n})n}}{\rho^n} \rho^0 \\ &\dot{\ge} \rho^{-r_p}. \end{aligned} \tag{89}$$

Therefore, from [19, Theorem 5.1], the code is DMT optimal for the parallel channel, i.e, the SNR exponent of the code equals the outage exponent of the parallel channel. It can be shown that the outage exponent of the parallel channel is (see [19])

$$d_p(r_p) = n - r_p, \quad 0 \le r_p \le n. \tag{90}$$

Now since one use of the induced parallel channel corresponds to $m = 2n-1$ uses of the two-hop network, we have $r_p = (2n-1)r$. By substituting for r_p in (90), we obtain a lower bound on the $d(r)$ of the code for the OAF protocol. The lower bound occurs because dropping some symbols from the received vector could conceivably increase the probability of error. Therefore

$$\begin{aligned} d(r) &\ge d_p((2n-1)r) \\ &\ge n - (2n-1)r, \ \ 0 \le r \le \frac{n}{2n-1}. \end{aligned} \tag{91}$$

□

The above bound equals the $d_{\mathrm{out}}(r)$ of the OAF protocol for $r \le \frac{1}{2}$. For $r > \frac{1}{2}$, the source will transmit continuously to the destination and the relays will not participate. In this case, transmitting uncoded QAM of size ρ^r achieves $d_{\mathrm{out}}(r) = 1 - r$ (see [19]).

Since we have identified a code whose SNR exponent $d(r)$ equals the outage exponent of the OAF protocol, we can refer to $d_{\mathrm{out}}(r)$ given by Lemma 5 as the DMT $d^*(r)$ of the OAF protocol. This concludes the proof of Theorem 1. □

Remarks: We make the following remarks on the class of OAF protocols and the proposed DMT optimal code.

1) In the construction of DMT optimal code for the OAF protocol, it is enough if $\mathbb{L}$ is an algebraic Galois extension (not necessarily cyclic) of $\mathbb{Q}(\imath)$ of degree n and ℓ_0 is as defined in (87). In this case, $\{\sigma(\ell_0), \ldots, \sigma^{n-1}(\ell_0)\}$ would be replaced by the appropriate conjugates of ℓ_0.

2) Among the class of OAF protocols, the best DMT is achieved when $\frac{p}{m} = \frac{n}{2n-1}$, for r in the range $0 \leq r \leq \frac{n}{2n-1}$. Since n and $2n-1$ are relatively prime, the proposed code has the minimum possible delay of $2n-1$ with the parameters $p = n$ and $q = n-1$.
3) For $p = n$ and $q = n-1$, the DMT of the OAF protocol, with the choice of $\{A_j\}$ mentioned in Lemma 4 coincides with the DMT of the NAF protocol [1].
4) When each node in the system has only one transmit and one receive antenna, the DMT optimal code for the NAF protocol proposed in [20] has delay $4(n-1)$ which is larger than the delay $2n-1$ incurred by our construction of DMT optimal code for the OAF protocol introduced here, while both NAF and OAF have the same DMT.

III. The Nonorthogonal Selection Decode and Forward Protocol

In this section, we consider an NSDF protocol in which the source transmits a signal to the destination and the relays for p channel uses in the first phase. All the relays, which are not in outage, [3] will decode the source message. In the second phase, the relays will separately encode and transmit a vector of length q. The source continues to transmit to the destination in the second phase.

We only consider the case when $p \geq q$. If $p < q$ and $r > .5$, all the relays are in outage and partly for this reason we restrict our analysis to $p \geq q$. To compute the best possible DMT, we allow (p, q) to vary with the multiplexing gain r and choose the pair that maximizes the outage exponent for a given r. Since $p + q = m$ is fixed, this is equivalent to making an optimal choice of the parameter $\kappa := \frac{p}{q}$. This version of the protocol will be called the variable-NSDF protocol. We also compute the DMT of the fixed-NSDF protocol, wherein the ratio $\kappa = \frac{p}{q}$ is fixed for all r.

A. DMT of NSDF Protocol

Theorem 7: The DMT of the variable-NSDF protocol is given by

$$d^*(r) = \begin{cases} n\left(1 - \frac{(n-1)(\kappa_n+1)}{n} r\right), & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{(n-r)(1-r)}{(n-2)r+1}, & \frac{1}{\kappa_n+1} \leq r \leq 1 \end{cases} \quad (92)$$

where $\kappa_n = \frac{1+\sqrt{1+4(n-1)^2}}{2(n-1)}$.

For a given multiplexing gain r, the optimal value of κ is given by

$$\kappa = \begin{cases} \kappa_n, & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{1+(n-2)r}{(n-1)(1-r)}, & \frac{1}{\kappa_n+1} < r \leq 1. \end{cases} \quad (93)$$

For a fixed choice $\kappa = \frac{p}{q}$ that is independent of r, the DMT of the fixed-NSDF protocol is given by

$$d^*(r) = (n-1)\left(1 - \frac{mr}{p}\right)^+ + (1-r), \quad 0 \leq r \leq 1 \quad (94)$$

if $1 \leq \kappa \leq \kappa_n$, and

$$d^*(r) = \begin{cases} n\left(1 - \frac{m(n-1)}{nq} r\right), & 0 \leq r \leq \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \leq r \leq \frac{np-m}{(n-2)m+p} \\ n\left(1 - \frac{(n-1)m+p}{np} r\right), & \frac{np-m}{(n-2)m+p} \leq r \leq \frac{p}{m} \\ 1-r & \frac{p}{m} \leq r \leq 1 \end{cases} \quad (95)$$

if $\kappa \geq \kappa_n$.

Proof: The proof will proceed as follows. We determine for a given value of (p, q) with $p + q = m$, the expression for the outage exponent of the fixed NSDF protocol. We use this expression to optimally choose (p, q) for a given multiplexing gain r, to obtain the outage exponent of the variable NSDF protocol. The optimization is carried out in Appendix C. We then show how the ST code constructed in [6] for the point–point MIMO channel, can be tailored to achieve the outage exponent of the fixed and variable NSDF protocol, thereby establishing their DMT given by the above theorem.

Let $\underline{x_1}$ of size $p \times 1$ and $\underline{x_1'}$ of size $q \times 1$ be the signals transmitted by the source in the first and second phase, respectively. The relays that are not in outage in the first phase shall participate in the cooperative protocol in the second phase. Let $(k-1)$ relays, where $1 \leq k \leq n$ participate in the second phase and let $\{\underline{x_j}\}_{j=2}^k$, each of size $q \times 1$ be the signal transmitted by the relays simultaneously.

Let E_k denote the event that precisely $(k-1)$ relays are not in outage and hence participate in the cooperative (second) phase. The events $\{E_1, E_2, \ldots, E_n\}$ are disjoint and their probabilities sum up to 1. There are totally $\binom{n-1}{k-1}$ ways by which $(k-1)$ relays participate in the cooperation. But as we shall see later, the outage exponent of the induced channel when $k-1$ relays participate is the same irrespective of which set of $k-1$ relays participate and also the number $\binom{n-1}{k-1}$ can be ignored in the scale of interest. Hence, we shall denote E_k, $2 \leq k \leq n$, to be the event that the relays $\{2, \ldots, k\}$ are not in outage.

1) Outage Probability Conditioned on E_k: Consider the case when the event E_k, $2 \leq k \leq n$, has occurred. The case $k = 1$ will be dealt with separately. Let the signals received by the destination in the two phases be $\underline{y_1}$ and $\underline{y_2}$, where

$$\underline{y_1} = g_1 \underline{x_1} + \underline{w_1} \quad (96)$$

and

$$\underline{y_2} = g_1 \underline{x_1'} + \sum_{j=2}^{k} h_j \underline{x_j} + \underline{w_2}, \quad (97)$$

with $\underline{w_1}$, $\underline{w_2}$ denoting the noise added at the destination in the respective phases. We impose an energy constraint by choosing $\mathcal{E}$ to be the average energy available for transmission of a symbol at either the source or a relay. Let σ_w^2 be the variance of the noise added at the destination. Let $\rho = \frac{\mathcal{E}}{\sigma_w^2}$ be the average received SNR at the destination.

The channel model for the NSDF protocol can be written as

$$\underline{y} = H_k \underline{x} + \underline{w} \quad (98)$$

[3] We say that a relay is not in outage if the corresponding source–relay channel is not in outage.

where

- $\underline{x}$ is the compound vector formed by concatenating the transmissions of the source and the participating relays
$$\underline{x}^t = [\,\underline{x_1}^t \quad \underline{x}_1'^{\,t} \quad \underline{x_2}^t \quad \cdots \quad \underline{x_k}^t\,]. \tag{99}$$
- $\underline{y}$ is the signal received at the destination
$$\underline{y} = \begin{bmatrix} \underline{y_1} \\ \underline{y_2} \end{bmatrix}. \tag{100}$$
- H_k is the induced MIMO channel model
$$H_k = \begin{bmatrix} g_1 I_p & 0 & 0 & \cdots & 0 \\ 0 & g_1 I_q & h_2 I_q & \cdots & h_k I_q \end{bmatrix}. \tag{101}$$
- $\underline{w}$ is the additive white Gaussian (AWG) noise seen at the destination
$$\underline{w} = \begin{bmatrix} \underline{w_1} \\ \underline{w_2} \end{bmatrix}. \tag{102}$$

The probability of outage, for the channel in (98), is given by

$$P_{\mathrm{out},k}(r\log\rho) = \inf_{\substack{\Sigma_x \succeq 0 \\ \mathrm{Tr}(\Sigma_x) \le m\mathcal{E}}} \Pr\left[I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H_k}=H_k) < mr\log\rho\right] \tag{103}$$

where $I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}}|\boldsymbol{H_k}=H_k)$ is mutual information between $\underline{x}$ and $\underline{y}$, conditioned on the knowledge of H_k at the receiver. The input distribution can be assumed to be circularly symmetric Gaussian with covariance matrix Σ_x, without loss of optimality

$$I(\underline{\boldsymbol{x}};\underline{\boldsymbol{y}} \mid \boldsymbol{H_k}=H_k) = \log \mid I_m + \frac{\mathcal{E}}{\sigma_w^2} H_k \Sigma_x {H_k}^\dagger \mid. \tag{104}$$

Arguing as in case of the OAF protocol, in the scale of interest, we have

$$\begin{aligned} P_{\mathrm{out},k}(r\log\rho) &\doteq \Pr\left(\log \mid I_m + \rho H_k {H_k}^\dagger \mid < mr\log\rho\right) \\ &:= \rho^{-d_{\mathrm{out},k}(r)}. \end{aligned} \tag{105}$$

Let,

$$|g_1|^2 \doteq \begin{cases} \rho^{-u}, & u \ge 0 \\ 0, & u < 0 \end{cases} \tag{106}$$

$$|h_j|^2 \doteq \begin{cases} \rho^{-v_j}, & v_j \ge 0 \\ 0, & v_j < 0, \end{cases} \qquad j = 2,\ldots,k \tag{107}$$

and

$$v = \min\{v_j\}_{j=2}^k. \tag{108}$$

Then

$$\begin{aligned} &\log \mid I_m + \rho H_k H_k^\dagger \mid \\ &= \log\left(1+\rho|g_1|^2\right)^p \left(1+\rho|g_1|^2 + \rho\sum_{j=2}^k |h_j|^2\right)^q \\ &\doteq \log \rho^{p(1-u)^+ + q\max\{(1-u),(1-v)\}^+}. \end{aligned} \tag{109}$$

Substituting (109) in (105) we get the outage exponent as

$$d_{\mathrm{out},k}(r) = \inf_{p(1-u)^+ + q\max\{(1-u),(1-v)\}^+ < mr} u + (k-1)v. \tag{110}$$

It is clear that it is enough to consider $0 \le u, v \le 1$. Hence, the above infimum must be calculated over the region

$$pu + q\min\{u,v\} > m(1-r), \quad 0 \le u, v \le 1. \tag{111}$$

For $r > 1$, we set $u = v = 0$ to obtain the infimum and hence $d_{\mathrm{out},k}(r) = 0$. Therefore, it is enough to consider $r \le 1$. We consider two separate cases to evaluate $d_{\mathrm{out},k}(r)$. Note that our version of the NSDF protocol restricts attention to the case $\kappa = \frac{p}{q} \ge 1$.

Case I: $\min\{u,v\} = u$: We get

$$u > (1-r) \quad \text{and} \quad v > u. \tag{112}$$

Substituting in (110) we get

$$d_{\mathrm{out},k}(r) = k(1-r), \quad 0 \le r \le 1. \tag{113}$$

Case II: $\min\{u,v\} = v$: We have

$$pu + qv > m(1-r). \tag{114}$$

As in the case of the OAF protocol, we solve the optimization problem to get

$$d_{\mathrm{out},k}(r) = \begin{cases} k - \frac{(k-1)m}{q} r, & 0 \le r \le \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \le r \le 1 \end{cases} \qquad 2 \le k \le n. \tag{115}$$

Now, we handle the case when event E_1 occurs, i.e., $k=1$. The induced MIMO channel model in this case is given by

$$\begin{aligned} \begin{bmatrix} \underline{y_1} \\ \underline{y_2} \end{bmatrix} &= \begin{bmatrix} g_1 I_p & 0 \\ 0 & g_1 I_q \end{bmatrix} \begin{bmatrix} \underline{x_1} \\ \underline{x_1'} \end{bmatrix} + \begin{bmatrix} \underline{w_1} \\ \underline{w_2} \end{bmatrix} \\ &= H_1 \underline{x} + \underline{w}. \end{aligned} \tag{116}$$

The probability of outage of the above channel is given by

$$P_{\mathrm{out},1}(r\log\rho) := \rho^{-d_{\mathrm{out},1}(r)} \doteq \rho^{-(1-r)^+}. \tag{117}$$

2) Probability of the Set of Participating Relays: In this subsection, we will compute the probability of the event E_k. The probability of E_k is the product of probabilities of two events

$$\begin{aligned} \Pr(E_k) = \Pr(&(n-k) \text{ relays are in outage}) \cdot \\ \Pr(&(k-1) \text{ relays are not in outage}). \end{aligned} \tag{118}$$

We shall evaluate the probabilities of the two events mentioned above separately. Also, we say that a relay is participating in the second phase if the corresponding source–relay channel is not in outage.

The signal received by the jth relay R_j, where $2 \le j \le n$, in the first phase is given by

$$\underline{r_j} = g_j \underline{x_1} + \underline{v_j} \tag{119}$$

where v_j is the noise vector. The maximum mutual information between $\underline{x_1}$ and $\underline{r_j}$, conditioned on the knowledge of g_j at the relay, is

$$\begin{aligned}\mathcal{I}_{\max} &= \max_{\Sigma_{x_1}\succeq 0,\ \mathrm{Tr}(\Sigma_{x_1})\leq p\mathcal{E}} I(\underline{\boldsymbol{x_1}};\underline{\boldsymbol{r_j}} \mid \boldsymbol{g_j}=g_j)\\ &\doteq \frac{p}{m}\log(1+\rho|g_j|^2). \end{aligned}\tag{120}$$

Now, the probability that R_j is in outage is

$$\begin{aligned}\Pr(R_j \text{ is in outage}) &\doteq \Pr\left(\frac{p}{m}\log(1+\rho|g_j|^2) < r\log\rho\right)\\ &\doteq \Pr\left((1-u_j)^+ < \frac{mr}{p}\right)\\ &\doteq \rho^{-(1-\frac{mr}{p})^+}\end{aligned}\tag{121}$$

where

$$|g_j|^2 \doteq \begin{cases}\rho^{-u_j}, & u_j\geq 0\\ 0, & u_j<0.\end{cases}\tag{122}$$

Since the fading coefficients corresponding to different source–relay channels are independent of each other, we have

$$\Pr\left((n-k) \text{ relays are in outage}\right) \doteq \rho^{-(n-k)(1-\frac{mr}{p})^+}.$$

The probability of a relay participating in the second phase will be determined separately for $r\leq\frac{p}{m}$ and $r>\frac{p}{m}$. When $0\leq r\leq \frac{p}{m}$

$$\begin{aligned}&\Pr\left((k-1) \text{ relays participate}\right)\\ &= \left(1-\Pr\left(\text{a relay is in outage}\right)\right)^{k-1}\\ &\doteq \left(1-\rho^{-(1-\frac{mr}{p})^+}\right)^{k-1}\\ &\doteq 1.\end{aligned}\tag{123}$$

When $r>\frac{p}{m}$, the probability that a particular relay participates is given by

$$\begin{aligned}&\Pr\left(R_j \text{ participates in second phase}\right)\\ &\doteq \Pr\left(\frac{p}{m}\log(1+\rho|g_j|^2) > \left(\frac{p}{m}+\epsilon\right)\log\rho\right)\\ &= \Pr\left(\log(1+\rho|g_j|^2) > \left(1+\frac{m}{p}\epsilon\right)\log\rho\right)\\ &\doteq \Pr\left(|g_j|^2 > \rho^{\frac{m}{p}\epsilon}\right)\\ &\doteq \rho^{-\infty}\\ &\doteq 0.\end{aligned}\tag{124}$$

Therefore, when $r>\frac{p}{m}$,

$$\Pr\left(k \text{ relays participate in second phase}\right)\doteq 0,\quad 2\leq k\leq n.\tag{125}$$

Hence, when $r>\frac{p}{m}$, all the relays are in outage with probability 1.

Consolidating the above facts, we get

$$\Pr(E_k)\doteq\begin{cases}\rho^{-(n-k)\left(1-\frac{mr}{p}\right)^+}, & 0\leq r\leq\frac{p}{m}\\ 0, & r>\frac{p}{m},\ 2\leq k\leq n\\ 1, & r>\frac{p}{m},\ k=1.\end{cases}\tag{126}$$

3) Outage Probability of the NSDF Protocol: The probability of outage of the NSDF protocol can be calculated as follows:

$$\begin{aligned}&P_{\text{out}}(r\log\rho)\\ &\quad=\sum_{k=1}^{n}\binom{n-1}{k-1}\Pr(E_k)\ \Pr(H_k \text{ in outage}|E_k).\end{aligned}\tag{127}$$

Let $P_{\text{out}}(r\log\rho) := \rho^{-d_{\text{out}}(r)}$. It follows from (126) that for $r>\frac{p}{m}$ no relay will participate in the second phase and the channel will be as shown in (116). Hence, from (117), we can see that

$$d_{\text{out}}(r)=(1-r)^+,\quad \frac{p}{m}<r\leq 1.\tag{128}$$

For $0\leq r\leq\frac{p}{m}$, by substituting (115) and (126) in (127), we get

$$\begin{aligned}d_{\text{out}}(r)=\min_{2\leq k\leq n}\Bigg\{&(n-1)\left(1-\frac{mr}{p}\right)+(1-r),\\ &(n-k)\left(1-\frac{mr}{p}\right)+d_{\text{out},k}(r)\Bigg\}.\end{aligned}\tag{129}$$

By solving the optimization problem, we get the outage exponent $d_{\text{out}}(r)$ of NSDF protocol that equals $d^*(r)$ in Theorem 7. Please see Appendix C for the details.

B. DMT Optimal Codes for NSDF Protocol

In this subsection, we show how the ST constructed in [6] for the point–point MIMO channel, can be tailored to achieve the outage exponent of the NSDF protocol. In order to do so, we use an important property that codes in [6] possess. These codes are approximately universal, i.e., they achieve the outage exponent of a MIMO channel irrespective of the statistical characterization of the channel. Code construction in [6] is based on cyclic division algebras (CDA) and for a primer on these algebraic objects see Appendix D. Also, Appendix E provides a brief outline of the code construction in [6].

The induced channel model for the NSDF protocol is given by

$$Y=H_kZ_k+W\tag{130}$$

where

$$\begin{aligned}H_k&=\begin{bmatrix}g_1I_p & 0 & 0 & \cdots & 0\\ 0 & g_1I_q & h_2I_q & \cdots & h_kI_q\end{bmatrix},\quad 2\leq k\leq n\\ &=\begin{bmatrix}g_1I_p & 0\\ 0 & g_1I_q\end{bmatrix},\quad k=1\end{aligned}\tag{131}$$

is the induced channel matrix when $k-1$ relays are participating in the cooperation phase of the protocol and W is the noise matrix added at the destination. The matrix Z_k is a code matrix of size $(p+kq)\times T$, transmitted by the source and the relays. However, as we shall see now, the code matrices Z_k for every k, is obtained from the same matrix of size $(p+nq)\times(p+nq)$.

Now, for a fixed $\kappa = \frac{p}{q}$, where p and q are relatively prime, we outline the DMT optimal code when there are $(n-1)$ relays. [4]

a) Distributed Code for NSDF Protocol: Consider a $T \times T$ approximately universal code as constructed in [6], whose code matrices we refer to as Z. We set $T = p + nq$. We allocate the set of first p rows and next q rows of Z to the source. Each relay is allocated a distinct set of q rows of Z, different from those allocated to the source.

- In the first phase, the source transmits the first set of p rows of Z, one by one for pT channel uses.
- At the end of the first phase, the jth relay R_j, $2 \leq j \leq n$, if not in outage, will decode Z and transmit q rows of Z allocated to it for qT channel uses. The relays will be able to decode Z even though the source transmitted only p rows of Z. This is due to the fact that Z is approximately universal. This will be shown in the next paragraph.
- In the second phase, when the relays are transmitting the source also transmits the set of q rows allocated to it.

The total delay of the code will equal $(p+q)T$ channel uses. We show that such a code will achieve the outage exponent of the NSDF protocol by an example and then the general case is straightforward.

b) Proof of DMT Optimality of the Code: Let $n=3, p=2$, $q = 1$. Then we choose Z to be the code matrices of a 5×5 ST code as constructed in [6]. In the first phase, the source transmits the first two rows of Z and in the second phase, it transmits the third row of Z. The relays R_2 and R_3 are allocated fourth and fifth rows of Z, respectively. A code achieves outage exponent of the NSDF protocol, if it achieves outage exponent of the induced channel matrix H_k irrespective of the number of relays participating in the cooperation, i.e., the code is DMT optimal for $H_k, 1 \leq k \leq n$. Consider the case $k = 2$, i.e., when relay R_3 is in outage and R_2 is not in outage. Then the relay R_2 should be able to decode Z which we shall explain in a short while now. Now the 3×4 induced channel H_2 in this case is given by (see (131))

$$H_2 = \begin{bmatrix} g_1 I_2 & 0 & 0 \\ 0 & g_1 & h_2 \end{bmatrix}. \tag{132}$$

We can write the induced channel model as

$$\begin{aligned} Y &= \begin{bmatrix} g_1 I_2 & 0 & 0 & 0 \\ 0 & g_1 & h_2 & 0 \end{bmatrix} Z + W & (133) \\ &= H_2' Z + W & (134) \end{aligned}$$

where an extra column of zeros has been added to induced channel H_2. However, it can be shown that the outage exponent of H_2 and H_2' are the same. Since the code Z is approximately universal, the SNR exponent of the code equals the outage exponent of H_2' and hence that of H_2. Hence, the code is DMT optimal for the channel H_2. In a similar manner it can be shown that the code is DMT optimal for the induced channel $H_k, k = 1, 3$ and also DMT optimal for the source–relay channel so that when the relay is not in outage it should be able to decode Z. Now since the code is DMT optimal for each one of the induced channels it is DMT optimal for the NSDF protocol. Proof for general (n, p, q) can be obtained along similar lines to those of the above example and we omit it here.

By constructing a DMT optimal code for each value of the ratio $\kappa = \frac{p}{q}$, we can construct DMT optimal codes for the variable-NSDF protocol and this concludes the proof of Theorem 7. □

Remarks: We mention the salient features of the results in this section below.

- The variable-NSDF protocol is shown to improve on the DMT of the best previously known static protocol when the number of relays is greater than two. In the case of single relay, the variable NSDF protocol has better DMT compared to E-SDF protocol, for $r \leq 0.5$.
- The DMT of the variable-NSDF protocol for the case of two relays is better than the tradeoff of the SAF protocol [21] for $r > 0.6$ (see Fig. 4).
- For κ in the range $1 < \kappa < \frac{n}{n-1}$, the fixed-NSDF protocol has a better DMT than that of the NAF protocol for any number of relays.
- For $\kappa = 1$, the fixed-NSDF protocol and the NAF protocol have the same DMT.
- When $p = q = 1$, the DMT of the fixed-NSDF protocol coincides with that of the NAF protocol. However, the DMT optimal code for the fixed-NSDF protocol has a delay $2(n+1)$, where $(n-1)$ is the total number of relays, which is considerably shorter than the delay $4(n-1)$, if $n > 2$, of the DMT optimal codes for the NAF protocol constructed in [20].
- Interestingly, in the case of a single relay, the ratio κ_2, which is optimal for multiplexing gain r in the range $r < \frac{1}{1+\kappa_2}$, turns out to be the Golden number, $\kappa_2 = \frac{1+\sqrt{5}}{2}$.

IV. The Orthogonal Selection Decode-and-Forward Protocol

In this section, we consider the orthogonal selection decode-and-forward (OSDF) protocol. The OSDF protocol is the same as the NSDF protocol, except that the source remains silent in the second phase.

We state the DMT of the variable-OSDF and fixed-OSDF protocol, but we omit the proof since the DMT can be obtained along similar lines to the derivation of the DMT for the NSDF protocol. Also, an approximately universal CDA code of size $(p+(n-1)q) \times (p+(n-1)q)$, where $(n-1)$ is the number of relays, will be DMT optimal for the OSDF protocol. The transmission of various rows of the code matrices by the source relays will be on similar lines to that mentioned in Section III-B for the NSDF protocol.

A. DMT of OSDF Protocol

Theorem 8: The DMT of the variable-OSDF protocol is given by

$$d^*(r) = \begin{cases} n\left(1 - \frac{2n-1}{n} r\right), & 0 \leq r \leq \frac{n-1}{2n-1} \\ \frac{n(1-r)}{(n-1)r+1}, & \frac{n-1}{2n-1} \leq r \leq 1. \end{cases} \tag{135}$$

For a given rate r, the optimal value of κ is given by

$$\kappa = \begin{cases} \frac{n}{n-1}, & 0 \leq r \leq \frac{n-1}{2n-1} \\ \frac{1+(n-1)r}{(n-1)(1-r)}, & \frac{n-1}{2n-1} < r \leq 1. \end{cases} \tag{136}$$

[4] We are considering only rational values of κ here although, while computing the outage exponent, κ was allowed to take on irrational values too.

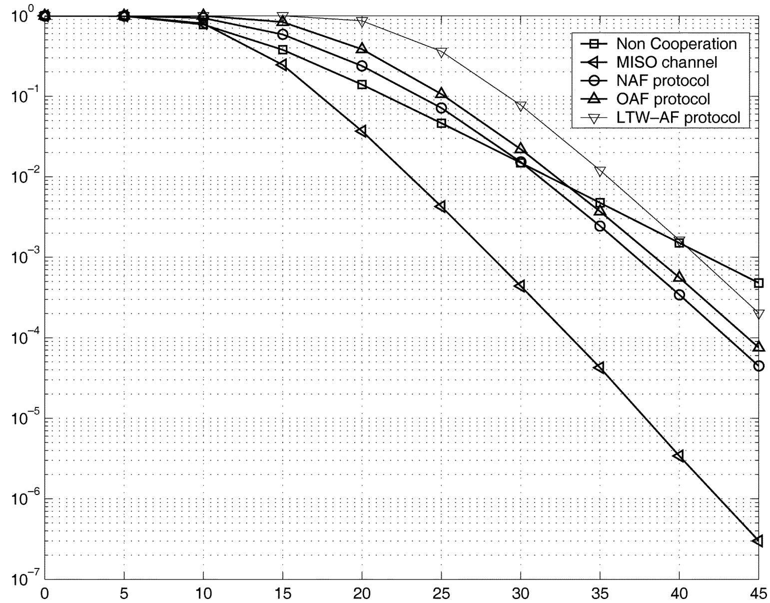

Fig. 8. Outage probability of amplify-and-forward protocols with one relay. Rate = 4 bits per channel use.

For a fixed choice $\kappa = \frac{p}{q}$ that is independent of r, the DMT of the fixed-OSDF protocol is given by

$$d^*(r) = \begin{cases} n\left(1 - \frac{mr}{p}\right), & 0 \le r \le \frac{(n-1)p}{nm-p} \\ (1-r), & \frac{(n-1)p}{nm-p} \le r \le 1. \end{cases} \quad (137)$$

If $1 \le \kappa \le \frac{n}{n-1}$

$$d^*(r) = \begin{cases} n\left(1 - \frac{m(n-1)}{nq} r\right), & 0 \le r \le \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \le r \le \frac{np-m}{m(n-1)} \\ n\left(1 - \frac{mr}{p}\right), & \frac{np-m}{m(n-1)} \le r \le \frac{(n-1)p}{nm-p} \\ (1-r), & \frac{(n-1)p}{nm-p} \le r \le 1. \end{cases} \quad (138)$$

if $\kappa \ge \frac{n}{n-1}$.

V. The Nonorthogonal Amplify-and-Forward Protocol

In this section, we construct a code that is DMT optimal for the nonorthogonal amplify-and-forward (NAF) protocol. For the sake of completeness, we will reproduce here the description of the NAF protocol. The DMT of the protocol was first computed in [1].

Under this protocol, the source S transmits at each time instant, and the relays take turns in transmitting an amplified version of a previously received signal. If the number of relays is $(n-1)$, the set of equations describing a $2(n-1)$-length frame are (see [1])

$$\begin{aligned} y_t &= g_1 x_t + w_t, \quad t \text{ odd} \\ y_t &= b_i h_i (g_i x_{t-1} + v_i) + g_1 x_t + w_t, \begin{cases} i = \frac{t}{2} + 1 \\ t \text{ even} \end{cases} \end{aligned} \quad (139)$$

where b_i is the amplification factor at relay R_i, x_t is the signal transmitted at time instant t, y_t is the signal received at the destination at time t, and w_t is the noise added at the destination at time t. v_i is the noise added at the relay R_i in this frame.

Theorem 9: [1, Theorem 4] The DMT of the NAF protocol is given by

$$d^*(r) = (1-r)^+ + (n-1)(1-2r)^+. \quad (140)$$

A. Explicit DMT Optimal Codes for NAF Protocol

We will present an explicit construction, based on CDA, which achieves the DMT of the NAF protocol.

Consider a $2(n-1) \times 2(n-1)$ DMT optimal CDA ST code. In accordance with the NAF protocol in [1], let the source continuously transmit the vector $[x_1, x_2, \ldots, x_{4(n-1)^2}]$, coming from a row-by-row vectorization of the code. Each intermediate relay $R_i, i = 2, \ldots, n$, forwards at time $t = 4(n-1)(i-2) + 2(n-1) + k$ what it received at time $t = 4(n-1)(i-2) + k$ where $k = 1, 2, \ldots, 2(n-1)$.

Theorem 10: The above code achieves the DMT of the NAF protocol.

Proof: For the single relay case, we use the equivalent representation of the channel for the NAF protocol in matrix form

$$\begin{bmatrix} y_1 & y_3 \\ y_2 & y_4 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ g_2 h_2 & g_1 \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} + \begin{bmatrix} w_1 & w_3 \\ h_2 v_{2,1} + w_2 & h_2 v_{2,3} + w_4 \end{bmatrix}.$$

Upon vectorizing the above channel, we can see that the noise vector is white in the scale of interest. Hence, the DMT of the above channel is met by the approximately universal 2×2 CDA code.

Proceeding as in the single-relay case, we can show that the DMT of the NAF protocol is achieved by the corresponding approximately universal $2(n-1) \times 2(n-1)$ CDA code. □

The above code for the NAF protocol was first presented in [8]. Around the same time, in [20], the authors constructed DMT

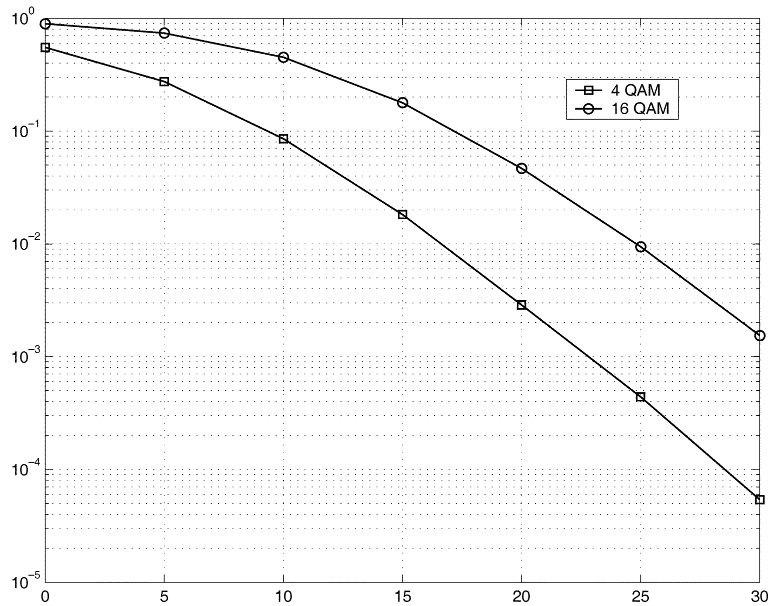

Fig. 9. Performance of OAF code with one relay.

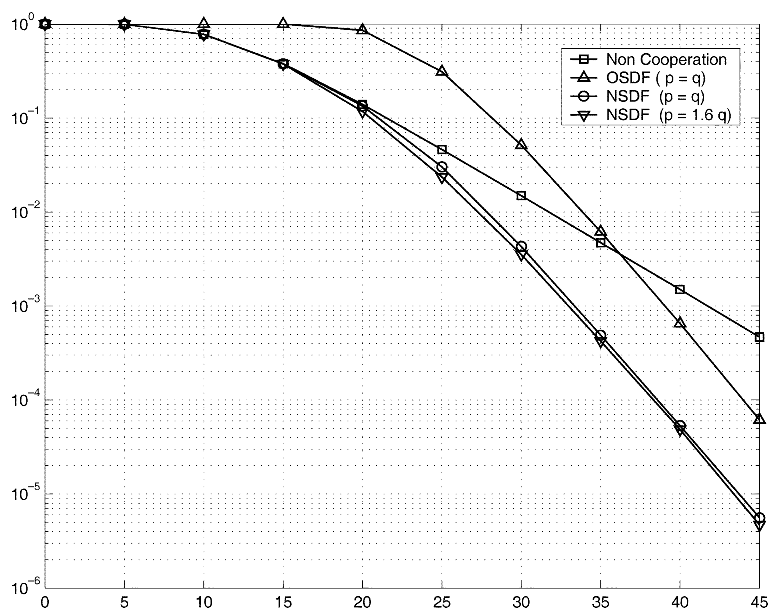

Fig. 10. Outage probability of decode-and-forward protocols with one relay, Rate = 4 bits per channel use.

optimal codes for the NAF protocol which have shorter delays than the codes presented here.

VI. Numerical Results

In this section, we present numerical results on the performance of the protocols discussed in this paper. We consider the two-hop network with one relay, $n = 2$. (see Fig. 1).

Fig. 8 compares the outage probabilities of different AF protocols with rate 4 bits per channel use. The OAF protocol with $p = 2, q = 1$ has a better performance when compared to the OAF protocol of [5] (referred as LTW-AF in the figure) wherein p is equal to q. However, while both the NAF and the OAF protocol have the same DMT, the NAF protocol has a slightly better performance by 1 dB compared to the OAF protocol. Fig. 9 plots the performance of the number theory based OAF code proposed in this paper, when the input for the channel is drawn from 4- and 16-QAM signal constellations. Fig. 10 plots the outage probabilities of NSDF and OSDF protocol with single relay. The NSDF protocol with unequal time durations ($p = 1.6q$) performs marginally better than that with equal time durations.

VII. Conclusion

In this paper, we have considered two-phase protocols for cooperative communication under two broad categories: amplify-and-forward (AF) and decode-and-forward (DF). In all of the protocols considered, we permitted the time durations of broadcast and cooperative phases to be unequal and also vary with the multiplexing gain r and optimized them for obtaining the best DMT.

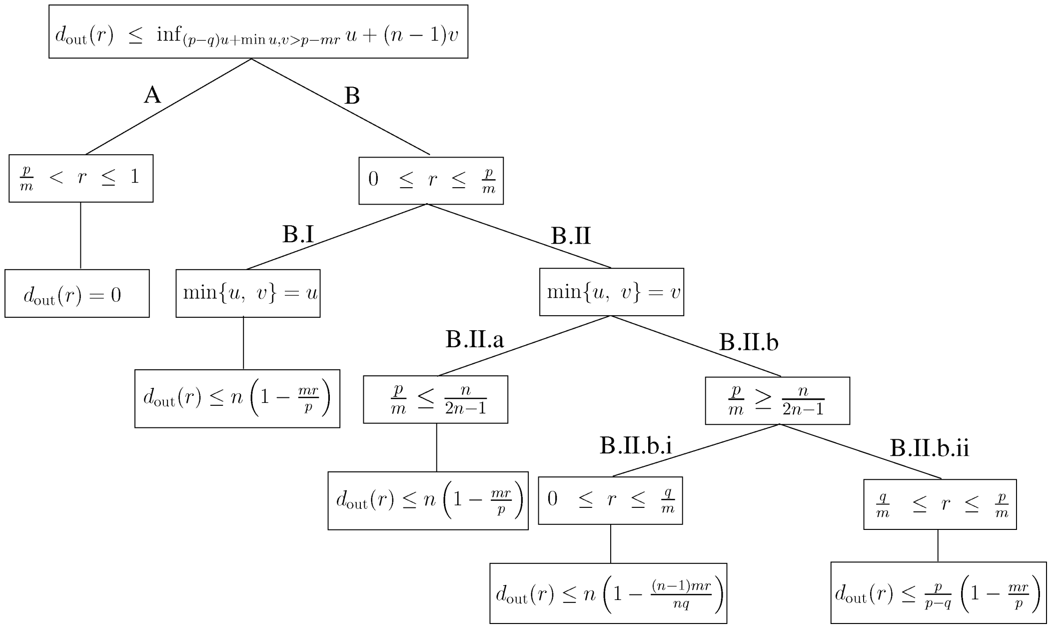

Fig. 11. Flowchart describing the computation of upper bound on $d_{\text{out}}(r)$ of OAF protocol with parameters (p, q).

We first studied a class of orthogonal AF (OAF) protocols and showed that a specific OAF protocol has same DMT as that of the nonorthogonal amplify-and-forward (NAF) protocol, while incurring lesser delay. A simple minimum-delay code based on elementary number theory was constructed that is DMT optimal for this specific OAF protocol. Then we studied nonorthogonal and orthogonal selection decode-and-forward protocols (NSDF and OSDF) and determined their DMT. The variable NSDF protocol, wherein the time durations of the broadcast phase and the cooperative phase is permitted to vary with the multiplexing gain, is shown to improve upon the DMT of the best previously known static protocol when the number of relays is greater than two. DMT optimal codes based on cyclic division algebras were constructed for both NSDF and OSDF protocols.

Appendix A
Proof of Lemma 2

Proof: Let $f_j = g_j h_j$. Then $B = \sum_{j=2}^{n} f_j A_j$. We have

$$\begin{aligned} \underline{x}^\dagger B B^\dagger \underline{x} &= \|B^\dagger \underline{x}\|^2 \\ &= \left\| \sum_{j=2}^{n} f_j^* A_j^\dagger \underline{x} \right\|^2. \end{aligned} \tag{141}$$

Let $f_j^* A_j^\dagger \underline{x} = \underline{y}_j = (y_{1j} \ y_{2j} \ \cdots \ y_{pj})$. Hence

$$\begin{aligned} \underline{x}^\dagger B B^\dagger \underline{x} &= \left\| \sum_{j=2}^{n} \underline{y}_j \right\|^2 \\ &= \sum_{i=1}^{p} \left| \sum_{j=2}^{n} y_{ij} \right|^2. \end{aligned} \tag{142}$$

By applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \underline{x}^\dagger B B^\dagger \underline{x} &\leq \sum_{i=1}^{p} \left(\sum_{j=2}^{n} |y_{ij}|^2 \right) \left(\sum_{j=2}^{n} 1^2 \right) \\ &= (n-1) \sum_{i=1}^{p} \sum_{j=2}^{n} |y_{ij}|^2 \\ &= (n-1) \sum_{j=2}^{n} \|y_j\|^2 \\ &= (n-1) \sum_{j=2}^{n} \|f_j^* A_j^\dagger \underline{x}\|^2 \\ &= (n-1) \underline{x}^\dagger \sum_{j=2}^{n} |f_j|^2 A_j A_j^\dagger \underline{x}. \end{aligned} \tag{143}$$

With this, we get the bound stated in the lemma. □

Appendix B
Solution to the Optimization Problem for the OAF Protocol

We need to solve

$$d_{\text{out}}(r) \leq \inf_{(p-q)u+q\min\{u,v\}>p-mr} u+(n-1)v. \tag{144}$$

We separately consider different ranges of r (see flowchart in Fig. 11).

Case A $r > \frac{p}{m}$*:* Since $r > \frac{p}{m}$, $p - mr$ is negative. Hence, the infimum is obtained by choosing $u = v = 0$ and

$$d_{\text{out}}(r) \leq 0. \tag{145}$$

But, by definition, $d_{\text{out}}(r)$ cannot be negative. Therefore

$$d_{\text{out}}(r) = 0, \quad \text{for } r > \frac{p}{m}. \tag{146}$$

Case B $0 \le r \le \frac{p}{m}$: As shown in Fig. 11, this case further breaks up into two cases.

Case B.I $\min\{u,v\} = u$: Consider the inequality

$$pu > p - mr \tag{147}$$

which leads to

$$u > 1 - \frac{mr}{p}. \tag{148}$$

Therefore, by substituting in (144), we get

$$d_{\text{out}}(r) \le n\left(1 - \frac{mr}{p}\right). \tag{149}$$

Case B.II $\min\{u,v\} = v$: Noting that one solution to the optimization problem in (144) is

$$u = v = 1 - \frac{mr}{p} \tag{150}$$

we consider the perturbations

$$u = 1 - \frac{mr}{p} + \delta \tag{151}$$

$$v = 1 - \frac{mr}{p} - \delta\left(\frac{p-q}{q}\right) \tag{152}$$

where δ is a small positive number. In this case

$$\begin{aligned} u + (n-1)v &= 1 - \frac{mr}{p} + \delta + (n-1)\left(1 - \frac{mr}{p} - \delta\frac{p-q}{q}\right) \\ &= n\left(1 - \frac{mr}{p}\right) + \delta\left(1 - (n-1)\frac{p-q}{q}\right). \end{aligned} \tag{153}$$

Now, we need to consider two separate cases depending on the ratio $\frac{p}{m}$.

Case B.II.a $\frac{p}{m} \le \frac{n}{2n-1}$: In this case, it follows that

$$1 - (n-1)\frac{p-q}{q} \ge 0. \tag{154}$$

We can thus set $\delta = 0$ in (153) to obtain the infimum. Hence

$$d_{\text{out}}(r) \le n\left(1 - \frac{mr}{p}\right). \tag{155}$$

Case B.II.b $\frac{p}{m} \ge \frac{n}{2n-1}$: In this case, we choose δ as large as possible under the constraints $u \le 1$ and $v \ge 0$. The condition $u \le 1$

$$\begin{aligned} &\Rightarrow 1 - \frac{mr}{p} + \delta \le 1 \\ &\qquad \Rightarrow \delta \le \frac{mr}{p}. \end{aligned} \tag{156}$$

The condition $v \ge 0$ implies

$$\begin{aligned} &1 - \frac{mr}{p} - \delta\left(\frac{p-q}{q}\right) \ge 0 \\ &\qquad \Rightarrow \delta \le \frac{q}{p-q}\left(1 - \frac{mr}{p}\right). \end{aligned} \tag{157}$$

Hence, δ is chosen to meet one of the upper bounds in (156) or (157) depending on r. By equating the two upper bounds, we get

$$r = \frac{q}{m}. \tag{158}$$

So, we need to consider two different ranges of r.

Case B.II.b.i $0 \le r \le \frac{q}{m}$: Here the largest possible value of δ is given by

$$\delta = \frac{mr}{p}. \tag{159}$$

Then

$$\begin{aligned} d_{\text{out}}(r) &\le n\left(1 - \frac{mr}{p}\right) + \frac{mr}{p}\left[1 - (n-1)\left(\frac{p-q}{q}\right)\right] \\ &= n\left(1 - \frac{(n-1)mr}{nq}\right). \end{aligned} \tag{160}$$

Case B.II.b.ii $\frac{q}{m} \le r \le \frac{p}{m}$: Here the largest possible value of δ is given by

$$\delta = \left(\frac{q}{p-q}\right)\left(1 - \frac{mr}{p}\right). \tag{161}$$

Then

$$\begin{aligned} d_{\text{out}}(r) &\le n\left(1 - \frac{mr}{p}\right) \\ &\quad + \left(\frac{q}{p-q}\right)\left(1 - \frac{mr}{p}\right)\left[1 - (n-1)\left(\frac{p-q}{q}\right)\right] \\ &= \frac{p}{p-q}\left(1 - \frac{mr}{p}\right). \end{aligned} \tag{162}$$

Selecting the Best Parameter Pair (p,q) for Given r: The above discussion has provided upper bounds on the outage exponent for a given value of (p,q). Our next step is to identify for a given value of r, $0 \le r \le 1$, the parameters (p,q) that will result in the least restrictive upper bound on outage exponent.

- Cases A, B.I, and B.II.a tell us that for $\frac{p}{m} \le \frac{n}{2n-1}$

$$d_{\text{out}}(r) \le \begin{cases} n\left(1 - \frac{mr}{p}\right), & 0 \le r \le \frac{p}{m} \\ 0, & \frac{p}{m} \le r \le 1. \end{cases} \tag{163}$$

  The uniformly least restrictive upper bound (over all possible choices of p with $\frac{p}{m} \le \frac{n}{2n-1}$) results from choosing $\frac{p}{m}$ to equal $\frac{n}{2n-1}$ leading to the upper bound

$$d_{\text{out}}(r) \le \begin{cases} n\left(1 - \frac{(2n-1)r}{n}\right), & 0 \le r \le \frac{n}{2n-1} \\ 0, & \frac{n}{2n-1} \le r \le 1. \end{cases} \tag{164}$$

- Similarly, Cases A, B.I, and B.II.b tell us that for $\frac{p}{m} \ge \frac{n}{2n-1}$

$$d_{\text{out}}(r) \le \begin{cases} n\left(1 - \frac{(n-1)mr}{nq}\right), & 0 \le r \le \frac{q}{m} \\ \frac{p}{p-q}\left(1 - \frac{mr}{p}\right), & \frac{q}{m} \le r \le \frac{p}{m} \\ 0, & \frac{p}{m} \le r \le 1. \end{cases} \tag{165}$$

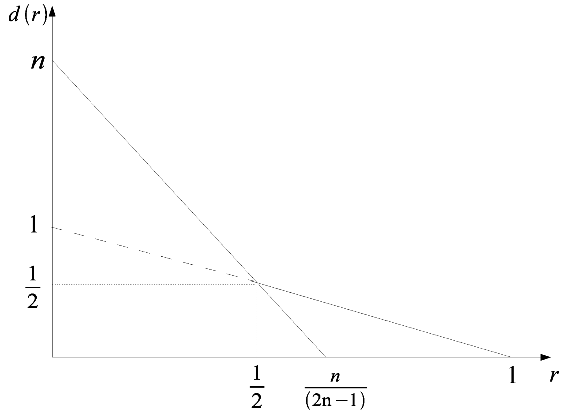

Fig. 12. The two competing straight-line bounds on the DMT of the OAF protocol. The lower straight line connecting $(0,1)$ to $(1,0)$ corresponds to noncooperation.

The uniformly least restrictive upper bound (over all possible choices of p with $\frac{p}{m} \geq \frac{n}{2n-1}$) results from choosing $\frac{p}{m}$ to equal 1 leading to the upper bound

$$d_{\text{out}}(r) = (1-r), \ \ 0 \leq r \leq 1. \tag{166}$$

It remains to determine for a given value of r, whether the least restrictive bound results from choosing $\frac{p}{m} = \frac{n}{2n-1}$ or else, $\frac{p}{m} = 1$. From the plot in Fig. 12 it can be verified that the least restrictive upper bounds results from choosing $\frac{p}{m} = \frac{n}{2n-1}$ for $0 \leq r \leq \frac{1}{2}$ and $\frac{p}{m} = 1$ for $\frac{1}{2} \leq r \leq 1$ leading to the final upper bound

$$d_{\text{out}}(r) \leq \begin{cases} n\left(1 - \frac{(2n-1)r}{n}\right), & 0 \leq r \leq \frac{1}{2} \\ (1-r), & \frac{1}{2} < r \leq 1. \end{cases} \tag{167}$$

The proof of Lemma 3 is now complete.

Appendix C
Solution to the Optimization Problem for the NSDF Protocol

In order to obtain the outage exponent of the fixed NSDF protocol we need to solve the following optimization problem for $0 \leq r \leq \frac{p}{m}$, since for $r > \frac{p}{m}$, $d_{\text{out}}(r) = (1-r)^+$:

$$d_{\text{out}}(r) = \min_{2 \leq k \leq n} \left\{ (n-1)\left(1 - \frac{mr}{p}\right) + (1-r), \right. \\ \left. (n-k)\left(1 - \frac{mr}{p}\right) + d_{\text{out},k}(r) \right\}. \tag{168}$$

For $0 \leq r \leq \frac{p}{m}$, let

$$\begin{aligned} d_a(r) &= (n-1)\left(1 - \frac{mr}{p}\right)^+ + (1-r)^+ \\ &= n\left(1 - \frac{(n-1)m + p}{np} r\right) \end{aligned} \tag{169}$$

and

$$d_b(r) = \min_{2 \leq k \leq n} \left((n-k)\left(1 - \frac{mr}{p}\right)^+ + d_{\text{out},k}(r) \right) \tag{170}$$

where $d_{\text{out},k}(r)$ is given by (see (115))

$$d_{\text{out},k}(r) = \begin{cases} k - \frac{(k-1)m}{q} r, & 0 \leq r \leq \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \leq r \leq 1 \end{cases} \quad 2 \leq k \leq n. \tag{171}$$

Therefore

$$d_{\text{out}}(r) = \min\left\{ d_a(r), d_b(r) \right\}, \quad 0 \leq r \leq \frac{p}{m}. \tag{172}$$

Substituting the value of $d_{\text{out},k}(r)$ in (170) and using the fact that $p \geq q$ we can see that the minimum occurs when $k = n$. Hence we have

$$d_b(r) = d_n(r) = \begin{cases} n\left(1 - \frac{(n-1)m}{nq} r\right), & 0 \leq r \leq \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \leq r \leq 1. \end{cases} \tag{173}$$

Depending on the choice of the ratio $\kappa = \frac{p}{q}$ and the multiplexing gain r, either $d_a(r)$ or $d_b(r)$ will determine the actual $d_{\text{out}}(r)$. It can be shown that there is a critical value of κ, which we shall denote by κ_n, below which $d_{\text{out}}(r) = d_a(r)$, i.e.,

$$d_{\text{out}}(r) = d_a(r), \quad 1 \leq \kappa \leq \kappa_n. \tag{174}$$

In order to compute κ_n, we compare the curves corresponding to $d_a(r)$ and $d_b(r)$. We can see that at $\kappa = \kappa_n$

$$\begin{aligned} \frac{(n-1)m + p}{np} &= \frac{(n-1)m}{nq} \\ pnm - pm &= qnm - qm + pq. \end{aligned} \tag{175}$$

By substituting $\kappa_n = \frac{p}{q}$ in the above equation, we get

$$(n-1)\kappa_n^2 - \kappa_n - (n-1) = 0. \tag{176}$$

Hence

$$\kappa_n = \frac{1 + \sqrt{1 + 4(n-1)^2}}{2(n-1)}. \tag{177}$$

Therefore, when $\kappa < \kappa_n$, we get $d_{\text{out}}(r)$ of the fixed NSDF protocol as

$$d_{\text{out}}(r) = (n-1)\left(1 - \frac{mr}{p}\right)^+ + (1-r), \quad 0 \leq r \leq 1. \tag{178}$$

When $\kappa \geq \kappa_n$, both $d_a(r)$ and $d_b(r)$ determine the $d_{\text{out}}(r)$ for different ranges of r. We can check that the point of intersection r_i of $d_a(r)$ and $d_b(r)$, is given by

$$r_i = \frac{np - m}{(n-2)m + p} \tag{179}$$

and, hence, when $\kappa \geq \kappa_n$, we get $d_{\text{out}}(r)$ of the fixed NSDF protocol as

$$d_{\text{out}}(r) = \begin{cases} n\left(1 - \frac{m(n-1)}{nq} r\right), & 0 \leq r \leq \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \leq r \leq \frac{np-m}{(n-2)m+p} \\ n\left(1 - \frac{(n-1)m+p}{np} r\right), & \frac{np-m}{(n-2)m+p} \leq r \leq \frac{p}{m} \\ 1 - r & \frac{p}{m} \leq r \leq 1. \end{cases} \tag{180}$$

Hence, we have computed the outage exponent $d_{\text{out}}(r)$ for the fixed-NSDF protocol.

Now, to get the best possible $d_{\text{out}}(r)$ in case of the variable-NSDF protocol, we vary κ with the multiplexing gain r, i.e., we choose the value of κ which maximizes the diversity at any given multiplexing gain. By comparing the outage exponents in (178) and (180), it is clear that we must choose $\kappa = \kappa_n$ for $r \leq \frac{1}{\kappa_n+1}$. For $r \geq \frac{1}{\kappa_n+1}$, we need to track the point of intersection of $d_a(r)$ and $d_b(r)$. This point would correspond to maximum diversity at a certain r. At the point of intersection of $d_a(r)$ and $d_b(r)$ we have

$$d_{\text{out}}(r_i) = \frac{m}{p}\left(1 - \frac{np-m}{(n-2)m+p}\right). \tag{181}$$

By substituting $\kappa = \frac{p}{q}$ in (179) and (181), and by eliminating κ from both the equations, we get

$$d_{\text{out}}(r) = \frac{(n-r)(1-r)}{(n-2)r+1}, \quad \frac{1}{\kappa_n+1} \leq r \leq 1. \tag{182}$$

Thus, we get the $d_{\text{out}}(r)$ of variable NSDF protocol as

$$d_{\text{out}}(r) = \begin{cases} n\left(1 - \frac{(n-1)(\kappa_n+1)}{n} r\right), & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{(n-r)(1-r)}{(n-2)r+1}, & \frac{1}{\kappa_n+1} \leq r \leq 1 \end{cases} \tag{183}$$

where $\kappa_n = \frac{1+\sqrt{1+4(n-1)^2}}{2(n-1)}$, with the optimal value of κ given by

$$\kappa = \begin{cases} \kappa_n, & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{1+(n-2)r}{(n-1)(1-r)}, & \frac{1}{\kappa_n+1} < r \leq 1. \end{cases} \tag{184}$$

Appendix D
Primer on Space–Time (ST) Codes From Cyclic Division Algebras (CDAs)

1) Division Algebras: Division algebras are rings with identity in which every nonzero element has a multiplicative inverse. The center $\mathbb{F}$ of any division algebra D, i.e., the subset comprising of all elements in D that commute with every element of D, is a field. The division algebra is a vector space over the center $\mathbb{F}$ of dimension n^2 for some integer n. A field $\mathbb{L}$ such that $\mathbb{F} \subset \mathbb{L} \subset D$ and such that no subfield of D contains $\mathbb{L}$ is called a *maximal subfield* of D (Fig. 13). Every division algebra is also a vector space over a maximal subfield and the dimension of this vector space is the same for all maximal subfields and equal to n. This common dimension n is known as the *index* of the division algebra.

2) Cyclic Division Algebras: Our interest is in CDA, i.e., division algebras in which the center $\mathbb{F}$ and a maximum subfield $\mathbb{L}$ are such that $\mathbb{L}/\mathbb{F}$ is a finite cyclic Galois extension. CDAs have a simple characterization that aids in their construction, see [18, Proposition 11], or [4, Theorem 1].

Let $\mathbb{F}, \mathbb{L}$ be number fields, with $\mathbb{L}$ a finite, cyclic Galois extension of $\mathbb{F}$ of degree n. Let σ denote the generator of the Galois group $\text{Gal}(\mathbb{L}/\mathbb{F})$. Let z be an indeterminate satisfying

$$\ell z = z\sigma(\ell), \quad \forall\ \ell \in \mathbb{L} \qquad \text{and} \qquad z^n = \gamma \tag{185}$$

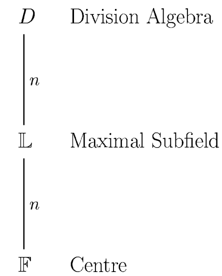

Fig. 13. Structure of a division algebra.

for some non-norm element $\gamma \in \mathbb{F}^*$, by which we mean some element γ having the property that the smallest positive integer t for which γ^t is the relative norm $N_{\mathbb{L}/\mathbb{F}}(u)$ of some element u in $\mathbb{L}^*$, is n. Then, a CDA $D(\mathbb{L}/\mathbb{F}, \sigma, \gamma)$ with index n, center $\mathbb{F}$ and maximal subfield $\mathbb{L}$ is the set of all elements of the form

$$\sum_{i=0}^{n-1} z^i \ell_i, \quad \ell_i \in \mathbb{L}. \tag{186}$$

Moreover, it is known that every CDA has this structure. It can be verified that D is a right vector space (i.e., scalars multiply vectors from the right) over the maximal subfield $\mathbb{L}$.

3) ST Codes From CDAs: An ST code $\mathcal{X}$ can be associated to D by selecting the set of matrices corresponding to the matrix representation of elements of a finite subset of D. Note that since these matrices are all square matrices, the resultant ST code necessarily has $T = n_t$.

The matrix corresponding to an element $d \in D$ corresponds to left multiplication by the element d in the division algebra. Let λ_d denote this operation $\lambda_d : D \to D$, defined by

$$\lambda_d(e) = de, \quad \forall\, e \in D. \tag{187}$$

It can be verified that λ_d is a $\mathbb{L}$-linear transformation of D. From (186), a natural choice of basis for the right-vector space D over $\mathbb{L}$ is $\{1, z, z^2, \ldots, z^{n-1}\}$. A typical element in the division algebra D is $d = \ell_0 + z\ell_1 + \cdots + z^{n-1}\ell_{n-1}$, where the $\ell_i \in \mathbb{L}$. By considering the effect of multiplying $d \times 1, d \times z, \ldots, d \times z^{n-1}$ one can show that the $\mathbb{L}$-linear transformation λ_d associated to the element d under this basis has the matrix representation

$$\begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{n-1}) & \gamma\sigma^2(\ell_{n-2}) & \ldots & \gamma\sigma^{n-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \gamma\sigma^2(\ell_{n-1}) & \ldots & \gamma\sigma^{n-1}(\ell_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n-1} & \sigma(\ell_{n-2}) & \sigma^2(\ell_{n-3}) & \ldots & \sigma^{n-1}(\ell_0) \end{bmatrix}. \tag{188}$$

A set of such matrices, obtained by choosing a finite subset of elements in D constitutes the CDA-based ST code $\mathcal{X}$. In [6], the authors have constructed CDA-based ST code for all values of n. For all the codes constructed in [6], the underlying constellation is quadrature amplitude modulation (QAM) and the center of the division algebra is $\mathbb{F} = \mathbb{Q}(\imath)$.

Appendix E
Construction of Approximately Universal Codes

Consider a CDA having center $\mathbb{F} = \mathbb{Q}(\imath)$ and maximum subfield $\mathbb{L}$ that is a degree-T cyclic Galois extension $\mathbb{L}/\mathbb{F}$ of $\mathbb{F}$. We

$$\begin{array}{ll} D & \\ | & \\ \mathbb{L} & \mathcal{O}_{\mathbb{L}} = <\beta_1, \ldots, \beta_n> \\ | & \quad | \\ \mathbb{F} = \mathbb{Q}(\imath) & \mathcal{O}_{\mathbb{F}} = \mathbb{Z}[\imath] \end{array}$$

Fig. 14. Structure of division algebra.

set $T = p + nq$ in the case of NSDF protocol. Let σ be the generator of the cyclic Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$. Let $\mathcal{O}_{\mathbb{F}}$ and $\mathcal{O}_{\mathbb{L}}$ denote the ring of algebraic integers in $\mathbb{F}$ and $\mathbb{L}$, respectively. It is known that $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}[\imath]$. Let $\{\beta_1, \ldots, \beta_n\}$ be an integral basis for $\mathcal{O}_{\mathbb{L}}/\mathcal{O}_{\mathbb{F}}$. Let $\mathrm{D}(\mathbb{L}/\mathbb{F}, \sigma, \gamma)$ denote the associated CDA. (See Fig. 14.)

For M even, let $\mathcal{A}_{\mathrm{QAM}}$ denote the M^2-QAM constellation given by

$$\mathcal{A}_{\mathrm{QAM}} = \{a + \imath b \mid |a|, |b| \leq M - 1,\ a, b \text{ odd}\} \tag{189}$$

and

$$\mathcal{A}_{\mathrm{QAM}}(\beta_1, \ldots, \beta_n) = \left\{ \sum_i a_i \beta_i \mid a_i \in \mathcal{A}_{\mathrm{QAM}} \right\}. \tag{190}$$

Consider the ST code $\mathcal{X}$ comprising of matrices associated to all the elements in the CDA D. Let $\mathcal{Z}$ denote the normalized code

$$\mathcal{Z} = \{\theta X \mid X \in \mathcal{X}\} \tag{191}$$

where θ is chosen to ensure that

$$\|\theta X\|_F^2 \leq T\rho, \quad \text{for all } X \in \mathcal{X}. \tag{192}$$

The transmitted code matrix, denoted by Z, will be of the form

$$\theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{T-1}) & \gamma\sigma^2(\ell_{T-2}) & \ldots & \gamma\sigma^{T-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \gamma\sigma^2(\ell_{T-1}) & \ldots & \gamma\sigma^{T-1}(\ell_2) \\ \ell_2 & \sigma(\ell_1) & \sigma^2(\ell_0) & \ldots & \gamma\sigma^{T-1}(\ell_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{T-1} & \sigma(\ell_{T-2}) & \sigma^2(\ell_{T-3}) & \ldots & \sigma^{T-1}(\ell_0) \end{bmatrix}$$
$$\text{where } \ell_0 \in \mathcal{A}_{\mathrm{QAM}}(\beta_1, \ldots, \beta_n). \tag{193}$$

The code $\mathcal{Z}$ as constructed above is approximately universal for any MIMO channel with T transmit antennas and any number of receive antennas.

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