Operating Point Selection in Multiple Access Rate Regions

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Abstract—We study the selection of the rate allocation in multiple access channels (MAC). We consider MACs with different rate regions. Namely, we investigate the polytope rate regions, convex non-polytope rate regions, and non-convex rate regions. Different operating points of the rate region possess different properties in terms of efficiency, fairness, stability, etc. Our goal is to provide guidelines for the choice of an operating point using the above-mentioned criteria. We use two methodological approaches: fairness function approach leading to an optimal system operation point and game theoretic approach leading to an equilibrium point. In particular, we use games with correlated constraints. All fairness concepts and normalized Nash equilibrium produce the same rate allocation in the case of a MAC with polytope rate region. In the case of MACs with convex non-polytope and non-convex rate regions this property does not hold and behavior becomes much more various. In the case of some non convex rate regions the max-min fair allocation may even not exist.

I. INTRODUCTION

In this contribution we consider capacity regions for different kind of multiple access channels (MAC). A capacity region consists of all n-tuples of achievable rates for a MAC with n users transmitting to a common destination. Typically, one has a very broad choice of achievable rates over the capacity region, however different operating points possess different properties in terms of efficiency, fairness, stability, etc. Our goal is to provide guidelines for the choice of an operating point using above-mentioned criteria. We classify MACs by geometrical characteristics of their capacity regions. Namely, we investigate the polytope capacity regions, convex non-polytope capacity region and non-convex capacity regions. An example of MAC with a polytope capacity region is the general time-invariant Gaussian MAC [22]. Examples of MACs with a convex non-polytope capacity region are the time division multiple access (TDMA) and frequency division multiple access (FDMA) channels [22]. An example of MAC with a non-convex capacity region is the collision channel without feedback [17]. For the selection of the operating point we use two methodological approaches: game theoretic approach leading to an equilibrium point and fairness function approach leading to an optimal system operation point.

The fairness function approach can be applied to any kind of capacity region. It turns out that in the case of the polytope capacity region some specific optimal fair points coincide with the equilibrium points attained in some games. The latter allows one to use distributed algorithms to achieve those operating points.

In particular, in the game theoretic framework, we consider games with correlated constraints arising in MACs with polytope capacity regions. In constrained games, each player is faced with a constrained optimization problem rather than a simple non-constrained optimization; the constraints may be independent of actions of other players, in which case they are called "orthogonal constraints" [12]. A more complex situation arises when the actions available to one player depend on those used by the others. Such games are called games with correlated constraints [12]. These games exhibit various characteristics that are very different than those without constraints or with orthogonal constraints. A central feature in these games is that they often possess a large number of equilibria. Natural questions that arise concern the selection of an equilibrium. Can we identify ones that are more fair, more efficient, or more stable than others? We study this type of selection criteria as well as other criteria related to decentralized implementations.

It turns out that in MAC with polytope capacity region the rate assignment corresponding to the α-fairness function optimization is the same for all values of the parameter α. It also coincides with the max-min fair allocation, which for this type of MAC, is the same as α-fair allocation when the parameter α goes to infinity. We prove that the α-fair allocation for the MAC with polytope capacity region also coincides with a unique normalized Nash equilibrium. We also show that in this case the maximization of the Jain's fairness index corresponds to maxmin fairness. The above properties are related to the particular polytope structure of the achievable rate region. They have very relevant practical implications. In fact, a normalized Nash equilibrium is especially appealing in terms of decentralized implementation, pricing and billing. The above mentioned properties imply that the design of distributed algorithms satisfying a very large family of fairness criteria reduces to the design of distributed algorithms for the normalized Nash equilibrium.

In the rest of this work we investigate MACs whose achievable regions are not a polymatroid; we study cases in which they are convex but not polytopes, and cases in which they are not convex. We show how the geometric properties of the rate capacity alter the properties of rate assignments

corresponding to various fairness concepts. In particular, we show that in the case of some non convex achievable rate regions the maxmin fair allocation may not exist.

II. POLYMATROID ACHIEVABLE RATE REGION

Consider n non-cooperative mobiles transmitting up-link to a base station. The square amplitude of the channel gain between mobile i and the base station is h_i. The variance of the additive white Gaussian noise is denoted by σ_0^2. We consider the setting in which the achievable rate region is given by the *convex* polytope C defined by the set of constraints:

$$\sum_{i \in S} R_i \leq \rho(S) \qquad \forall S \subset \{1, ..., n\} \qquad (1)$$

where

$$\rho(S) = \log\left(1 + \frac{\sum_{i \in S} P_i h_i}{\sigma_0^2}\right)$$

and P_i is the transmitted power of user i.

We assume that no mobile (player) can tolerate the losses that would occur if transmission rates were chosen outside the achievable region. This assumption places these games in the category of games with common constraints [12]. In our particular game, the achievable rates represent the strategies of players. Each player wishes to maxiomize some concave increasing function U_i of its rate. This game has coupled constraints: the choice of strategies of a player depends on the strategies chosen by other players.

The equilibrium notion here, is that of finding a rate allocation vector R within C such that no mobile i can gain by deviating from R_i to another S_i for which $(S_i, R^{-i}) \in C$ (here R^{-i} is the vector of strategies R_j for $j \neq i$, where R_j is the entry of R corresponding to player j). This equilibrium concept is a special case of the so-called "generalized equilibrium" or "social equilibrium". They have been introduced already in [4], [5] (for more recent papers see [6], [8]).

As shown recently in [3], to define games with constraints it is not sufficient to know the utility and constraints of each given player. One should also specify how does a player value the fact that constraints of another player are satisfied or violated. Some extreme cases are (i) a player is indifferent to satisfaction of constraints of other players, (ii) common constraints: if a constraint is violated for one player then it is violated for all players. The equilibrium point selections at hand are modelled as games with common constraints. In particular, we shall study the properties of a subset of the equilibria of such games known as normalized equilibria introduced by Rosen [12]. They have properties that are quite appealing in terms of decentralized implementation, pricing and billing.

Games with coupled constraints along with the generalized equilibrium has been applied to many networking problems, see e.g. [1], [2], [3], [7].

It is straightforward to see that we have:

Lemma 1. *All rates satisfying*

$$\sum_{i=1}^{n} R_i = \rho(\overline{S}), \ where \ \overline{S} = \{1, ..., n\},$$

subject to (1) are Nash equilibria and are Pareto-efficient. Any other point is not an equilibrium.

In view of the large number of Nash equilibria we address next the problem of selecting one which has desirable properties: the *normalized* Nash equilibrium. The motivation for proposing that one is related to pricing and billing issues.
Pricing We are interested in pricing mechanisms that induce equilibria strategies and that can be implemented in a scalable and decentralized way. Let λ_i be the per packet price for mobile i and let $\underline{\lambda}$ be the vector whose ith entry is λ_i. Then the payoff of mobile i with the additional pricing becomes

$$L_i^{\lambda}(R) := U_i(R_i) + \lambda_i(\sum_{i=1}^{n} R_i - \rho(\overline{S})).$$

Define

$$\widehat{C} \triangleq \left\{ R \ \middle| \ \sum_{i \in S} R_i \leq \rho(S), \ \forall S : ((S \subset \{1, ...n\}) \wedge (S \neq \overline{S})) \right\}.$$

Consider now the following relaxed game. Find $R^* \in \widehat{C}$ such that for each i and $R \in \widehat{C}$,

$$L_i^{\lambda}(R^*) \geq L_i^{\lambda}(R_i, (R^*)^{(-i)}).$$

If it has a solution then $L_i^{\lambda}(R^*)$ can be viewed as the Lagrangian that corresponds to the constrained optimization problem faced by mobile i when the other mobiles play $(R^*)^{(-i)}$ obtained by relaxing the single constraint $\sum_{i \in S} R_i \leq \rho(\overline{S})$.

From Karush-Kuhn-Tucker (KKT) Theorem we know that there exists a vector $\underline{\lambda}$ (whose entries are not necessarily equal) such that a rate vector R^* is an equilibrium in the original game only if for each i, R_i^* maximises $L_i^{\lambda}(R_i, (R^*)^{(-i)})$ obtained by relaxing the single constraint $\sum_{i \in S} R_i \leq \rho(\overline{S})$.

Thus, $\underline{\lambda}$ can be used to define a non-scalable pricing. It is non-scalable since the price per packet will depend on i.

Let x^* be the vector whose ith component is given by the solution to

$$\frac{dU_i(x)}{dx} + \lambda_i = 0. \qquad (2)$$

If $x^* \in \widehat{C}$ then it is the unique equilibrium of the relaxed game.

Denote the solution by $R^*(\underline{\lambda})$.

Problem P1: Consider a constant λ and let $\underline{\lambda}$ be a vector of dimension n with all its entries λ. Then, we wish to find λ such that $R^*(\underline{\lambda})$ is an equilibrium of the original game.

If we find such a λ then it defines indeed a scalable distributed pricing since the billing can be performed per packet and can be implemented without any knowledge to which mobile the packet belongs to.

An equilibrium associated with some constant λ that solves Problem P1 is a special case of the *normalized equilibrium* concept introduced in [12].

Theorem 1. *There exists a unique normalizied equilibrium to the original problem associated with some λ as defined in Problem P1.*

Proof: Define G to be the n dimensional square matrix whose ijth entry is $\frac{\partial^2 L_i^\lambda(R)}{\partial R_i \partial R_j}$. Then, since all the utility functions U_i are strictly concave increasing functions, $G + G^T$ is strictly negative definite. The Theorem then follows from [12, Thm 4].

A. Equal utility functions and maxmin fair assignment

Fix λ as given in Theorem 1. Assume that U_i is the same for all i. Then the Lagrangians corresponding to each player i have all the same dependence on their argument:

$$L_i^\lambda(R) := U(R_i) + \lambda(\sum_{j=1}^n R_j - \rho(\overline{S})).$$

Thus x^* defined in equation (2) has the form $(x, ...x)$. Note that x does not depend on $\rho(S)$ and thus on P_i and h_i, $i = 1, ...n$. Thus for any P_i and h_i such that $x^* \in \widehat{\mathcal{C}}$, the scalable pricing λ from Theorem 1 yields a Pareto equilibrium which is fair in the sense that all users receive the same rates. This is a special case of the maxmin fairness which we define next.

Definition 1. *A rate allocation is called maxmin fair if we cannot increase the rate R_i of user i without decreasing the rate R_j for some user j with $R_j \leq R_i$, while maintaining feasibility.*

Next we show that
(i) when maximizing the sum of utilities over C, the unique solution corresponds to the maxmin throughput assignment; and
(ii) the normalized equilibrium is always maxmin fair, even if $x^* \notin \widehat{\mathcal{C}}$. In the latter case, the equilibrium will of course not be symmetric.

Figures 1 and 2 are examples of achievable rate regions for the case of two users. Figure 1 exhibits a case where the maxmin fair rates are equal. In Figure 2 the point on the boundary (the segment S_1 parallel to the x-axis) for which the rates are equal is not Pareto optimal and therefore is not maxmin fair. The maxmin fair rate assignment is in that case the intersection between the segment S_1 and the diagonal segment.

Remark 1. There are achievable rate regions of other access networks that have the following property
Π_{Pareto} : **all the points on the boundaries other than those on the axis, are Pareto optimal**.
An example is given in Section III. Let x be the intersection of the diagonal segment with the Pareto frontier. By construction, all coordinates of x are equal. Since x is Pareto optimal, then for any direction i, the only way to increase x_i is by decreasing x_j for some j such that $x_j = x_i$. Hence, x is a max-min fair assignment. Thus when Π_{Pareto} holds, the maxmin assignment always symmetric even if the data are not.

To establish the above mentioned results we shall use the majorization order and Schur concavity which we define next.
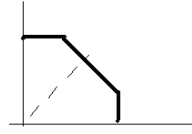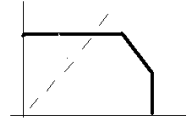

Fig. 1. Rate region: equal rates are maxmin fair

Fig. 2. Rate region: equal rates are not maxmin fair

Definition 2. *(Majorization and Schur-Concave Function [10])*
Consider two n-dimensional vectors $d(1), d(2)$. $d(2)$ majorizes $d(1)$, which we denote by $d(1) \prec d(2)$, if

$$\sum_{i=1}^k d_{[i]}(1) \leq \sum_{i=1}^k d_{[i]}(2), \quad k = 1, ..., n-1,$$

and

$$\sum_{i=1}^n d_{[i]}(1) = \sum_{i=1}^n d_{[i]}(2),$$

where $d_{[i]}(m)$ is a permutation of $d_i(m)$ satisfying $d_{[1]}(m) \geq d_{[2]}(m) \geq ... \geq d_{[n]}(m)$, $m = 1, 2$.
A function $f : R^n \to R$ is Schur concave if $d(1) \prec d(2)$ implies $f(d(1)) \geq f(d(2))$.

Lemma 2. [10, Proposition C.1 on p. 64] *Assume that a function $g : R^n \to R$ can be written as the sum $g(d) = \sum_{i=1}^n \psi(d_i)$ where ψ is a concave function from R to R. Then g is Schur concave.*

The following result from [14] characterizes the maxmin fair rate assignement in terms of majorization.

Theorem 2. *The maxmin fair rate assignment belongs to the dominant face $\overline{S}$ and is majorized by any other point on the dominant face.*

We would like to note that there is an algorithm with $O(n^2)$ complexity for determination of the maxmin fair rate assignment [14].

B. Other fairness concepts and their relation to the Normalized equilibrium

1) Global optimization: Consider the global optimization problem of maximizing the sum of the utilities $\sum_{i=1}^n U(R_i)$ over the achievable rate region C. We relax the constraint $\sum_i R_i \leq \rho(\overline{S})$ and write the Lagrangian

$$L^\lambda(R) := \sum_i U(R_i) + \lambda(\sum_{j=1}^n R_j - \rho(\overline{S})).$$

We can rewrite it as

$$L^\lambda(R) := \sum_i \psi(R_i) \text{ where } \psi(x) := U(x) + \lambda x - \frac{\rho(\overline{S})}{n}. \quad (3)$$

Using Lemma 2 we conclude that $L^\lambda(R)$ is Schur concave. We conclude from Lemma 2 the following:

Theorem 3. *The sum of utilities over the achievable rate region is maximized at the maxmin rate assignment.*

Proof: According to Theorem 2, the maxmin fair rate assignment is majorized by any other point on the dominant face. It therefore maximizes any Schur concave function on the dominant face. In particular, it maximizes $L^\lambda(R)$ for any λ since by Lemma 2 we conclude that $L^\lambda(R)$ is Schur concave. This implies the Theorem since by Karush-Kuhn-Tucker (KKT) Theorem, there exists λ such that the sum of utilities is maximized by the rate assignment that maximizes the Lagrangian $L^\lambda(R)$. ◇

2) α-fairness: We next recall the concept of α-fairness and of general α-fairness. Mo and Walrand [11] introduced the family of utility functions indexed by a real non-negative parameter α:

$$V_\alpha(x) = \frac{x^{1-\alpha}}{1-\alpha}$$

for $\alpha \neq 1$. Consider the problem of maximizing $\sum_{i=1}^n V_\alpha(x_i)$ where $(x_1, ..., x_n)$ lies in a closed convex set X. It is shown in [11] that the solution of the maximization problem converges to the maxmin fair assignment over X as $\alpha \to \infty$, it provides the harmonic fairness for $\alpha = 2$, it converges to the proportional fair assignment as $\alpha \to 1$ and it provides the globally optimal assignment over X for $\alpha = 0$.

Now, instead of assigning fairly $x \in X$ we can assign fairly some utility of x. We thus define

$$W_\alpha(x) = \frac{U(x)^{1-\alpha}}{1-\alpha}$$

and maximize $\sum_{i=1}^n W_\alpha(x_i)$ over X. The solution is called the generalized α-fairness [15].

Corollary 1. *(i) For all $\alpha \geq 0$, the alpha-fair rate assignment coincides with the unique maxmin rate assignment.*
(ii) Consider some strictly concave increasing utility function U. For all $\alpha \geq 0$, the generalized alpha-fair rate assignment coincides with the unique maxmin rate assignment.

Proof: Both the α-utility V_α as well as the generalized α-utility W_α are concave. It follows from Lemma 2 that $\sum_{i=1}^n V_\alpha(R_i)$ and $\sum_{i=1}^n W_\alpha(R_i)$ are Schur concave. Since these are concave functions defined on a convex compact set, they have a solution (which is unique for all $\alpha > 0$). The solution is thus the same for all $\alpha > 0$ and is majorized by any other point on the dominant face of the rate region. In particular since this solution does not depend on α, it is the limit of the α-fair solutions as $\alpha \to \infty$ and therefore it follows from [11] that the solution is maxmin fair. ◇

Remark 2. *The proof of the Corollary provides an alternative proof to Theorem 2.*

Remark 3. *It may seem astonishing that all fairness concepts provide the same rate allocation. In fact, this is not the case in general fair assignment (see examples in [15]) although the utilities V_α and W_α are always Schur concave. To understand this, note that both α-fair and generalized α-fair assignments are Pareto optimal. Whenever the Pareto optimal set has the property that the sum rates are constant then indeed any Schur concave function will have the same maximizer and thus all the fairness concepts would coincide. In the ensuing*

Sections III and IV we give examples of MACs where different fairness concepts result in different rate allocations.

3) Jain's fairness: We shall call a rate allocation *Jain's fair* if it maximizes the Jain's fairness index [9]:

$$J = \frac{(\sum_{i=1}^n R_i)^2}{n \sum_{i=1}^n R_i^2}.$$

The Jain's fairness index ranges from $1/n$ (worst case) to 1 (best case).

Since the maxmin fair rate allocation minimizes $\sum_{i=1}^n R_i^2$ [14], we conclude that the maxmin fair rate allocation is also the Jain's fair rate allocation.

4) The normalized equilibrium: We know from Theorem 1 that there exists a unique normalized equilibrium R^* to the original problem associated with some λ as defined in Problem P1. R^* is thus such that for each i, its ith component maximizes $L_i^\lambda(R) := U(R_i) + \lambda(\sum_{j=1}^n R_j - \rho(\overline{S}))$. This implies that R^* is the unique vector that maximizes

$$\sum_j U(R_j) + \lambda(\sum_{j=1}^n R_j - \rho(\overline{S}))$$

over $\widehat{C}$. This is the Lagrangian that corresponds to the global optimization problem. We conclude that R^* is the globally optimal solution. Applying Theorem 3 we conclude:

Theorem 4. *The unique normalized equilibrium is the maxmin fair assignment.*

C. Differential services and weighted α-fairness

We next raise the question of providing differentiation between the mobiles. We do that by defining K priority classes. Let $k(i)$ be the priority class corresponding to mobile i. We associate to class k priority some positive constant w_k. Introduce the following problem.

Problem P2. Let λ be some constant and define the price per traffic of mobile i as $\lambda_i(w_k) = \lambda/w_k$ if mobile i belongs to priority class k. We wish to find the constant λ such that $\lambda_i(w_{k(i)})$ defines an equilibrium in our problem.

If such pricing exists then it is indeed scalable since billing is done not according to the exact identity of the source of each packet but according to the priority class of the transmitted packets.

An equilibrium obtained as in problem P2 indeed exists and is unique [12]. This is the general form of a normalized equilibrium. The choice of w_k now allows to determine the rate that each priority class would get.

Furthermore, according to [12], finding normalized equilibrium is equivalent to the solution of the following optimization problem:

$$U(R) = \sum_{i=1}^n w_{k(i)} U(R_i), \qquad (4)$$

subject to constraints (1). In the important particular case when the optimal solution to (4) lies on the dominant face and the

choice of utility function corresponds to α-fairness, we have explicit expressions for λ and for the optimal rates.

Theorem 5. *Let $U(x)$ be α-fairness utility function. If the optimal rate allocation lies on the dominant face, the Lagrange multiplier and the optimal rates are given by*

$$\lambda = \left(\frac{\sum_{j=1}^{n} (w_{k(j)})^{1/\alpha}}{\rho(\bar{S})} \right)^{\alpha},$$

and

$$R_i = \frac{\rho(\bar{S})}{\sum_{j=1}^{n} (w_{k(j)})^{1/\alpha}} (w_{k(i)})^{1/\alpha}.$$

Proof: The formulae of the theorem's statement are derived with the help of the Lagrangian

$$L = \sum_{i=1}^{n} w_{k(i)} \frac{R_i^{1-\alpha}}{1-\alpha} + \lambda(\rho(\bar{S}) - \sum_{i=1}^{n} R_i).$$

We note that if we take $\alpha = 1$ the prices become proportional to the rates. In the case of $\alpha = 1$ the utility function is logarithmic, which corresponds to the standard utility function for elastic traffic. This provides justification to price elastic traffic proportionally to required rates.

We also note that for any choice of weights we have that $R_i \rightarrow \rho(\bar{S})/n$ as $\alpha \rightarrow \infty$ whenever the point $(\rho(\bar{S})/n, ..., \rho(\bar{S})/n)$ belongs to the dominant face. In fact, since the case $\alpha \rightarrow \infty$ corresponds to the lexicographic optimization for any choice of weights, the following more general statement holds.

Theorem 6. *As $\alpha \rightarrow \infty$, the weighted α-fair rate allocation converges to maxmin fairness.*

We would like to note that the lexicographic optimization and maxmin fairness concepts coincide for polymatroid capacity regions but can differ in the case of non-convex regions. We shall discuss this issue in more detail in the special section on non-convex capacity regions.

There are several kinds of multiple access systems whose capacity region is characterized by a polytope. Some examples are listed below:

- MAC systems with single transmit antennas at both sides and time invariant channel [22].
- MAC systems with single transmit and receive antennas and flat fading, when the channel state information is known only at the receiver but not at the transmitter [13].
- MAC systems with multiple transmit and receive antennas and unbiased flat fading, when the channel state information is known only at the receiver but not at the transmitter [23].

III. Convex Non-polytope Achievable Rate Region

Let us consider the Gaussian multiple access orthogonal channel. Classical example of orthogonal channels are TDMA and FDMA. In this case, the achievable rate region is given by

$$R_i \leq \theta_i \ln\left(1 + \frac{P_i h_i}{\theta_i \sigma^2}\right) \qquad \forall i = 1, ..., n \qquad (5)$$

where $0 \leq \sum_{i=1}^{K} \theta_i \leq 1$.

Given P_i and h_i, $i = 1...K$ the achievable rate region (5) is convex but it is not a polytope and it is strictly contained in (1). All points on the capacity region border are Pareto efficient and Nash equilibria. Therefore, here again we need to choose among many Pareto efficient points and Nash equilibria. We suggest to use α-fairness utility function to select a particular equilibrium point. In the following theorem we characterize the α-fair equilibrium selection.

Theorem 7. *The α-fair rate allocation in the Gaussian multiple access orthogonal channel, which maximizes the utility function*

$$U(R) = \sum_{i=1}^{n} \frac{R_i^{1-\alpha}}{1-\alpha},$$

is unique for any value of the parameter α and is given by θ's which solve the following system of equations

$$\left[\theta_i \ln(1 + \frac{P_i h_i}{\theta_i \sigma^2}) \right]^{-\alpha} \left[\ln(1 + \frac{P_i h_i}{\theta_i \sigma^2}) - \frac{P_i h_i}{\theta_i \sigma^2 + P_i h_i} \right] = \lambda_0,$$

$$\sum_{i=1}^{n} \theta_i = 1.$$
(6)

The above system has a solution in explicit form in the case of total rate maximization ($\alpha = 0$):

$$\theta_i^* = \frac{P_i h_i}{\sum_{k=1}^{n} P_k h_k}.$$
(7)

When $\alpha \rightarrow \infty$, the α-fair rate allocation provides maxmin fair rate allocation ($R_1^* = R_2^* = ... = R_n^*$).

Proof. The uniqueness of the α-fair rate allocation follows from the fact that we deal with the convex optimization problem. The equations (6) follow from the KKT conditions for the Lagrangian

$$L = \sum_{i=1}^{n} \frac{R_i^{1-\alpha}}{1-\alpha} + \lambda_i \left[\theta_i \ln(1 + \frac{P_i h_i}{\theta_i \sigma^2}) - R_i \right] + \lambda_0 \left[1 - \sum_{i=1}^{n} \theta_i \right].$$

For the particular case $\alpha = 0$, it follows from system (6) that

$$\ln(1 + \frac{P_i h_i}{\theta_i \sigma^2}) - \frac{P_i h_i}{\theta_i \sigma^2 + P_i h_i} = \ln(1 + \frac{P_j h_j}{\theta_j \sigma^2}) - \frac{P_j h_j}{\theta_j \sigma^2 + P_j h_j},$$

or

$$\ln(1 + \frac{P_i h_i}{\theta_i \sigma^2}) - \ln(1 + \frac{P_j h_j}{\theta_j \sigma^2}) = \frac{P_i h_i}{\theta_i \sigma^2 + P_i h_i} - \frac{P_j h_j}{\theta_j \sigma^2 + P_j h_j}.$$

Now it is easy to see that the values of θ's provided by (7) make zero both sides of the above equation. Also, it is clear that the values of θ's provided by (7) satisfy the normalization condition.

◇

Let us calculate the Jain's fairness index for the rate allocation corresponding to the total rate maximization

$$J = \frac{(\sum_{i=1}^{n} R_i)^2}{n \sum_{i=1}^{n} R_i^2} = \frac{(\sum_{i=1}^{n} P_i h_i)^2}{n \sum_{i=1}^{n} P_i^2 h_i^2} \geq \frac{1}{n}.$$

We note that the lower bound $1/n$ can be achieved and hence the allocation corresponding to the total rate maximization can be extremely unfair judging by the Jain's fairness index. Thus, the α-fairness concept provides a continuous spectrum of rate allocations from the possibly very unfair total rate maximization to the completely fair maxmin rate allocation. It is easy to calculate the maximal total rate corresponding to the total rate maximization problem

$$\sum_{i=1}^{n} R_i^* = \ln\left(1 + \frac{\sum_{i=1}^{n} P_i h_i}{\sigma^2}\right).$$

MAC systems which enable time-sharing of the coding schemes are in general characterized by a convex capacity region. Examples of MACs having a convex non-polytope capacity region with non linear boundary surfaces are:

- Time invariant MAC systems with multiple antennas and without intersymbol interference (ISI) [21].
- Time invariant MAC with single [20] or multiple antennas [21] and ISI.
- Flat fading channel with channel state information known at the transmitter and the receiver and single [26] or multiple transmitting and receiving antennas.
- In all previous cases with fading channels we considered the *ergodic* capacity regions, i.e. we assumed that the codewords have a duration such that the channel is ergodic in such time interval. However, it is possible to consider a second situation when codewords have a duration independent of the scale of the channel variation. In general, the channel is not stationary and ergodic in the timeframe of a codeword. In such a case, it is possible to have reliable communications only if the channel is known at both ends. These kind of capacity region for MAC with single transmit and receive antennas are investigated in [27] and the maximum achievable rates are referred to as delay limited capacities.

The approach of Theorem 7 equally applies to the above MACs. In particular, in the case of any strictly convex capacity region, there is a unique α-fair rate allocation. We also note that in the present case, similarly to the polymatroid capacity region case, the Jain's fair allocation coincides with the maxmin fair allocation.

IV. NON-CONVEX ACHIEVABLE RATE REGION

Shannon pointed out in [16] that all capacity and zero-capacity regions are convex if it is possible to time-share the coding schemes applied to attain individual points of the region. However, time sharing is not applicable to all systems. When some system characteristic prevents the use of time sharing the capacity region is not *necessarily* convex. As an example, time sharing is not applicable if the system nodes do not share a common time reference.

We present below some examples of systems with non-convex rate regions.

A. Collision Channel Without Feedback

The collision channel without feedback has been studied first in [17]. The absence of a feedback channel does not enable synchronization for applying time sharing of the code-books. The interested reader is referred to [17] for a detailed description of the system. Shortly, we recall here that the two cases of slot-synchronized and asynchronous collision channel are investigated in [17]. The source for each user is Q-ary symmetric, independent of the others and generating information at a rate of C_i packet/slot. The duty factor p_i of user i is the fraction of time during which user i is actually transmitting packets. Thus, the transmitting rate of the user when she is actually using the channel is $\frac{C_i}{p_i}$. The signals at the receiver is impaired, eventually, only by colliding signals but not by additive noise. The capacity region is the set of all source rates that are approachable in the sense that there exist coding schemes with rate $\frac{C_i}{p_i}$ with enable reliable communications[1].

Let $\mathcal{C}_{u,0}$, $\mathcal{C}_u$, $\mathcal{C}_{s,0}$, and $\mathcal{C}_s$ be the zero and non zero capacity regions of asynchronous collision channels, and the zero and non zero capacity regions of the slot-synchronous collision channels, respectively. All these regions coincide and the boundary of the common capacity region $\mathcal{C}$ is the set of all points [17] $C = (C_1, C_2, \ldots, C_n)$ such that

$$C_i = p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - p_j) \tag{8}$$

being $p = (p_1, \ldots p_n)$ a probability vector whose elements are nonnegative and satisfying the constraint $\sum_{i=1}^{n} p_i = 1$.

Interestingly, $\mathcal{C}$ is not convex but its complement in the non-negative orthant is convex [18]. Figure 3 shows the capacity region for the two users case.

In the next theorem we characterize the α-fair rate allocations for the non-convex capacity region described by (8).

Theorem 8. *The α-fair rate allocation for the capacity region described by (8) has three distinct cases with respect to the value of α:*

- **Case $\alpha > \alpha_*$:** *The α-fair rate allocation corresponds to the maxmin rate allocation.*
- **Case $\alpha = \alpha_*$:** *There are several α_*-fair rate allocations. Among α_*-fair rate allocations one corresponds to the maxmin fair rate allocation. Moreover, if $n = 2$, any rate allocation on the capacity boundary is α_*-fair. If $n > 2$, one α_*-fair rate allocation corresponds to the maxmin fair allocation and the other α_*-fair rate allocations correspond to total rate optimization and are given by the points $R_i = 1$ and $R_j = 0$ for $j \neq i$.*

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[1]The communications systems analyzed in the previous sections were characterized by the same source rate and transmission rate, while in this case the two quantity differs. To keep clear this feature, we adopt a different notation for the source rate in the collision channel and in the following recovery channel.

- **Case** $\alpha < \alpha_*$**:** *The α-fair rate allocations correspond to total rate optimization and are given by the points $R_i = 1$ and $R_j = 0$ for $j \neq i$.*

The threshold value α_* is given by

$$\alpha_* = \frac{(n-1)\ln(n/(n-1))}{\ln(n) + (n-1)\ln(n/(n-1))}. \tag{9}$$

Proof. To determine the threshold value α_*, we equate the values of the α-fairness utility function evaluated at the maxmin fair point $(1/n, ..., 1/n)$ and the corner point $(1, 0, ..., 0)$. Namely, we need to solve the following equation with respect to α:

$$\frac{n}{1-\alpha}\left[\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\right]^{1-\alpha} = \frac{1}{1-\alpha}.$$

If $n = 2$, the boundary of the capacity region has an explicit form $R_1^{1/2} + R_2^{1/2} = 1$ and the statement of the theorem for $n = 2$ follows immediately. If $n > 2$, we consider small perturbations around the maxmin fair point

$$p_1 = \frac{1}{n} + \varepsilon, \quad p_i = \frac{1}{n} - \frac{1}{n-1}\varepsilon, \; i = 2, ..., n,$$

and around the corner points

$$p_1 = 1 - \varepsilon, \quad p_i = \frac{1}{n-1}\varepsilon, \; i = 2, ..., n,$$

in order to conclude that we deal with isolated maxima.

$\diamond$

Since the point $p = (1/n, ..., 1/n)$ provides the maximal value to the Jain's fairness index, in the case of the non-convex capacity region described by (8) the maxmin fairness coincides with the Jain's fairness in this example of MAC with non convex capacity region.

The capacity of the slot-synchronized collision channel has been extended in [19] to the case when a multiuser decoder is utilized at the receiver and the colliding messages can be recovered with probability q. This channel is referred to as recovery channel. The capacity of the n-user recovery channel with recovery probability q is the closure of all rate points $(R_1, R_2, \ldots, R_n)$ satisfying the following for all $i = 1, \ldots n$:

$$R_i \leq p_i \prod_{j=1, j \neq i} (1-p_j) + qp_i\left(1 - \prod_{j=1, j\neq i}(1-p_j)\right) \tag{10}$$

where $0 \leq p_i \leq 1$.

In contrast to the previously investigated rate regions, the region (10) is neither convex, as the cases investigated in Section II and III, nor concave as for the collision channel without feedback. Examples of this capacity region are provided in [19]. In the next theorem we characterize the α-fair rate allocations for the capacity region (10).

Theorem 9. *The α-fair rate allocation for the capacity region described by (10) has three distinct cases with respect to the value of α:*

- **Case** $\alpha > \alpha_*$**:** *The set $\mathcal{S}_{\alpha^+}$ of the α-fair rate allocations contains the single point $R_1 = \ldots R_n = q$ corresponding to the maxmin fair rate allocation.*

- **Case:** $\alpha < \alpha_*$**:** *The set $\mathcal{S}_{\alpha^-}$ of the α-fair rate allocations contains n points corresponding to total rate optimization points, namely $\mathcal{S}_{\alpha^-} = \{R | (R_i = 1)\bigcap_{j=1, j\neq i}^n (R_j = 0), \text{ for } i = 1, \ldots n\}$.*
- **Case** $\alpha = \alpha_*$**:** *The set $\mathcal{S}_{\alpha^*}$ of the α-fair rate allocations contains contains $n+1$ points. More specifically, $\mathcal{S}_{\alpha^*} = \mathcal{S}_{\alpha^+} \bigcup \mathcal{S}_{\alpha^-}$.*

The threshold value α_* is given by

$$\alpha_* = 1 + \frac{\ln(n)}{\ln(q)}. \tag{11}$$

Proof. We focus on the case $q > 0$ since for $q = 0$ the recovery channel reduces to the collision channel. All the extreme points for the optimization problem $\max \sum_{i=1}^n \frac{R_i^{1-\alpha}}{1-\alpha}$ with $(R_1, R_2, \ldots R_n)$ belonging to the rate region (10) are obtained as solution of the KKT conditions for the Lagrangian

$$L = -\sum_{i=1}^n \frac{R_i^{1-\alpha}}{1-\alpha} + \sum_{i=1}^n \lambda_i \left(R_i - p_i \prod_{i\neq j}(1-p_j)\right.$$

$$\left. -qp_i\left(1 - \prod_{i\neq j}(1-p_j)\right)\right) + \sum_{i=1}^n \nu_i(p_i - 1).$$

The points satisfying the KKT conditions belong to the union set $\mathcal{S}_{\alpha^-} \bigcup_{i=0, i\neq 1}^n \mathcal{K}_i$ where $\mathcal{K}_i$ is the set of all distinct permutations of $(\underbrace{q, \ldots q}_{i \text{ times}}, \underbrace{0, \ldots 0}_{n-i})$. Potential maximum values for the utility function are reached in $\mathcal{S}_{\alpha^-}$ where $\sum_{i=1}^n \frac{R_i^{1-\alpha}}{1-\alpha} = \frac{1}{1-\alpha}$ and in $\mathcal{S}_{\alpha^+} = \mathcal{K}_n$ with $\sum_{i=1}^n \frac{R_i^{1-\alpha}}{1-\alpha} = \frac{nq^{1-\alpha}}{1-\alpha}$ (the other sets correspond to obvious local maxima and minima). To determine the threshold value α_*, we equate the values of the α-fairness utility function evaluated at $\mathcal{S}_{\alpha^-}$ and $\mathcal{S}_{\alpha^+}$. Namely, α^* is the solution of the following equation with respect to α:

$$\frac{nq^{1-\alpha}}{1-\alpha} = \frac{1}{1-\alpha}.$$

$\diamond$

With the allocation rate $(q, q, \ldots q)$ all users transmit continuously ignoring the presence of the interfering users and relying only on the multiuser decoding capabilities of the receiver. It could be surprising that there exists a single threshold for the parameter α which determines a transition from the set of the allocation rates $\mathcal{S}_{\alpha_-}$ to the set $\mathcal{S}_{\alpha_+}$ while the points in $\mathcal{K}_i$ are not α-fair allocations for any value of α. This phenomenon is due to system model adopted to describe the recovery channel. In fact, the recovery probability is assumed constant and independent of the number of colliding users. In practical systems the recovery probability q is a decreasing function of the system load, i.e. the number of active users in the system. Taking into account this aspect could induce a smoother transition from $\mathcal{S}_{\alpha_-}$ to $\mathcal{S}_{\alpha_+}$ through the sets $\mathcal{K}_i$ when the parameter α increases.

B. Implication on max-min fairness

Consider the utility region given in Figure 4. It has two points, A and B, which are both lexicographically maxima: the

value of the smallest coordinate of any point in the region is maximized there; moreover among all points that maximize the smallest coordinate, points A and B also maximize the value of the largest coordinate. For the case where the achievable region is convex, lexicographic maxima are also maxmin fair. The region of Figure 4 however has no maxmin assignment! For example, at point A, any way to increase the assignment in the vertical axis direction results in a decrease in the assignment in the horizontal direction.
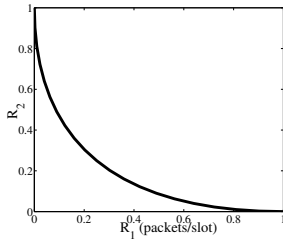

Fig. 3.   Rate region of a collision channel with two users
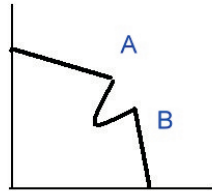

Fig. 4.   Rate region: equal rates are maxmin fair

We note that in this case the Jain's fair allocation is well defined (it is the intersection point of the central line pointed by $(1/n, ..., 1/n)$ and the boundary of the capacity region) and is different from the lexicographic fair allocations.

The Figure 4 is taken from a rate region in interference channel with random access [25]. To our knowledge, this is the first time that this phenomenon is identified in our context. A similar phenomenon has been discovered in the context of assignment of discrete objects, see [24].

V. CONCLUSION

We have studied MACs in the context of non-cooperative games with correlated constraints. In the non-cooperative games with correlated constraints the actions available to one player depend on those used by the others. A typical feature in these games is that they often possess infinitely many equilibria. To select among the equilibria we have proposed to use the Normalized Nash equilibrium and different fairness concepts such as maxmin fairness, lexicographic fairness, α-fairness and Jain's fairness. We have considered three main types of MACs: polymatroid regions, convex non polytope regions, and non convex capacity regions. In the case of Gaussian MAC (an example of a channel with polymatroid achievable rate region), the normalized Nash equilibrium and all fairness concepts select the same equilibrium. In the case of Gaussian multiple access orthogonal channel (an example of a channel with convex non-polytope achievable rate region) different fairness concepts can select different equilibria. We have characterized these equilibria. Finally, the analysis pointed out very unexpected behaviors of channels with non-convex achievable rate regions. It turns out that in some channels with non-convex achievable rate regions the maxmin fair allocation might not even exist. An interesting future research direction is to study equilibria selection in the case of several base stations.

REFERENCES

[1] E. Altman and Z. Altman, "S-modular games and power control in wireless networks", IEEE Trans. on Automatic Control, Vol. 48, No. 5, pp. 839–842, May, 2003.

[2] E. Altman and A. Shwartz, "Constrained Markov Games: Nash Equilibria", Annals of Dynamic Games, vol. 5, Birkhauser, V. Gaitsgory, J. Filar and K. Mizukami, editors, pp. 213-221, 2000.

[3] E. Altman and E. Solan, "Games with constraints with networking applications", manuscript, available from the authors. Submitted, March 2007.

[4] K.J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, Econometrica 22 (1954), pp. 265-290.

[5] G. Debreu, "A social equilibrium existence theorem", Proceedings of the National Academy of Sciences of the U.S.A, 38:886-893, 1952.

[6] G. Debreu, "Existence of competitive equilibrium", Handbook of Mathematical Economics, vol 2, K. Arrow and M. Intrilligator, eds, New-York, North-Holland, 1982.

[7] R. El Azouzi and E. Altman, "Constrained traffic equilibrium in routing", IEEE Automatic Control Vol. 48, No. 9, pp. 1656–1660, Sept. 2003.

[8] F. Facchinei, A. Fischer and V. Piccialli, "On generalized Nash games and variational inequalities", Operations Research Letters Volume 35, Issue 2, March 2007, Pages 159-164

[9] R. Jain, D. Chiu and W. Hawe, "A Quantitative Measure of Fairness and Discrimination for Resource Allocation in Shared Systems", DEC Research Report TR-301, 1984.

[10] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*.   Academic Press, 1979, vol. 143 of Mathematics in Science and Engineering.

[11] J. Mo and J. Walrand, "Fair End-to-End Window-based Congestion Control", SPIE '98, International Symposium on Voice, Video and Data Communications, 1998.

[12] J. B. Rosen, "Existence and Uniqueuness of Equilibrium Points for Concave N-person games", Econometrica, Vol 33, pp 520-534, 1965.

[13] S. Shamai and A.D. Wyner, "Information theoretic considerations for symmetric, cellular, multiple-access channels– Part I," *IEEE Transactions on Information Theory*, vol. 43, pp. 1877-1894, Nov. 1997.

[14] K.W. Shum and C.W. Sung, "Fair Rate Allocation in Some Gaussian Multiaccess Channels", in *Proceedings of ISIT 2006*, Seattle, pp.163-167.

[15] C. Touati, E. Altman and J. Galtier, "Generalized Nash Bargaining Solution for bandwidth allocation", Computer Networks, Vol 50, pp 3242-3263, 2006

[16] C.E. Shannon, "Two Way Communication Channels", in *Proc. 4th Berkeley Symposium Mathematics, Stastistics and Probability*, vol. 1, pp. 611-644, 1961

[17] J. L. Massey and P. Mathys, "The Collision Channel without Feedback," *IEEE Transactions on Inforfmation Theory*, vol. 31, pp. 192-204, March 1985

[18] K. A. Post, "Convexity of the Non-achievable Rate Region for the Collision Channel without Feedback," *IEEE Transactions on Information Theory*, vol. 31, pp. 192-204, March 1985

[19] S. Tinguely, M. Rezaeian, and A. Grant, "The Collision Channel with Recovery," *IEEE Transactions on Information Theory*, vol. 51, pp. 3631-3638, October 2005

[20] Roger S. Cheng and Sergio Verdú, "Gaussian Multiaccess Channels with ISI: Capacity Region and Multiuser Water–Filling", *IEEE Transactions on Inforfmation Theory*, vol. 39, pp. 773-785, May 1993.

[21] W. Yu and W. Rhee and S. Boyd and J. M. Cioffi, "Iterative Water-filling for Gaussian Vector Multiple Access Channels", *IEEE Transactions on Information Theory*, vol. 50, pp. 145–152, January 2004.

[22] Thomas M. Cover and Joy A. Thomas, *Elements of Information Theory*, wiley, 1991, New York.

[23] Wonjong Rhee and J.M. Cioffi, "On the capacity of multiuser wireless channels with multiple antennas", *IEEE Transactions on Inforfmation Theory*, vol. 49, pp. 2580-2595, October 2003.

[24] Saswati Sarkar, Leandros Tassiulas: Fair Bandwidth Allocation for Multicasting in Networks with Discrete Feasible Set. IEEE Trans. Computers 53(7): 785-814 (2004).

[25] O. Simeone and Y. Bar-Ness, "A game-theoretic view on the interference channel with random access", Proc. IEEE DySPAN 2007.

[26] D. N.C. Tse and S. Hanly, "Multiaccess Fading Channel– Part I: Polymadroid Structure, Optimal Resource Allocation and Throughput Capacity," *IEEE Transactions on Inforfmation Theory*, vol. 44, pp. 2796-2815, Nov 1998

[27] D. N.C. Tse and S. Hanly, "Multiaccess Fading Channel– Part II: Delay Limited Capacity," *IEEE Transactions on Inforfmation Theory*, vol. 44, pp. 2816-2831, Nov 1998