Transmission of Correlated Sources Over Gaussian Multiple Access Channel with Phase Shifts

Fadi Abi Abdallah, Raymond Knopp
Mobile Communications Department
Eurecom
Sophia-Antipolis Cedex - France
Fadi.Abi-Abdallah,Raymond.Knopp@eurecom.fr

Giuseppe Caire
Department of Electrical Engineering
University of Sourthern California
Los Angeles CA, 90089 USA
caire@usc.edu

***Abstract*— In this paper, we address the problem of separate encoding of correlated sources observed by sensor nodes that send their encoded information through Gaussian multiple access channel (GMAC) with phase shifts. We suppose that the phases are perfectly known at the receiver and unknown to the transmitters. For discrete sources with finite-cardinality alphabets, we prove that the separation theorem holds for both random ergodic and arbitrary non-random models for the phase shifts, and consequently, the strategy of combining Slepian-Wolf coding to capacity achieving channel encoders is optimal for both.**

I. Introduction

In many sensor network applications, the observations collected by the sensor nodes are spatially correlated, for instance in scenarios where distributed sensing of a random field is performed (e.g. geological exploration, environmental sensing, electromagnetic sensing, etc.). With low-cost radio-equipped sensors, the observations are further encoded and sent through a noisy channel to a collector node where the information is extracted and processed. The main question that arises is how to efficiently encode the data at each node and how to benefit from the correlation between the observed sources. Shannon proved in [1] that, in a point-to-point communication scenario, an optimal way to send a random source through a noisy channel is to compress the source at a rate slightly greater than its entropy, in bits per source letter, and then to encode it at a rate slightly less than the capacity of the channel, in bits per channel use, prior to sending it across the channel. This coding strategy, known as the source-channel coding theorem or the separation theorem, is very useful because it permits one to split the encoder into two separate entities, the first being the source coding block and the second the channel encoder. Unfortunately, this strategy does not lead to optimal system performances in general network scenarios. An example of the latter is considered in [2], where the authors provide bounds on the capacity region for the MAC with arbitrarily correlated sources; they provide sufficient conditions for the correlated sources to be sent over the channel with an arbitrarily small probability of error. Although the resultant rate region contains the one achieved by separation between the source and the channel encoders, it is shown in [3] that it is not the capacity region for reliable transmission. All these results with others in [4] show the sub-optimality of the separation-based coding strategy and open the door toward cooperative coding strategies that try to map the correlation between the sources into correlation between the transmitted signals. One recent example of this is the scheme described in [5].

The coding problem that we consider here is a variation on the same theme. We consider M sensor nodes deployed in a certain area where each of them senses a single spatial dimension of the source and sends a representation of its measurement through a GMAC corrupted by phase shifts. In contrast to the work of El Gamal[6], we assume that each node does not have side information with respect to its own channel phase shift, and as a result cannot align its phase at the receiver in order to benefit from some generalized form of coherent combining which exploits the source correlation structure. As a side note, any wireless sensor network problem using a real-valued GMAC implicitly assumes this form of synchronization. In removing the assumption on phase synchronization we focus on the most pertinent channel model in a pragmatic sense. This is especially true in wireless sensor network applications where we often deal with low-cost components, at least when it comes to the link between the sensors and the collector node. Even in relatively high-cost cellular basestations, feedback-based combining schemes are very difficult (and costly) to achieve even for a centralized antenna array, let alone for distributed spatial processing across several basestations. Furthermore, it is conceivable for future low-end sensor networks that the sensors may not even be equipped with radio receivers in order to limit power consumption which is often dominated by the receiver electronics. This, of course, would rule out the possibility of any form of closed-loop synchronisation and necessarily result in phase differences at the receiver.

In our problem formulation, we assume that the source is discrete and of finite-cardinality per dimension and the goal is to reconstruct the vector source as reliably as possible at the collector node. What remains is to define a set of necessary and sufficient conditions under which the source can be sent and reconstructed with an arbitrarily small probability of decoding error. We consider two cases for phases variation: ergodic random phase sequences and deterministic but arbitrarily-varying phase sequences. By deriving a converse in both cases, we prove that the separation theorem holds for any number M of sensor nodes. Hence, the set of the achievable rates is the intersection of two rate-regions, the first being the Slepian-Wolf rate region [7] and the second, being the capacity region of the GMAC [8]. Another closely-related work is that of Barros and Servetto [9], [10]; in their model the uplink channel is a set parrallel non-interfering channels instead of a MAC. They proved that the separation is also optimal in that case and conclude that in the absence of interference, there is nothing to lose by compressing the source dimensions to their most efficient representation (Slepian-Wolf coding) and separately adding capacity-attaining channel codes.

The paper is organised as follows: in section II, we describe our system model. In section III, we state the two theorems that constitute the main contribution of the paper, provide proofs of the converse for both models for phase-variation and show the optimality of a separation-based coding scheme. In Section IV, we discuss several points concerning the two theorems and section V is dedicated for the conclusion and ongoing works, specifically for the case of continuous-valued sources.

II. Model

The system model is depicted in Fig.1. We consider M discrete correlated sources $U_1, \dots, U_M$ of respectively finite alphabets $\mathcal{U}_1, \dots, \mathcal{U}_M$ following the joint probability distribution $p(u_1, \dots, u_M)$. Source vectors $\mathbf{U}_1, \dots, \mathbf{U}_M$ of dimension K are generated by collecting K i.i.d samples of the sources $U_1, \dots, U_M$ respectively. Before being transmitted, these source vectors are encoded separately by M encoders $f_1, \dots, f_M$. The encoder f_m is a function that maps $\mathbf{U}_m$ onto a sequence of N channel symbols $\mathbf{X}_m \triangleq \{X_{m,n}; n = 1, \dots, N\}$, each of which taken from a finite alphabet $\mathcal{X}_m$. Thus

$$\begin{aligned} f_m : \quad \mathcal{U}_m^K \quad &\longrightarrow \quad \mathcal{X}_m^N \\ \mathbf{u}_m \in \mathcal{U}_m^K \quad &\longrightarrow \quad \mathbf{x}_m = f_m(\mathbf{u}_m) \in \mathcal{X}_m^N \end{aligned}$$

Let $\mathbf{Z} = \{Z_i; i = 1, \dots, N\}$ denote an i.i.d. sequence drawn according to a Gaussian distribution representing the channel noise where $Z_i \sim \mathcal{N}_C(0, N_0)$, and $\mathbf{\Phi}_m = \{\Phi_{m,i}; i = 1, \dots, N\}$ denote the set of random phases induced by the channel and associated to the encoder f_m. Let $\mathbf{\Phi} \triangleq \{\mathbf{\Phi}_m; m = 1, \dots, M\}$ be perfectly known to the decoder. The received signal is $\mathbf{Y} \triangleq \{Y_i; i = 1, \dots, N\}$ which belongs to the infinite alphabet $\mathcal{Y}^N$, and Y_i can be written as

$$Y_i = \sum_{m=1}^{M} X_{m,i} e^{j\Phi_{m,i}} + Z_i. \tag{1}$$

We consider the following power constraint

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[|X_{m,i}|^2\right] \leq E_m \tag{2}$$

for $m = 1, \dots, M$, where E_m represents the mean energy allowed per transmission for sensor m. For the channel phase sequences $\mathbf{\Phi}_m$, we shall consider the following different cases:

1) $\mathbf{\Phi}_m$ are random, perfectly known to the receiver and unknown to the transmitters, extracted from a jointly stationary and ergodic process $\{\Phi_{1,i}, \cdots, \Phi_{M,i}\}$. Furthermore, we assume that $\Phi_{m,i}$ (the i-th marginals of the process) are individually uniformly distributed over $[-\pi, \pi]$ and that the i-th marginal distribution of the phase difference $\Delta\Phi_{m,m',i} \triangleq \Phi_{m,i} - \Phi_{m',i}$ is also uniformly distributed over $[-\pi, \pi]$.
2) $\mathbf{\Phi}_m$ are *arbitrary* sequences, denoted by $\boldsymbol{\phi}_m$ since they are non-random. The transmitters have no knowledge of the phase sequences.
3) $\mathbf{\Phi}_m$ are *arbitrary and constant* sequences, that is, $\Phi_{m,i} = \phi_m$ for all $i = 1, \dots, N$, where ϕ_m is an arbitrary value in $[-\pi, \pi]$. In this case, the phases are constant for the whole duration of transmission but the transmitters have no knowledge about their values.

In section III, one coding theorem will be dedicated to the first phase sequences case, and another one for the last two cases, their corresponding proof being quite similar. After receiving $\mathbf{Y}$, the decoder generates an estimate $\widehat{\mathbf{U}}_m$ on each source $\mathbf{U}_m$ given the full knowledge on $\mathbf{\Phi}$. Thus, we have

$$g : \mathcal{Y}^N \times [-\pi; \pi]^{NM} \quad \longrightarrow \quad \mathcal{U}_1^K \times \cdots \times \mathcal{U}_M^K \tag{3}$$

$$(\mathbf{y}, \boldsymbol{\phi}) \quad \longrightarrow \quad g(\mathbf{y}, \boldsymbol{\phi}) = (\widehat{\mathbf{U}}_1, \cdots, \mathbf{U}_M). \tag{4}$$

Given a code, i.e., mapping functions $f_1, \dots, f_M$ and g, we define the error probability as

$$P^K(e) = \Pr\left((\mathbf{U}_1, \cdots, \mathbf{U}_M) \neq (\widehat{\mathbf{U}}_1, \cdots, \widehat{\mathbf{U}}_M)\right).$$

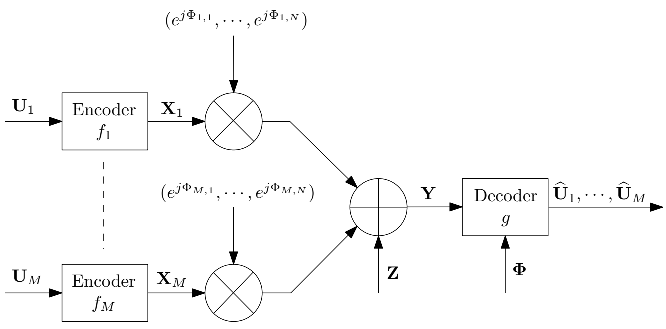

Fig. 1. Correlated sources over GMAC with phase shifts perfectly known at the receiver

III. Coding Theorems for Correlated Sources Over GMAC with Phase Shifts

A. Ergodic Phase Sequences

For the ergodic phase sequences, we have the following coding theorem:

Theorem 1: M discrete correlated sources $U_1, \dots, U_M$ of finite alphabets drawn according to $p(u_1, \dots, u_M)$ can be transmitted with an arbitrarily small probability of error over a GMAC with ergordic phases perfectly known at the receiver and with source-channel rate $r \triangleq K/N$ if, and only if

$$H(U_S|U_{S^c}) \leq \frac{N}{K}\log\left(1+\sum_{m\in S}\frac{E_m}{N_0}\right), \forall S \subseteq \{1,2,\cdots,M\}, \tag{5}$$

where the shorthand notation A_S represents the set of random variables $\{A_i, i \in S\}$.

Proof: The proof of this theorem can be divided in two parts: direct part and converse.

For the direct part, we have to prove that if the conditions in (5) are satisfied, the sources can be transmitted with an arbitrarily small probability of error. In fact, it is clear that, when the bounds on the joint entropy of the sources are satisfied, a simple separated approach that makes use of Slepian-Wolf coding and standard Gaussian superposition coding allows the transmission and the reconstruction of the sources at the receiver point with a vanishing probability of error.

The converse proof of the theorem is put in Appendix A. It is shown that if the sources are transmitted with an arbitrarily small probability of error, then they must verify the joint entropy conditions in (5). This theorem shows that in the case of ergodic phases, a separation-based scheme is optimal. In other words, compressing the sources to their most efficient representations by performing Slepian-Wolf coding, and separately adding capacity-achieving channel encoders is an optimal coding scheme. Moreover, it shows that coding cooperation between the transmitters does not buy anything. ■

B. Arbitrary Phase Sequences

Assuming that the phase sequences are not random, unknow at the transmitters, and perfectly known at the receiver, we have the following theorem:

Theorem 2: For the arbitrary phase sequences (or arbitrary and constant phases), M discrete correlated sources $U_1, \dots, U_M$ of finite alphabets can be transmitted reliably over a GMAC with a given source-channel rate r if, and only if their joint entropies satisfy the inequalities in (5).

Proof: [Proof for arbitrary phase sequences] The direct part proof is the same as for Theorem 1. The converse proof of the theorem is put in Appendix B. ■

Proof: [Proof for arbitrary and constant phase sequences] It is important to point out that this case cannot be considered as a special case of the arbitrary phase sequences. Although the necessary and sufficient transmissibility conditions for the case of arbitrary phase sequences are also necessary and sufficient for the case of arbitrary constant phases, this fact is not immediately evident. Notice also that constant phases reduces the possibility with respect to arbitrary sequences, therefore, the capacity region may be larger (certainly, not smaller). Hence, we have only to show the converse. In fact, by repeating the derivations in Appendix B while taking into account that the phase sequences are arbitrary and constant, we obtain the following necessary conditions

$$H(U_S|U_{S^c}) \leq \frac{N}{K}\log\left(1+\sum_{m\in S}\frac{E_m}{N_0}+ \frac{1}{NN_0}\inf_{\phi_1,\dots,\phi_M}\left\{\sum_{i=1}^{N}T_i\right\}\right) \tag{6}$$

where

$$\begin{aligned}\sum_{i=1}^{N}T_i &= \sum_{\substack{m,m'\in S\\ m'>m}}\sum_{i=1}^{N}|\rho_{m,m',i}|\cos(\Delta\phi_{m,m'}+\theta_{m,m',i}) \\ &= \sum_{\substack{m,m'\in S\\ m'>m}}\mathrm{Re}\left[\sum_{i=1}^{N}|\rho_{m,m',i}|e^{j(\Delta\phi_{m,m'}+\theta_{m,m',i})}\right]\end{aligned} \tag{7}$$

By defining the complex number $\rho_{m,m'}$ as

$$\rho_{m,m'} \triangleq \sum_{i=1}^{N}|\rho_{m,m',i}|e^{j\theta_{m,m',i}}, \tag{8}$$

(7) becomes

$$\sum_{i=1}^{N}T_i = \sum_{\substack{m,m'\in S\\ m'>m}}\mathrm{Re}\left[|\rho_{m,m'}|e^{j(\Delta\phi_{m,m'}+\theta_{m,m'})}\right]. \tag{9}$$

Using Eq. (9), it can be shown, as in Appendix B, that

$$\inf_{\phi_1,\dots,\phi_M}\sum_{i=1}^{N}T_i \leq 0. \tag{10}$$

Therefore, we deduce that the following conditions must be verified for reliable transmission of the sources

$$H(U_S|U_{S^c}) \leq \frac{N}{K}\log\left(1+\sum_{m\in S}\frac{E_m}{N_0}\right)\ \forall S \subseteq \{1,\cdots,M\}. \tag{11}$$

■

IV. Discussions

In this section, we discuss several important points that concern the two theorems stated above. For the converse of Theorem 2, we mention here two notes: first, other than this converse provides us with necessary conditions for reliable communication, it gives in addition some constraints or some properties about the family of codes that achieves optimality. In fact, for the case of arbitrary phase sequences, channel symbols at time i corresponding to an optimal code must be uncorrelated, otherwise the code is suboptimal. To see it clearer, let's take two blocks of channel symbols of length N corresponding to sensors m and m'. Without loss of generality, we'll assume $m = 1$ and $m' = 2$. From (28) and (37), we conclude that any code allowing the reconstruction of the sources with vanishing probability of errors must verify

$$H(U_1, U_2|U_{S^c}) \leq \frac{N}{K}\log\left(1 + \frac{E_1+E_2}{N_0} - \frac{2}{NN_0}\sum_{i=1}^{N}|\rho_{1,2,i}|\right) \quad (12)$$

Knowing that the inequality

$$H(U_1, U_2|U_{S^c}) \leq \frac{N}{K}\log\left(1 + \frac{E_1+E_2}{N_0}\right) \quad (13)$$

can be achieved by a separate-based coding scheme, we deduce that if $\rho_{1,2,i} \neq 0$ for at least one i, the optimal system performance cannot be reached. Therefore, any optimal code, including the one based on the source-channel separation, must verify $\rho_{m,m',i} = 0 \quad \forall m, m' \in \{1, \ldots, M\}, m \neq m', i = 1, \ldots, N$. Similarly, in the case of arbitrary and constant phases, it can be shown that optimal codes should verify $\rho_{m,m'} = 0 \ \ \forall m, m' \in \{1, \ldots, M\}, m \neq m'$.

The second note on this converse is the one related to the interval of values that can take the phase shifts. Until now, we have considered that phase shifts can take values in $[-\pi; \pi]$. In fact, Theorem 2 can be extended to the case where the phase shifts belong to the interval $[-\pi/2; \pi/2]$. Therefore, restraining the interval of phase values does not break imperatively the separation optimality. To prove this, it suffices to show that the following inequality

$$\inf_{\underline{\phi}_{M_S}} Q_{M_S}(\underline{\phi}_{M_S}) \leq 0 \quad (14)$$

still holds. To this end, by choosing

$$\Delta\phi_{1,2,i} = \begin{cases} \pi/2 - \theta_{1,2,i} & \text{if } \theta_{1,2,i} \in [0; \pi] \\ -\pi/2 - \theta_{1,2,i} & \text{if } \theta_{1,2,i} \in [-\pi; 0[\end{cases} \quad (15)$$

it becomes obvious to see that the infimum over the phase sequences in (37) is less or equal to zero. Then, by making similar modifications to the phases in (39), it can be easily shown that (14) still holds. Notice that, although the separation remains optimal when the phases belong to $[-\pi/2; \pi/2]$, there exist other possibilitites of phase intervals for which this optimality still holds (as an example, when the phases take just two different values α and $\alpha + \pi$).

Another important point concerns the fact that in many wireless sensor networks, the sensors may not be at the same distance from the collector node. In that case, we should consider an attenuation factor $\sqrt{\alpha_m}$ associated to each encoder that reflects the quality of the channel between each sender and the receiver. If we assume that these attenuation factors are known at the receiver point, Theorems 1 and 2 can be easily generalised to include this type of model.

The last point we would like to discuss is the utility of information exchange between the sensor nodes under a sum-energy constraint. The question here is to see if we can gain more if the sensors have the possibility of communicating between each others. The sum-energy constraint is described by the following inequality

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{m=1}^{M}\mathbb{E}[||X_{m,i}||^2] \leq \sum_{m=1}^{M} E_m \quad (16)$$

In fact, under (16), any kind of communication or information exchange between the sensors is useless and a separate-based coding scheme is optimal. To see it clearer, assume that the sensors can commmunicate in a free manner between themselves; therefore, each sensor knows perfectly the realisations of all the sources. The converse for this resultant model contains obviously all the achievable performances resulting from any kind of collaboration between the nodes. In that case, one can simply verify that the necessary condition

$$H(U_1, \cdots, U_M) \leq \frac{N}{K}\log\left(1 + \sum_{m=1}^{M}\frac{E_m}{N_0}\right) \quad (17)$$

must hold, and this for any kind of phase sequences considered in this paper. Knowing that the above inequality can be achieved by a separate-based coding scheme without involving any type of collaboration between the sensor nodes, shows that there is no gain in exchanging information using inter-sensor connections.

V. Conclusion and Ongoing Work

In this paper, we extended the separation theorem to the case of separately encoded correlated discrete sources sent over a GMAC with phase shifts perfectly known at the receiver and unknown to the transmitters. Hence, for different assumptions on the phase shifts, we proved that a set of two-stage encoders performing distributed source coding in the Slepian-Wolf sense and capacity-achieving channel coding leads to an optimal system performance. The presented model constitutes one of the rare scenarios in network information theory where the separation theorem holds. While previous works in the litterature concerned by sending correlated sources over MAC channels were more focused on cooperative coding strategies and on trying unsuccessfully to find necessary and sufficient conditions for optimality, we showed, by introducing a small and practical variation to the model (which is that of phase shifts unknown at the transmitters and Gaussian channel noise), that the optimal performance can be simply reached with a separate source-channel coding scheme. As for our ongoing work, the utility of information exchange between the separate encoders in the presence of different attenuation factors is under investigation while some new results extending the source-channel separation to the case of Gaussian sources has been obtained.

APPENDIX

A. Proof of the Converse of Theorem 1

Given a code with a fixed source-channel rate r, Fano's inequality yields $\frac{1}{K}H(\mathbf{U}_1,\cdots,\mathbf{U}_M|\mathbf{Y},\mathbf{\Phi}) \le \lambda_K$, where, for a family of codes of increasing block length and achieving vanishing error probability, we have $\lambda_K \to 0$ as $K \to \infty$. To simplify notations, let $\mathbf{V} \triangleq (\mathbf{U}_{S^c}, \mathbf{X}_{S^c}, \mathbf{\Phi})$, $\mathbf{\Phi}_{*i} \triangleq (\Phi_{1,i},\ldots,\Phi_{M,i})$ and

$$A_i(\boldsymbol{\phi}_{*i}) \triangleq (Y_i - \sum_{m\in S^c} e^{j\Phi_{m,i}} X_{m,i}|\boldsymbol{\phi}_{*i}) \qquad (18)$$

for any subset $S \subseteq \{1,2,\cdots,M\}$. Now, we can write $\forall S \subseteq \{1,2,\cdots,M\}$,

$$\begin{aligned}
H(U_S|U_{S^c}) &= \frac{1}{K}H(\mathbf{U}_S|\mathbf{U}_{S^c}) \\
&= \frac{1}{K}H(\mathbf{U}_S|\mathbf{U}_{S^c},\mathbf{\Phi}) \\
&= \frac{1}{K}H(\mathbf{U}_S|\mathbf{V}) \\
&= \frac{1}{K}I(\mathbf{U}_S;\mathbf{Y}|\mathbf{V}) + \frac{1}{K}H(\mathbf{U}_S|\mathbf{V},\mathbf{Y}) \\
&\overset{(a)}{\le} \frac{1}{K}I(\mathbf{U}_S;\mathbf{Y}|\mathbf{V}) + \lambda_K \\
&\le \frac{1}{K}\sum_{i=1}^{N} H(Y_i|\mathbf{V}) - \frac{1}{K}H(\mathbf{Y}|\mathbf{U}_S,\mathbf{V}) + \lambda_K \\
&= \frac{1}{K}\sum_{i=1}^{N} H(Y_i - \sum_{m\in S^c} e^{j\Phi_{m,i}}X_{m,i}|\mathbf{V}) - \\
&\quad \frac{1}{K}H(\mathbf{Z}) + \lambda_K \\
&\overset{(b)}{\le} \frac{1}{K}\sum_{i=1}^{N} \mathbb{E}_{\mathbf{\Phi}_{*i}}[\log \mathrm{Var}(A_i(\mathbf{\Phi}_{*i}))] - \\
&\quad \frac{N}{K}\log N_0 + \lambda_K \\
&\le \frac{N}{K}\log\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{\mathbf{\Phi}_{*i}}[\mathrm{Var}(A_i(\mathbf{\Phi}_{*i}))]\right) - \\
&\quad \frac{N}{K}\log N_0 + \lambda_K \qquad (19)
\end{aligned}$$

where (a) follows from

$$\begin{aligned}
H(\mathbf{U}_S|\mathbf{U}_{S^c},\mathbf{X}_{S^c},\mathbf{Y},\mathbf{\Phi}) &\le H(\mathbf{U}_S,\mathbf{U}_{S^c}|\mathbf{X}_{S^c},\mathbf{Y},\mathbf{\Phi}) \qquad (20)\\
&\le H(\mathbf{U}_S,\mathbf{U}_{S^c}|\mathbf{Y},\mathbf{\Phi}) \\
&\le \lambda_K,
\end{aligned}$$

and $\mathbb{E}_{\mathbf{\Phi}_{*i}}[.]$ in (b) denotes the expectation with respect to the probability distribution $p(\phi_{1,i},\cdots,\phi_{M,i})$. Without loss of generality, we can restrict the code to have mean zero on all components. Therefore,

$$\begin{aligned}
\mathrm{Var}\left(A_i(\boldsymbol{\phi}_{*i})\right) &= \mathrm{Var}\left(\sum_{m\in S} e^{j\phi_{m,i}}X_{m,i} + Z_i\right) \\
&= N_0 + \sum_{m,m'\in S}\mathbb{E}\left[X_{m,i}X^*_{m',i}e^{j(\phi_{m,i}-\phi_{m',i})}\right] \\
&= N_0 + \sum_{m\in S}\mathbb{E}\left[X_{m,i}X^*_{m,i}\right] + \\
&\quad 2\sum_{\substack{m,m'\in S\\ m'>m}}\mathrm{Re}\left\{\mathbb{E}\left[X_{m,i}X^*_{m',i}\right]e^{j\Delta\phi_{m,m',i}}\right\}.
\end{aligned} \qquad (21)$$

$\mathbb{E}\left[X_{m,i}X^*_{m',i}\right]$ is a complex number depending on m,m' and i; we shall call this number $\rho_{m,m',i} = |\rho_{m,m',i}|e^{j\theta_{m,m',i}}$. Letting the average energy of the i-th symbol be denoted by $E_{m,i}$, we can rewrite (21) as

$$\mathrm{Var}\left(A_i(\boldsymbol{\phi}_{*i})\right) = N_0 + \sum_{m\in S} E_{m,i} + T_i \qquad (22)$$

where

$$T_i = 2\sum_{\substack{m,m'\in S\\ m'>m}} |\rho_{m,m',i}|\cos(\Delta\phi_{m,m',i} + \theta_{m,m',i}). \qquad (23)$$

Notice that $\mathbb{E}_{\mathbf{\Phi}_{*i}}[T_i] = 0$, which is due to the fact that $\Delta\Phi_{m,m',i}$ is uniformly distributed on $[-\pi;\pi]$. Therefore, we can proceed with (19) and write

$$\begin{aligned}
H(U_S|U_{S^c}) &\le \frac{N}{K}\log\left[N_0 + \sum_{m\in S}\frac{1}{N}\sum_{i=1}^{N}E_{m,i} + \mathbb{E}_{\mathbf{\Phi}_{*i}}[T_i]\right] - \\
&\quad \frac{N}{K}\log N_0 + \lambda_K \qquad (24)\\
&\le \frac{N}{K}\log\left(1 + \sum_{m\in S}\frac{E_m}{N_0}\right) + \lambda_K \qquad (25)
\end{aligned}$$

Letting $K \to \infty$, we find the necessary conditions for reliable transmission: $\forall S \subseteq \{1,2,\cdots,M\}$,

$$H(U_S|U_{S^c}) \le \frac{N}{K}\log\left(1 + \sum_{m\in S}\frac{E_m}{N_0}\right). \qquad (26)$$

B. Proof of the Converse of Theorem 2: Arbitrary Phase Sequences

Given a code with a fixed source-channel rate r and a fixed $\boldsymbol{\phi}$, Fano's inequality yields $\frac{1}{K}H_{\boldsymbol{\phi}}(\mathbf{U}_1,\cdots,\mathbf{U}_M|\mathbf{Y}) \le \lambda_K(\boldsymbol{\phi})$. We require that a family of codes of increasing block length achieves vanishing error probability for all possible $\boldsymbol{\phi}$ since they are unknown at the transmitters, i.e., that $\lambda_K(\boldsymbol{\phi}) \to$

0 as $K \to \infty$. Now, we can write,

$$\begin{aligned} H(U_S|U_{S^c}) &= \frac{1}{K} H(\mathbf{U}_S|\mathbf{U}_{S^c}) \\ &= \frac{1}{K} H(\mathbf{U}_S|\mathbf{U}_{S^c}, \mathbf{X}_{S^c}) \\ &= \frac{1}{K} I_{\boldsymbol{\phi}}(\mathbf{U}_S; \mathbf{Y}|\mathbf{U}_{S^c}, \mathbf{X}_{S^c}) + \\ & \quad \frac{1}{K} H_{\boldsymbol{\phi}}(\mathbf{U}_S|\mathbf{U}_{S^c}, \mathbf{X}_{S^c}, \mathbf{Y}) \\ &\leq \frac{1}{K} I_{\boldsymbol{\phi}}(\mathbf{U}_S; \mathbf{Y}|\mathbf{U}_{S^c}, \mathbf{X}_{S^c}) + \lambda_K(\boldsymbol{\phi}) \\ &\leq \frac{1}{K} \sum_{i=1}^{N} H_{\boldsymbol{\phi}}(Y_i|\mathbf{U}_{S^c}, \mathbf{X}_{S^c}) - \\ & \quad \frac{1}{K} H_{\boldsymbol{\phi}}(\mathbf{Y}|\mathbf{X}_1, \cdots, \mathbf{X}_M) + \lambda_K(\boldsymbol{\phi}) \\ &\leq \frac{1}{K} \sum_{i=1}^{N} \log \mathrm{Var}\left(A_i(\boldsymbol{\phi}_{*i})\right) - \\ & \quad \frac{N}{K} \log N_0 + \lambda_K(\boldsymbol{\phi}) \end{aligned} \tag{27}$$

where $H_{\boldsymbol{\phi}}(.)$ and $I_{\boldsymbol{\phi}}(.)$ denote respectively the entropy and the mutual information corresponding to a given arbitrary $\boldsymbol{\phi}$. Since these inequalities must hold for every $\boldsymbol{\phi}$, we obtain the tightest conditions by taking the infimum of the RSH term in (27) with respect to $\boldsymbol{\phi}$. Therefore, letting K goes to ∞, we can write

$$\begin{aligned} H(U_S|U_{S^c}) &\leq \inf_{\boldsymbol{\phi}} \left\{ \frac{1}{K} \sum_{i=1}^{N} \log \left[1 + \sum_{m \in S} \frac{E_{m,i}}{N_0} + \frac{T_i}{N_0} \right] \right\} \\ &\leq \frac{N}{K} \inf_{\boldsymbol{\phi}} \left\{ \log \left[1 + \sum_{m \in S} \frac{E_m}{N_0} + \frac{1}{N N_0} \sum_{i=1}^{N} T_i \right] \right\} \\ &= \frac{N}{K} \log \left[1 + \sum_{m \in S} \frac{E_m}{N_0} + \frac{1}{N N_0} \inf_{\boldsymbol{\phi}} \left\{ \sum_{i=1}^{N} T_i \right\} \right] \end{aligned} \tag{28}$$

where we have used again Jensen's inequality and the monotonicity of the logarithm in order to take the infimum inside the log. Now, we will prove that the infimum term in (28) cannot be positive, i.e.,

$$\inf_{\boldsymbol{\phi}} \left\{ \sum_{i=1}^{N} T_i \right\} \leq 0. \tag{29}$$

Notice that if the chosen code satisfies $\rho_{m,m',i} = 0 \ \forall m, m', i$, then the equality is achieved in (29) for all phase sequences; this point will be discussed in more details in section IV.

Returning back to the proof of (29), let's take $S = \{1, \ldots, l\}$ with $2 \leq l \leq M$; note that specifying the subset S is just to simplify notations and the following proof holds $\forall S \subseteq \{1, \ldots, M\}$. Define the matrix

$$\underline{\boldsymbol{\phi}}_l \triangleq [\phi_{m,i}] \quad m = 1, \cdots, l \quad i = 1, \cdots, N \tag{30}$$

and

$$Q_l(\underline{\boldsymbol{\phi}}_l) \triangleq \sum_{i=1}^{N} T_i \tag{31}$$

$$= \sum_{i=1}^{N} \sum_{m=1}^{l-1} \sum_{m'>m}^{l} \mathrm{Re}\left[|\rho_{m,m',i}| e^{j(\Delta\phi_{m,m',i} + \theta_{m,m',i})} \right]. \tag{32}$$

Consequently, proving (29) reduces to prove that $\inf_{\underline{\boldsymbol{\phi}}_L} Q_l(\underline{\boldsymbol{\phi}}_l) \leq 0$. To this end, we can first derive a relation between $Q_{l-1}(\underline{\boldsymbol{\phi}}_{l-1})$ and $Q_l(\underline{\boldsymbol{\phi}}_l)$ like the following

$$\begin{aligned} Q_l(\underline{\boldsymbol{\phi}}_l) - Q_{l-1}(\underline{\boldsymbol{\phi}}_{l-1}) & \qquad\qquad (33) \\ &= \sum_{i=1}^{N} \sum_{m=1}^{l-1} \mathrm{Re}\left[|\rho_{m,l,i}| e^{j(\phi_{m,i} - \phi_{l,i} + \theta_{m,l,i})} \right] \\ &= \sum_{i=1}^{N} \mathrm{Re}\left[e^{-j\phi_{l,i}} \sum_{m=1}^{l-1} |\rho_{m,l,i}| e^{j(\phi_{m,i} + \theta_{m,l,i})} \right] \\ &= \sum_{i=1}^{N} \mathrm{Re}\left[e^{-j\phi_{l,i}} |\rho_{l,i}| e^{j\theta_{l,i}} \right] \end{aligned} \tag{34}$$

where

$$\rho_{l,i} = |\rho_{l,i}| e^{j\theta_{l,i}} \triangleq \sum_{m=1}^{l-1} |\rho_{m,l,i}| e^{j(\phi_{m,i} + \theta_{m,l,i})}. \tag{35}$$

Note that for a given code and a fixed $\underline{\boldsymbol{\phi}}_{l-1}$, the complex number $\rho_{l,i}$ is fixed and is independant from $\boldsymbol{\phi}_l$. Now, it becomes easy to prove that $\inf_{\underline{\boldsymbol{\phi}}_l} Q_l(\underline{\boldsymbol{\phi}}_l) \leq 0$. In fact, for $l = 2$ we have

$$Q_2(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = \sum_{i=1}^{N} \mathrm{Re}\left[|\rho_{1,2,i}| e^{j(\Delta\phi_{1,2,i} + \theta_{1,2,i})} \right]. \tag{36}$$

By taking $\Delta\phi_{1,2,i} = \pi - \theta_{1,2,i}$, we obtain that

$$\inf_{\underline{\boldsymbol{\phi}}_2} Q_2(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = -\sum_{i=1}^{N} |\rho_{1,2,i}| \leq 0. \tag{37}$$

Suppose now that

$$\inf_{\underline{\boldsymbol{\phi}}_{l-1}} Q_{l-1}(\underline{\boldsymbol{\phi}}_{l-1}) \leq 0$$

and that this infimum is attained for a certain value $\underline{\boldsymbol{\phi}}_{l-1} = \underline{\boldsymbol{\phi}}^*_{l-1}$; using the recurrence relation in (34), we can write

$$\begin{aligned} \inf_{\underline{\boldsymbol{\phi}}_l} Q_l(\underline{\boldsymbol{\phi}}_l) &\leq Q_l(\underline{\boldsymbol{\phi}}^*_{l-1}, \boldsymbol{\phi}^*_l) \\ &= Q_{l-1}(\underline{\boldsymbol{\phi}}^*_{l-1}) - \sum_{i=1}^{N} |\rho_{l,i}| \\ &\leq 0. \end{aligned} \tag{38}$$

where the entries of $\boldsymbol{\phi}^*_l = (\phi^*_{l,1}, \ldots, \phi^*_{l,N})$ are chosing like the following

$$\phi^*_{l,i} = \theta_{l,i} - \pi \quad \text{for} \quad i = 1, \ldots, N. \tag{39}$$

Using this result in (28) completes the proof of Theorem 2 for arbitrary phase sequences.

REFERENCES

[1] C.E. Shannon, ”A mathematical theory of communication”, *Bell System Technical Journal*, 27:379-423, 623-656, 1948.

[2] T.M. Cover, A. El Gamal and M. Salehi ”Multiple access channels with arbitrarily correlated sources”, *IEEE Transactions on Information Theory*, vol.26, issue 6, Nov 1980, pp. 648-657.

[3] G. Dueck, ”A note on the multiple access channel with correlated sources”, *IEEE Transactions on Information Theory*, vol.27, issue 2, Mar 1981, pp. 232-235.

[4] T.M. Cover and J.A. Thomas, *Elements of Information Theory*, 1991.

[5] A.D. Murugan, P.K. Gopala and H. El Gamal ”Correlated sources over wireless channels: cooperative source-channel coding”, *IEEE Journal on Selected Areas in Communications*, vol.22, Issue 6, Nov 2004, pp. 988-998.

[6] H. El Gamal, ”On the scaling laws of dense wireless sensor networks: the data gathering channel”, *IEEE Transactions on Information Theory*, vol.51, issue 3, Mar 2005, pp. 1229-1234.

[7] D. Slepian and J.K. Wolf, ”Noiseless coding of correlated information sources”, *IEEE Transactions on Information Theory*, vol.19, issue 4, Jul 1973, pp. 471-480.

[8] R. Ahlswede, ”Multi-way communication channels”, in *Proc. 2nd In. Symp. on Information Theory*, Tsahkadsor, Armenian S.S.R., 1971, pp. 23-52, Publishing House of the Hungarian Academy of Sciences, 1973.

[9] J. Barros and S.D. Servetto, ”Reachback capacity with non-interfering nodes”, *IEEE Symposium on Information Theory (ISIT)*, Yokohama, Japan, 2003.

[10] J. Barros and S.D. Servetto, ”Network information flow with correlated sources”, *IEEE Transactions on Information Theory*, vol.52, issue 1, Jan 2006, pp. 155-170.