BRIEF ANNOUNCEMENT: A DYNAMIC EXCHANGE GAME

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ABSTRACT. Our work aims to study a model based on an extended variant of the stable fixtures problem where multiple matches can be established between pairs of players, moreover preference orders are subject to alteration due to player strategies.

1. INTRODUCTION

We present a distributed system model, incorporating an extended version of the stable fixtures problem into a general game theoretic framework and present a dynamic game which arises upon it. Our work is motivated by our current research on peer-to-peer applications, but it can be easily applied to any distributed system in which selfish players are required to build bilateral bonds among themselves (e.g. neighborhoods) guided by a global preference order (see Section 4).

2. BUILDING UP THE PROBLEM DEFINITION

We consider a distributed game where $\mathcal{I}$ denotes the player set ($|\mathcal{I}| = n$ is the number of players), $\mathcal{S}$ depicts the collection of strategy sets ($\mathcal{S} = (\mathcal{S}_i)$ for $\forall i \in \mathcal{I}$), $\mathcal{X}$ function gives the player consequences ($\mathcal{X} = (\mathcal{X}_i)$ for $\forall i \in \mathcal{I}$) on the combination of strategy sets ($\mathcal{X} : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{R}^n$). In the following if we say that player i prefers one of her strategies to an other, it is because her *strict* preference order over the consequences $\mathcal{X}_i$ for the *given* best response strategy set yields so. In the following we define our problem starting from its roots, i.e traditional matching problems.

2.1. Stable roommate problem. The formal definition of the stable roommate (SR) problem is to find a matching $\mathcal{M}$ on the setting presented above, $\mathcal{M}$ being a set of $\frac{|\mathcal{I}|}{2}$ disjoint pairs of players, which is stable if there are no two players, each of whom prefers the other to his partner in $\mathcal{M}$. Such a pair is said to block $\mathcal{M}$. Player i's strategy is $s_i \in \mathcal{S}_i$, $\mathcal{S}_i$ being the set $\{\{i,j\} : j \in \mathcal{I} \setminus \{i\}\}$, and $\mathcal{X}_i$ is assumed to give strict order on i's possible pairs, termed *preference list* in the literature. Following the statement of the SR problem by Gale and Shapley in [2], Irving's [4] presents a polynomial-time algorithm to determine whether a stable matching exists for a given SR instance, and if so to find one such matching.

2.2. Stable fixtures problem. Irving and Scott present in [5] the stable fixtures (SF) problem, which is a generalization of the SR problem. Formally, the notion of *capacity* is introduced such that for each $i \in \mathcal{I}$ a positive integer c_i, which is player i's capacity, denotes the maximum number of matches, i.e. pairs (i,j) in which player i can appear. i's strategy is $s_i \subseteq \mathcal{S}_i = \{\{i,j\} : j \in \{\mathcal{I} \setminus i\}\}$ and $\mathcal{X}_i$ gives again the strictly ordered *preference list* on i's matches. It is straightforward to see that the SR problem is a special case of the SF problem when $c_i = 1$ $\forall i \in \mathcal{I}$, i.e. each player may have 1 match at most. A matching $\mathcal{M}$ here is a set of acceptable pairs $\{i,j\}$ such that for $\forall i \in \mathcal{I}$ $|\{j : \{i,j\} \in \mathcal{M}\}| \leq c_i$, where a pair $\{i,j\}$ is acceptable if i appears in s_j and j appears in s_i. $\mathcal{M}$ is stable if there is no blocking pair, i.e. an acceptable pair $\{i,j\} \notin \mathcal{M}$ such that

- either i has fewer matches than c_i or prefers j to at least one of his matches in $\mathcal{M}$; *and*
- either j has fewer matches than c_j or prefers i to at least one of his matches in $\mathcal{M}$.

[5] describes a linear-time algorithm that determines whether a stable matching exists, and if so, returns one such matching.

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2.3. **Stable exchange problem.** We further extend the SF problem with the possibility of multiple matches between two given players. Therefore player i's strategy is $s_i \subseteq \mathcal{S}_i = \{\{i,j,c_{ij}\} : j \in \{\mathcal{I} \setminus i\}, 0 \leq c_{ij} \leq \min(c_i, c_j)\}$, where c_{ij} (resp. c_i) denotes player i's number of matches towards player j (resp. towards all the players). A matching $\mathcal{M}$ is a set of matches $\{i,j,c_{ij}\}$ such that $\{i,j,c_{ij}\} \in s_i$, $\{j,i,c_{ij}\} \in s_j$ and $\sum_{j:\{i,j,c_{ij}\}\in\mathcal{M}} c_{ij} \leq c_i$ holds for $\forall i \in \mathcal{I}$; moreover $\mathcal{M}$ is stable if, likewise in the SF problem's case, there is no blocking match, i.e. no match $\{i,j,c'\} \notin \mathcal{M}$, thus $c' > c_{ij}$ for $\forall i,j : (i,j,c_{ij}) \in \mathcal{M}$, such that

- either i has fewer matches than c_i or $\mathcal{X}_i(i,j,c')$ is greater than $\mathcal{X}_i(\cdots)$ of at least one of his matches in $\mathcal{M}$; *and*
- either j has fewer matches than c_j or $\mathcal{X}_j(j,i,c')$ is greater than $\mathcal{X}_j(\cdots)$ of at least one of his matches in $\mathcal{M}$;

where we denote players i and j's c'th *pairwise* match's consequence for i by $\mathcal{X}_i(i,j,c')$. In other words, in a stable matching no two players could have a new match between themselves which is preferred by both of them to any of their existing matches. To avoid inconsistency in the consequence order of consecutive matches between given players, we make the following assumption:

Assumption 1. *$\mathcal{X}_i(i,j,c') > \mathcal{X}_i(i,j,c'')$ holds for any pair of matches between players i and j if $c' < c''$ for $\forall i,j$.*

2.4. **Uniform stable exchange problem.** We investigate a special case of the stable exchange problem, specifically the case with uniformity on $\mathcal{I}$.

Assumption 2. *$c_i = c$ for $\forall i \in \mathcal{I}$.*

Let us suppose that the consequence function $\mathcal{X}$ and thus the preference order on $\mathcal{S}$ are defined based on a player parameter set denoted by $\alpha = (\alpha_i)$, such that α_i for $\forall i \in \mathcal{I}$ is a positive scalar of $[0,1]$. The implications of the α parameter vector on $\mathcal{X}$ are compacted in the following assumption for the uniform case, i.e. when Assumption 2 holds.

Assumption 3. *For $\forall i \in \mathcal{I}$, $\mathcal{X}_i(i,j,c') > \mathcal{X}_i(i,k,c')$ holds for a given $c' \leq c$ for any given pair $j,k \in \{\mathcal{I} \setminus i\}$ if, and only if $\alpha_j > \alpha_k$. For the case $\alpha_j = \alpha_k$, $\mathcal{X}_i(i,j,c') = \mathcal{X}_i(i,k,c')$ for any $c' \leq c$.*

Proposition 1. *At least one stable matching exists for a given exchange problem instance corresponding to Assumptions 2 and 3, and a slightly extended version of Irving's algorithm (presented in* [5]*) finds it in polynomial time.*

3. Dynamic exchange game

We now introduce a more elaborate setting in which the parameter vector α, introduced previously, is considered as a *strategy* variable vector the players can decide on, hence influence the $\mathcal{X}$ function. In this setting, the traditional matching problem becomes a game.

3.1. **Problem definition.** The setting relates to the uniform exchange problem with Assumptions 2 and 3.[1] With the above introduced notations, a *joint* strategy for player i is a scalar value α_i of $[0,1]$ *and* an instance $s_i \subseteq \mathcal{S}_i = \{(i,j,c_{ij}) : j \in \{\mathcal{I} \setminus i\}, 0 \leq c_{ij} \leq \min(c_i, c_j)\}$.

Since α is now part of the user strategy set, a dynamic game arises, which we describe in non-cooperative game theoretic terms. Every player $i \in \mathcal{I}$ selfishly maximizes her payoff given by $\mathcal{P}_i$, in this case constructed by the $\alpha_i \in \alpha$ strategy implied cost *and* the consequence of strategy s_i, i.e. $\mathcal{P} : \alpha \times \mathcal{S} \to \mathcal{R}^n$.

Conjecture 1. *Showing the existence of the pairwise Nash equilibrium and finding it in a given game instance is NP complete. For a Nash equilibrium, which must be a stable matching, the $\mathcal{P}_i(\{\alpha_i^*, \alpha_{-i}^*\}, \{s_i^*, s_{-i}^*\}) \geq \mathcal{P}_i(\{\alpha_i, \alpha_{-i}^*\}, \{s_i, s_{-i}^*\})$ holds for any α_i, s_i and for $\forall i \in \mathcal{I}$, where α_{-i}^* and s_{-i}^* depict the best response counter strategy sets.*

[1] In an extended version of the model we plan to relax the uniform c assumption, moreover we plan to consider it as a strategy variable (along with α) by defining a joint payoff function on the whole strategy set.

3.2. **Optimal joint strategy.** The optimal player strategy tuple is $(\alpha_i^*, s_i^*) = \arg_i(\max(\mathcal{P}_i(\alpha, \mathcal{S})))$ for $\forall i \in \mathcal{I}$ with the constraint that stable matching is symmetric in $s^* = (s_i^*)$ for $\forall i \in \mathcal{I}$, since every match is pairwise. The social welfare is given by $\max(\sum_{i \in \mathcal{I}} \mathcal{P}_i(\alpha, \mathcal{S}))$ also with the stable matching constraint.

Conjecture 2. *The joint optimization problems defined above are NP complete.*

3.3. **Heuristics.** We sketch a distributed algorithm which would approximate the optimal strategies in polynomial time. Our algorithm builds upon the evolutionary game theoretic framework [3], where player decisions regarding their 2 strategic variables are *interleaved*.

$k = 0$, initial strategy set α^k, initial fitness set $\mathcal{P}^k$
repeat
  compute stable matching $\mathcal{M}^k$ by Irving's algorithm's extended version based on $s_i^k = \arg\max \mathcal{P}_i(\mathcal{S})|_{\alpha^k}$ for $\forall i \in \mathcal{I}$, where $\mathcal{M}^k = \bigcup_{i \in \mathcal{I}} \mathcal{M}_i^k$, i.e. player i's matches in $\mathcal{M}^k$
  compute $\mathcal{P}_i^k$ given α^k and $\mathcal{M}^k$ for $\forall i \in \mathcal{I}$
  compute $\bar{\mathcal{P}}_{-i}^k = \sum_{j \in \mathcal{M}_i^k} \mathcal{P}_j^k$ for $\forall i \in \mathcal{I}$
  for all $i \in \mathcal{I}$ **do**
    if $\mathcal{P}_i^k < \bar{\mathcal{P}}_{-i}^k$ **then**
      $\alpha_i^{k+1} := \frac{1}{|\mathcal{M}_i^k|} \sum_{j \in \mathcal{M}_i^k} \alpha_j^k$
    else
      $\alpha_i^{k+1} := \alpha_i^k$
    end if
  end for
  $k := k + 1$
until $\alpha^k = \alpha^{k-1}$

Conjecture 3. *The algorithm above converges to a stable state irrespectively to the initial state, i.e* α^0.

4. Applications

The motivation behind this work comes from peer-to-peer backup and storage applications. In this context our players are peers that are characterized by their storage capacities (c) that they share with other peers willing to reciprocate. A globally known peer profile (corresponding to the α vector) indicates e.g. peer reliability: it is assumed to be observed, maintained and advertised by the peer set for all participants. Here, matches represent the peer selection.

An other possible application is peer-to-peer content distribution, where α records the players' contribution levels (e.g. ratio of uploaded and downloaded bytes) and their capacities (c) reflect their uplink bandwidth. The matching relates to the overlay neighbor set and the uplink bandwidth allocation. A similar approach, though without a game theoretic flavor, has been studied in [1], where the authors relate the peer selection algorithm of BitTorrent to a *b-matching* problem.

References

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