Source-Channel Coding for Very-Low Bandwidth Sources

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***Abstract*— We address the source-channel coding problem of a sensor observing a slowly time-varying Gaussian source and communicating its information to a receiver through a Gaussian channel. Due to the slowly time-varying characteristic of the source, we consider that the sensor is capable of using many channel dimensions per source symbol. Under an energy constraint per source realisation, we derive a theoretical lower bound on the MSE distortion as well as an analytical upper bound based on a practical coding scheme involving a linear uniform quantizer followed by an orthogonal modulation and a MAP receiver. Other coding schemes coupled with an MMSE estimator are also proposed and their performances are compared. An extension to the case where the sensor has the capability of encoding a sequence of N source components is studied and a general upper bound in that case is obtained.**

I. Introduction

We consider one sensor tracking a slowly time-varying random sequence and sending its observations over a wireless channel to a receiver. The source is represented by a Gaussian random variable U, and the observations are assumed to be noiseless. The sensor is in general a tiny device with strict energy constraints. The communication channel between the sender and the receiver is an additive white Gaussian noise channel.

An important question is how to efficiently encode the random source, and what performance can be achieved. The slowly time-varying characteristic of the source has two main impacts on the way the coding problem should be addressed : firstly, the time between two observations is long, and the sensor will not wait for a sequence of observations to encode it. Therefore, the sensor will encode only one observation before sending it through the channel. Secondly, for each source realisation the channel can be used a large number of times, hence, there is no constraint on the dimensionality of the channel codebook. The latter condition amounts to saying that very low-rate codes should be used.

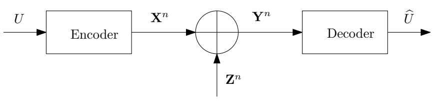

Fig. 1. System model

The model is depicted in Fig. 1. The encoder maps one realisation of the source $U \sim \mathcal{N}(0,1)$ into $\mathbf{X}^n$ where n is the dimension of the channel input. $\mathbf{X}^n$ is then sent across the channel corrupted by a white Gaussian noise sequence $\mathbf{Z}^n$, and is received as $\mathbf{Y}^n$. The receiver is a mapping function which tries to construct an estimate $\widehat{U}$ of U given $\mathbf{Y}^n$. The fidelity criterion that we wish to minimize is the MSE distortion defined as

$$D \triangleq \mathbb{E}[(U-\widehat{U})^2], \tag{1}$$

under the mean energy constraint $\mathbb{E}[||\mathbf{X}^n||^2] \leq E$. It is well-known that the linear encoder (i.e. $X = \sqrt{E}U$) achieves the best performance under the mean energy constraint for the special case $n = 1$ [1], [2], [3]. This case does not reflect the case of low-bandwidth sources where the best we can do is to bound the MSE distortion by deriving a lower and an upper bound and trying to minimize the gap between these two bounds.

Regardless of its achievability using practical coding schemes, a lower bound is easily derived over all possible encoders and decoders using classical information theory. For an upper bound, we propose several achievable schemes based on separated source-channel encoders combined with a MAP receiver or an MMSE estimator. Note that an MMSE estimator is the one which minimizes the MSE distortion, but, from a practical point of view, it is too complex to implement. The separate source-channel encoder is based on a quantizer followed by a modulator. Here, the Gaussian source is quantized in b bits which are mapped onto an appropriate modulation before being transmitted over the channel. The distortion is caused by the quantization process and the noisy channel. Increasing the number of quantization bits per source component has the effect of reducing the quantization error and simultanously increasing the error induced by the channel ; decreasing it will have the opposite effect. Thus, the number of quantization bits has to be optimized as a function of the energy. Such optimization can be found in the literature for example in [4] and [5], where the authors try to bound the optimal number of quantization bits that minimizes distortion ; the main difference with our model remains in the power constraint they are considering. The choice of the quantizer and the modulator has a great impact on the upper bound and is discussed in the third section.

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The paper is organised as follows. The theoretical lower bound on the distortion is derived in section II. In section III, we propose an achievable coding scheme and derive an analytical upper bound on the distortion. Other achievable coding schemes combined with an MMSE estimator are studied in section IV. An extension to the sequence coding case is analysed in section V and a more general upper bound is derived. Section VI contains simulations and numerical illustrations to the analytic results as well as comparisons and discussions.

II. Lower Bound

Let us take first the more general case where the encoder maps N source components $\mathbf{U}^N = (U_1, \dots, U_N)$ into $\mathbf{X}^{nN} = (\mathbf{X}_1^n, \dots, \mathbf{X}_N^n)$, and the decoder maps $\mathbf{Y}^{nN} = (\mathbf{Y}_1^n, \dots, \mathbf{Y}_N^n)$ into $\widehat{\mathbf{U}}^N$. Define the distortion as $D_N \triangleq \frac{1}{N}\mathbb{E}[||\mathbf{U}^N - \widehat{\mathbf{U}}^N||^2]$ and the mean energy constraint as

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[||\mathbf{X}_i^n||] \leq E. \qquad (2)$$

Clearly, it suffices to put $N = 1$ to return to our special model stated above in the introduction section and depicted in Fig. 1. Now, let us find a lower bound on the distortion D_N over all possible encoders and decoders satisfying (2). We have these standard inequalities [6]

$$\begin{aligned} I(\mathbf{U}^N;\widehat{\mathbf{U}}^N) &= h(\mathbf{U}^N) - h(\mathbf{U}^N/\widehat{\mathbf{U}}^N) \\ &= h(\mathbf{U}^N) - h(\mathbf{U}^N - \widehat{\mathbf{U}}^N/\widehat{\mathbf{U}}^N) \\ &\geq h(\mathbf{U}^N) - h(\mathbf{U}^N - \widehat{\mathbf{U}}^N) \\ &\geq \frac{N}{2}\log(2\pi e) - \sum_{i=1}^{N} h(U_i - \widehat{U}_i) \\ &\geq -\sum_{i=1}^{N}\frac{1}{2}\log(\mathbb{E}[(U_i - \widehat{U}_i)^2]) \\ &\geq \frac{N}{2}\log(\frac{1}{D_N}), \end{aligned}$$

and

$$\begin{aligned} I(\mathbf{U}^N;\widehat{\mathbf{U}}^N) &\leq I(\mathbf{X}^{nN}, \mathbf{Y}^{nN}) \\ &= h(\mathbf{Y}^{nN}) - h(\mathbf{Y}^{nN}/\mathbf{X}^{nN}) \\ &\leq \sum_{i=1}^{N}\sum_{j=1}^{n} h(Y_{ij}) - h(\mathbf{Z}^{nN}) \\ &\leq \sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{2}\log(\mathbb{E}[Y_{ij}^2]) - \frac{nN}{2}\log(\sigma_z^2) \\ &\leq \frac{nN}{2}\log(\frac{\sum_{i=1}^{N}\sum_{j=1}^{n}\mathbb{E}[Y_{ij}^2]}{nN\sigma_z^2}) \\ &= \frac{nN}{2}\log(\frac{E + n\sigma_z^2}{n\sigma_z^2}). \end{aligned}$$

From these inequalities, we obtain

$$D_N \geq \frac{1}{(1 + \frac{E}{n\sigma_z^2})^n} \qquad (3)$$

Note that this inequality holds for all N and especially when $N = 1$. Therefore,

$$D = D_1 \geq \frac{1}{(1 + \frac{E}{n\sigma_z^2})^n}. \qquad (4)$$

The RHS of that inequality is a decreasing function of n. Since that latter number is unconstrained in our model specifications, therefore

$$D \geq \lim_{n\to\infty}\frac{1}{(1 + \frac{E}{n\sigma_z^2})^n} = e^{-E/\sigma_z^2}. \qquad (5)$$

The RHS term in (5) constitutes a lower bound over D and coincides with $D(C)$ where $D(R)$ is the rate distortion function of the source U, $C = \lim_{n\to\infty} C_n$, and C_n is the capacity of the n-dimensional channel defined by

$$C_n \triangleq \max_{p(\mathbf{x}^n):\mathbb{E}[||\mathbf{X}^n||^2]\leq E} I(\mathbf{X}^n, \mathbf{Y}^n). \qquad (6)$$

III. Analytical Upper Bound

The performance of a linear encoding scheme that forward the realisation of the source into the channel is suboptimal; its resultant distortion [3]

$$D = \frac{\sigma_z^2}{\sigma_z^2 + E} \qquad (7)$$

decreases linearly with the energy E while in the case of very-low bandwidth sources, the lower bound is exponentially decreasing in E. In order to minimize this gap, we propose a separate source-channel coding scheme and derive an analytical upper bound on the minimal achievable distortion. As presented in Fig. 2, the encoder is formed by a uniform linear quantizer followed by an orthogonal modulator. The choice of an orthogonal modulation is motivated by the fact that the probability of correct detection approaches that of the regular simplex constellation when the size of the modulation becomes large [7, p. 381]. The optimality of the regular simplex is proved in [8] and [9] for different energy constraints and assumptions on apriori probabilities, although not those arising in our model (i.e. average energy constraint and non-uniform priors). However, it is also shown in [10] that, under an average energy constraint, the regular simplex still optimizes the union bound even if its 'strong' optimality does not hold.

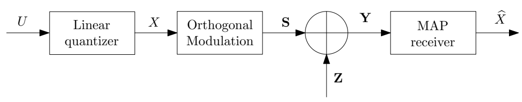

Fig. 2. The proposed coding scheme based on a linear quantizer followed by an orthogonal modulation and a MAP receiver

The quantizer is a function $f : \mathcal{U} \to \mathcal{X} \triangleq \{x_1, \dots x_M\}$ that assigns a value x_i to each $u \in I_i$ for $i = 1, \dots, M$, where x_i and I_i are respectively the quantization levels and intervals. The partition of $\mathcal{U}$ is as follows : $I_1 =]-\infty; \Delta[$, $I_M = [\Delta; \infty[$, and for $i = 2, \dots, M-1$,

$$I_i = [-\Delta + \frac{\Delta(i-2)}{2^{b-1}-1}; -\Delta + \frac{\Delta(i-1)}{2^{b-1}-1}[, \qquad (8)$$

where $\Delta = 2\sqrt{b\log 2}$, $M = 2^b$ and $b \geq 2$ is an integer representing the quantization bits per source component. The quantization levels are chosen as follows : $x_1 = -\Delta$, $x_M = \Delta$ and for $i = 2, \ldots, M-1$, x_i is the value in the middle of I_i. This quantizer is easy to build and its corresponding quantization distortion achieves the exponential rate of decay of $2^{-2b(1+O(b))}$, where $O(b)$ is a function that goes to zero when $b \rightarrow \infty$. It is illustrated in Fig. 3. The output X of the quantizer is assigned to a signal $\mathbf{S}$ chosen from an orthogonal modulation of size M that is sent through the Gaussian channel.

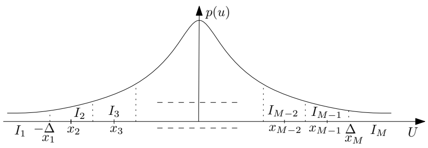

Fig. 3. The linear quantizer

Given the received signal, a MAP receiver makes the decision on the signal that has been sent and decodes $\mathbf{s}_j$ given that $\mathbf{s}_i$ has been transmitted according to a probability distribution $P_{ij} \triangleq p(\mathbf{s}_j/\mathbf{s}_i)$. Let $p_i = \int_{I_i} p(u)\,du$ for $i = 1, \ldots, M$, and $Q(t) \triangleq \int_t^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$; the expression of P_{ij} can be easily found : if $i \neq j$

$$P_{ij} = \mathbb{E}_A \left[(1 - Q(T_{ij} - \sqrt{E}/\sigma_z^2)) \prod_{\substack{k=1 \\ k \neq i,j}}^{M} (1 - Q(T_{kj})) \right]$$

and $$P_{ii} = \mathbb{E}_A \left[\prod_{\substack{k=1 \\ k \neq i}}^{M} (1 - Q(T_{ki} + \sqrt{E}/\sigma_z^2)) \right]$$

where $T_{ij} = A/\sigma_z - \frac{\sigma_z}{E} \log(p_i/p_j)$, and $A \sim \mathcal{N}(0, \sigma_z^2)$. Hence, the exact expression of the distortion is

$$D = \sum_{i,j=1}^{M} P_{ij} \int_{I_i} (u - x_j)^2 p(u)\,du. \tag{9}$$

In fact, we desire to bound the distortion in order to be able to optimize the number of quantization bits given a certain amount of energy. The following bound holds :

$$\begin{aligned} D &= D_Q(1 - P_e) + D_e P_e && (10) \\ &< D_Q + D_e P_e && (11) \end{aligned}$$

where D_e is the MSE distortion given that an error decision has been made, P_e is the probability of making an error and D_Q represents the quantization distortion. Using the inequality $Q(\Delta) < \frac{e^{-\Delta^2/2}}{\sqrt{2\pi}\Delta}$, we can write

$$\begin{aligned} D_Q &= 2\int_{\Delta}^{\infty} (u - \Delta)^2 p(u)\,du + \sum_{i=2}^{M-1} \int_{I_i} (u - x_i)^2 p(u)\,du \\ &< \frac{2e^{-\Delta^2/2}}{\sqrt{2\pi}\Delta} + \frac{\Delta^2}{(2^b - 2)^2}. && (12) \end{aligned}$$

Again, using the same upper bound on the function $Q(\Delta)$ and the fact that $D_e < 4\Delta^2$ when $|u| \leq \Delta$, we obtain

$$\begin{aligned} D_e &< 4\Delta^2 + 2\int_{\Delta}^{\infty} (u + \Delta)^2 p(u)\,du \\ &< 4\Delta^2 + \frac{2(4\Delta^2 + 1)}{\sqrt{2\pi}\Delta} e^{-\Delta^2/2}. && (13) \end{aligned}$$

The probability of error can be bounded by

$$P_e \leq M^{\rho} e^{[-\frac{E}{2\sigma_z^2}(\frac{\rho}{\rho+1})]} \tag{14}$$

For the derivation, see the Appendix. Combining all these bounds in (11),(12),(13) and (14), we obtain

$$D < 2^{-2b(1+O(b))} + 2^{\rho b(1+O'(b)) - \frac{E ln(2)}{2\sigma_z^2}(\frac{\rho}{\rho+1})} \tag{15}$$

where $O'(b)$ is a function that goes to zero when $b \rightarrow \infty$. This bound is a sum of two exponential terms : the first represents the quantization distortion and is independant from the energy while the second represents the distortion due to the channel error. When we increase the amount of energy E, the second term decreases and becomes less than the first one ; in order to minimize the upper bound, the number of quantization bits should be increased so that the two terms will have the same decreasing behavior. Thus, when E is sufficiently large, optimizing the bound in (15) over ρ and b gives $D < e^{-2b_{opt}\log(2)}$, with $b_{opt} = \lfloor \frac{E}{12\sigma_z^2 \log(2)} \rfloor$ and $\rho_{opt} = 1$. Thus, the upper bound approaches the value $e^{-E/(6\sigma_z^2)}$ for large E compared to the value e^{-E/σ_z^2} of the lower bound.

IV. An MMSE-Based Scheme

Since the MAP receiver is the decoder which minimizes the probability of error and not the MSE distortion, it is interesting to see the distortion gain that could be obtained by using an MMSE estimator. To this end, we propose a scheme where the range of the Gaussian source is partitioned into M intervals $I_1, \ldots, I_M$ defined as in the previous section ; each of the intervals is mapped onto a signal chosen from a biorthogonal modulation of size M : for $i = 1, \ldots, M$, the intervals I_i and I_{M+1-i} are assigned respectively to the two signals $\mathbf{s}_i$ and $\mathbf{s}_{M+1-i}$ which belong to the same axis in the bi-orthogonal constellation. For $i, j = 1, \ldots, M$, let

$$J_i = \int_{I_i} u p(u)\,du, \qquad K_{ij} = \int_{\infty}^{\infty} \frac{p(\mathbf{y}/\mathbf{s}_i) p(\mathbf{y}/\mathbf{s}_j)}{\sum_{k=1}^{M} p_k p(\mathbf{y}/\mathbf{s}_k)}\,d\mathbf{y}.$$

Due to the symmetry in the construction of the encoder, we have that $J_j = -J_{M+1-j}$ and $K_{i,j} = K_{i,M+1-j}$ for all $i = 1, \ldots, M$, $j = 1, \ldots, M/2$ and $j \neq i, M+1-i$.
The receiver is an MMSE estimator which minimizes the mean square error distortion. The estimate of u is

$$\widehat{u}(\mathbf{y}) = \mathbb{E}[U/\mathbf{y}] = \frac{\sum_{i=1}^{M} J_i p(\mathbf{y}/\mathbf{s}_i)}{\sum_{i=1}^{M} p_i p(\mathbf{y}/\mathbf{s}_i)} \tag{16}$$

The MSE distortion could be written as

$$D = E[(U - \widehat{U}(\mathbf{Y}))^2] = 1 - \mathbb{E}[\widehat{U}^2(\mathbf{Y})] \quad (17)$$

$$= 1 - \sum_{i=1}^{M} \sum_{j=1}^{M} J_i J_j K_{ij} \quad (18)$$

$$= 1 + \sum_{i=1}^{M} J_i^2 K_{i,M+1-i} - \sum_{i=1}^{M} J_i^2 K_{i,i}. \quad (19)$$

We have that

$$K_{i,M+1-i} = \int_{-\infty}^{\infty} \frac{\frac{1}{(2\pi\sigma_z^2)^{M/4}} e^{\frac{-1}{2\sigma_z^2} \sum_{k=1}^{M/2} y_k^2} e^{-E/2\sigma_z^2}}{\sum_{k=1}^{M/2} p_k (e^{-\frac{\sqrt{E}}{\sigma_z^2} y_k} + e^{\frac{\sqrt{E}}{\sigma_z^2} y_k})} d\mathbf{y}$$

$$< e^{-E/2\sigma_z^2} \int_{-\infty}^{\infty} \frac{\frac{1}{(2\pi\sigma_z^2)^{M/4}} e^{\frac{-1}{2\sigma_z^2} \sum_{k=1}^{M/2} y_k^2}}{\sum_{k=1}^{M/2} 2p_k} d\mathbf{y}$$

$$= e^{-E/2\sigma_z^2},$$

and $K_{ii} = \mathbb{E}_{\mathbf{Y}/\mathbf{s}_i} \left[\frac{p(\mathbf{y}/\mathbf{s}_i)}{\sum_{k=1}^{M} p_k p(\mathbf{y}/\mathbf{s}_k)} \right]$

where $\mathbf{Y}/\mathbf{s}_i$ is a multivariate Gaussian of mean $\mathbf{s}_i$ and covariance matrix $\sigma_z^2 \mathbf{I}_{M/2}$. Thus,

$$D < 1 + 2e^{-E/2\sigma_z^2} \sum_{i=1}^{M/2} J_i^2 - 2 \sum_{i=1}^{M/2} J_i^2 \mathbb{E}_{\mathbf{Y}/\mathbf{s}_i} \left[\frac{e^{\frac{\sqrt{E}}{\sigma_z^2} y_i}}{\sum_{k=1}^{M/2} p_k (e^{-\frac{\sqrt{E}}{\sigma_z^2} y_k} + e^{\frac{\sqrt{E}}{\sigma_z^2} y_k})} \right] \quad (20)$$

Now, suppose that we use an orthogonal modulation instead of the biorthogonal one; for $i = 1, \ldots, M$, every interval I_i is mapped into a signal $\mathbf{s}_i$. Doing similar calculations as for the biorthogonal case, we obtain

$$D = 1 - \sum_{i=1}^{M} J_i^2 \mathbb{E}_{\mathbf{Y}/\mathbf{s}_i} \left[\frac{e^{\frac{\sqrt{E}}{\sigma_z^2} y_i} - e^{\frac{\sqrt{E}}{\sigma_z^2} y_{M+1-i}}}{\sum_{k=1}^{M} p_k e^{\frac{\sqrt{E}}{\sigma_z^2} y_k}} \right] \quad (21)$$

where $\mathbf{Y} = (Y_1, \ldots, Y_M)/\mathbf{s}_i$ is a multivariate Gaussian of mean $\mathbf{s}_i$ and covariance matrix $\sigma_z^2 \mathbf{I}_M$. We currently do not have asymptotic expressions for (20) and (21) as $E \to \infty$.

V. Extention to Sequence Coding

Due to the slowly time-varying characteristic of the source, we have assumed that just one source component is available to be encoded and then transmitted. We now extend to the case where the sensor can wait until having a sequence of length N of i.i.d. source realisations at the encoder input. Under the mean energy constraint in (2), we are interested to see how much the upper bound can be improved by coding a sequence of N source components. Using the same linear quantizer as in section III, every source component is quantized in b representing bits. The quantizer is followed by an orthogonal modulation of size $M_N = 2^{Nb}$ that takes the Nb bits available at the input and maps them into an Nb-dimensional signal of fixed energy equal to NE. Performing MAP decoding, we have that

$$D_N < D_{QN} + D_{eN} P_{eN} \quad (22)$$

where D_{QN} represents the quantization error, D_{eN} the MSE distortion when a error decision has been made and P_{eN} the probability of making an error. Clearly, we have $D_{QN} = D_{Q1} = D_Q$, $D_{eN} \le D_{e1} = D_e$ and $P_{eN} \le M_N^{\rho} e^{[-\frac{NE}{2\sigma_z^2}(\frac{\rho}{\rho+1})]}$. Doing the same optimization procedure as in section III, we obtain $D_N < e^{-2b_{Nopt}\log(2)}$ for large amount of energy E with $\rho_{opt} = 1$ and

$$b_{Nopt} = \left\lfloor \frac{NE}{4(N+2)\sigma_z^2 \log(2)} \right\rfloor. \quad (23)$$

Letting $N \to \infty$, the upper bound approaches asymptotically the value $e^{-E/2\sigma_z^2}$ and the gap with the lower bound is reduced to $3dB$.

VI. Numerical Results and Conclusions

In all the numerical results, the distortion is plotted versus the energy, and the variance of the channel noise is taken equal to one. Fig. 4 shows the inefficiency of the linear encoder compared with the theoretical upper and lower bounds. Note that the curve representing the upper bound is obtained like the following : for each value of E, we find b_{opt}, then we calculate the upper bound over D using the terms in (12),(13) and (27). Also the model studied in section III is simulated for different number of quantization bits and compared to the other curves. Fig. 5 and Fig. 6 show that the different types of encoders and decoders studied in section III and IV have comparable performance; therefore, using a MAP decoder instead of an MMSE estimator has practically no effects on the MSE distortion especially when $b > 2$. The analytical upper bound on the distortion D_N (eq.(22)) is plotted in Fig. 7 for several values of N. This plot shows the improvement that can be made to the upper bound, and consequently to the performance of the system when we code sequences; it shows that even with small length sequences, significant gain can be obtained.

As a conclusion, we have derived theoretical lower and upper bounds on the distortion for very-low bandwidth sources. The proposed source-channel coding schemes outperform the linear coding performance and lead to an exponentially decreasing behavior of the distortion in E. Also, we have shown that the difference in the performance between a MAP decoder and an MMSE estimator is negligible. Finally, we proved that the gap between the lower and the upper bound can be significantly reduced by coding relatively short sequences.

Appendix

By analogy to what has been done in [11], we obtain for the general case of unequal apriori probabilities that

$$P_{e_i} \le p_i^{-\frac{\rho}{\rho+1}} \int_{\mathbf{Y}} p(\mathbf{y}/x_i)^{\frac{1}{1+\rho}} \left[\sum_{j \ne i} (p_j p(\mathbf{y}/x_j))^{\frac{1}{1+\rho}} \right]^{\rho} d\mathbf{y}, \quad (24)$$

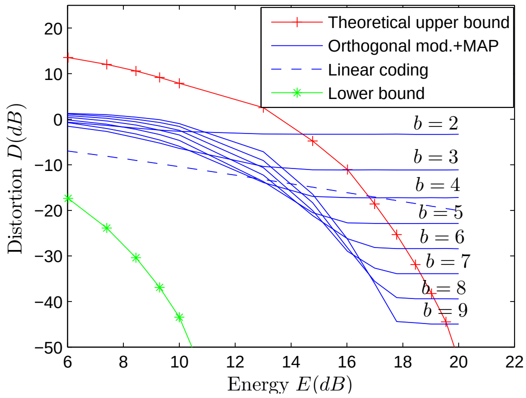

Fig. 4. Performances of the MAP-based scheme compared to the linear encoder and the theoretical lower bound.

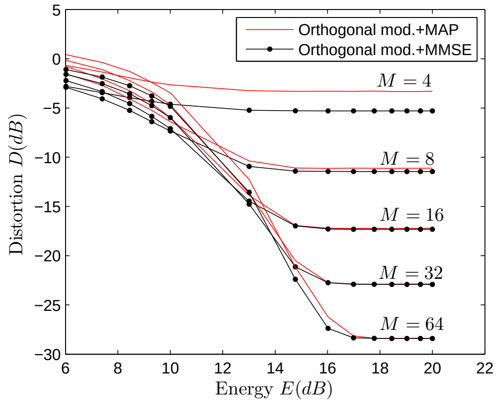

Fig. 5. Comparison between the MAP and the MMSE based scheme.

ρ being any positive number and P_{e_i} representing the probability of error given that x_i bas been sent. Following the same way of derivation as in [12, p. 65], (24) becomes

$$P_{e_i} \leq p_i^{-\frac{\rho}{\rho+1}} e^{-\frac{E}{2\sigma_z^2}\left(\frac{\rho}{1+\rho}\right)} \left(\sum_{j \neq i} p_j^{\frac{1}{1+\rho}} \right)^{\rho} \tag{25}$$

provided that $0 \leq \rho \leq 1$. Now Let V be a discrete random variable that takes the value $1/p_i$ with probability p_i for $i = 1, \ldots, M$. Using Jensen inequality, we have that

$$E[(V)^{\frac{\rho}{(\rho+1)}}] \leq (E[V])^{\frac{\rho}{(\rho+1)}} = M^{\frac{\rho}{(\rho+1)}} \tag{26}$$

for any $0 \leq \rho \leq 1$. Thus,

$$P_e \leq e^{-\frac{E}{2\sigma_z^2}\left(\frac{\rho}{1+\rho}\right)} \left(\sum_{i=1}^{M} p_i^{\frac{1}{1+\rho}} \right)^{\rho+1} \tag{27}$$

$$= e^{-\frac{E}{2\sigma_z^2}\left(\frac{\rho}{1+\rho}\right)} \left(\sum_{i=1}^{M} p_i \left(\frac{1}{p_i}\right)^{\frac{\rho}{1+\rho}} \right)^{\rho+1} \tag{28}$$

$$\leq M^{\rho} e^{-\frac{E}{2\sigma_z^2}\left(\frac{\rho}{1+\rho}\right)} \tag{29}$$

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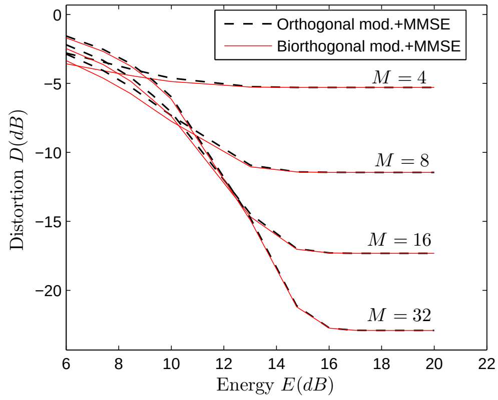

Fig. 6. Comparison between the orthogonal and the biorthogonal modulation performances combined with the MMSE estimator.

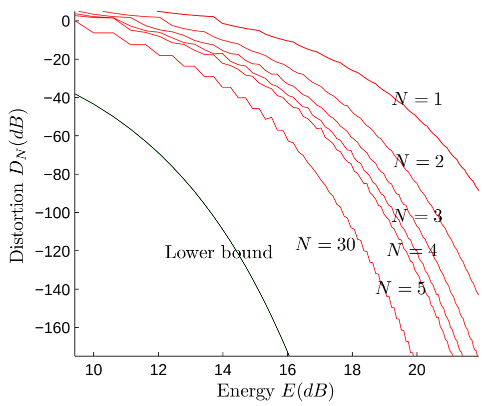

Fig. 7. The lower bound compared to the upper bound corresponding to sequence coding of length N.

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