Diversity-Multiplexing-Delay Tradeoff in Half-Duplex ARQ Relay Channels

Tarik Tabet, *Student Member, IEEE*,
Sanket Dusad, *Student Member, IEEE*, and
Raymond Knopp, *Member, IEEE*

***Abstract*—In this correspondence, we present an efficient protocol for the delay-limited fading ARQ single relay half-duplex channel. The source is using an Automatic Retransmission reQuest (ARQ) retransmission protocol to send data to the relay and the destination. When the relay is able to decode, both the relay and the source send the same data to the destination providing additional gains. The proposed protocol exploits two kinds of diversity: 1) space diversity available through the cooperative (relay) terminal, which retransmits the source's signals and 2) ARQ diversity obtained by leveraging the retransmission delay to enhance the reliability. The performance characterization is in terms of the achievable diversity, multiplexing gain and delay tradeoff for a high signal-to-noise ratio (SNR) regime. Finally, we show the benefits of power control on the diversity by controlling the source's power level over the retransmission rounds.**

***Index Terms*—Automatic Retransmission reQuest (ARQ), cooperative diversity, diversity multiplexing delay tradeoff, relay channel, wireless networks.**

I. Introduction

Distributed antennas can be used to provide a mean to combat fading with a similar flavor as that of space diversity. This could be used in *ad hoc* wireless networks where the constraints on the size of the terminals mitigates the presence of multiple antennas and full duplex transmissions. Another application scenario that has great potential is cellular networks. For uplink transmission, from an end user to a base station (access point), a relay can forward the end user message to the base station. The motivation comes from the fact that the end user is close to the cell boundary, and direct transmission requires high-power transmission and from the fact that the radio frequency (RF) technology used is kept simple by using one antenna at the end user preventing the benefit of the promising space–time techniques. This kind of reliability obtained by the creation of virtual antennas is referred to as cooperative diversity because the terminals share their resources to get the information across to the destination.

Cooperative schemes have attracted significant attention recently, and a variety of cooperation protocols have been studied and analyzed in various papers. The information-theoretic relay channel was first studied by van der Meulen [2], and some of the most important

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T. Tabet is with the Mobile Communications Laboratory, School of Computer and Communication Sciences, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland (e-mail: tarik.tabet@epfl.ch).

S. Dusad is with LTHC-LICOS, School of Computer and Communication Sciences, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland (e-mail: sanket.dusad@epfl.ch).

R. Knopp is with the Mobile Communications Department, Eurécom Institute, Sophia-Antipolis, France (e-mail: raymond.knopp@eurecom.fr.)

capacity results on relaying were published in [3]. A comprehensive review of past work on the relay channel and related problems appears in [4], and new information theoretic results could be found in [4]–[6]. The idea of cooperative diversity was pioneered in [7], [8] where the transmitters repeat detected symbols from each other to increase their rate region. Therein, the feasibility of user cooperation in a wireless network is demonstrated by an information theoretic exposition of the gains and a practical CDMA implementation. Taking into account practical constraints such as half-duplex transmission, and channel state information available only at the receiver (preventing from exploiting coherent transmission and combining), low-complexity cooperative diversity protocols are analyzed in [9], and extended to the case of multiple relays in [10].

Recently, the authors of [14] extended the Zheng–Tse formulation [13] and characterized the three dimensional diversity-multiplexing-delay tradeoff in multiple-input multiple-output (MIMO) Automatic Retransmission reQuest (ARQ) channels. They established that delay can be exploited as a potential source for diversity. Thus, retransmission protocol is an appealing scheme to combat fading and its performance has been studied in decentralized ad hoc networks [16]. Inspired by [14], we propose a new scheme for transmission in relay channel utilizing the ARQ to increase the diversity gain. We look at the tradeoff in the high signal-to-noise ratio (SNR) regime and point out the gain achieved by the ARQ.

Following the setup in [9], the terminals are constrained to employ half-duplex transmission, i.e., they cannot transmit and receive simultaneously. The source and the relay are allowed to transmit in the same channel using cooperative protocols not relying on orthogonal subspaces, allowing for a more efficient use of resources as in [10]. This is in contrast to [9], where the available bandwidth is divided into orthogonal channels allocated to the transmitting terminals. In the dynamic decode and forward scheme proposed in [10] the communication is across one block of length l, where l is asymptotically large. In our setting introduced in [12] and [11], the ARQ permits the use of communication over a variable number of blocks (henceforth referred to as number of rounds) of fixed length where the number of blocks used depend on the quality of the channel and are upper bounded by a fixed number L. If the destination is not able to decode at the end of these L blocks an outage is declared. Similarly, the relay accumulates enough information before it starts cooperating with the source. If the source-relay channel is always in outage, the relay may not be able to decode and thus it will not forward the source message. It is in contrast to the ARQ-DDF protocol used in [15] in a multiple access channel with two users, where once a user transmits its message successfully, it can cooperate with the other user *on each round*, using the DDF strategy. On each round the relay (in this case the user that successfully transmitted its message) will be able to decode the source message in $l^{'} < l$ symbols (l being asymptotically large) and will transmit the encoded message using an independent code-book during the rest of the codeword. In our case the number of rounds the relay will collaborate with the source by repeating its signal is random and depends on the quality of the source relay channel. The scheme proposed in this correspondence assesses the role of ARQ temporal diversity (by considering two dynamics of the channel: long-term static and long-term static channel) and cooperative diversity and the results are derived in terms of diversity multiplexing delay tradeoff at high SNR. Finally, a long-term power constraint is assumed in order to highlight the potential gain from a deterministic power control strategy based on [14].

The outline of this correspondence is as follows. Section II contains a summary of the useful results and notations used in the rest of the correspondence. We introduce the channel model and the details of the algorithm in Section III. The actual tradeoff for this protocol is analyzed and presented in Section IV for both long-term and short-term quasi-static channels. Section V proposes a power control scheme for ARQ relay protocol. Finally we summarize and present a few concluding remarks and future directions in Section VI.

II. Background

A. Notation

The symbol $\doteq$ will be used to denote the exponential quality, i.e., $f(\text{SNR}) \doteq \text{SNR}^b$ to denote

$$\lim_{\text{SNR}\to\infty} \frac{\log f(\text{SNR})}{\log \text{SNR}} = b$$

and similarly for $\dot{\leq}$ and $\dot{\geq}$. $(x)^+$ means $\max(0,x)$. $\mathcal{R}^{n+}$ denotes the set of real n-vectors with nonnegative elements, and $\mathcal{A}^+ = \mathcal{A} \cap \mathcal{R}^{n+}$. Let h be a circularly symmetric complex Gaussian with zero mean and unit variance. $\gamma = |h|^2$ is exponentially distributed with unit mean. Defining $\mu = -\frac{\log\gamma}{\log\text{SNR}}$ we note that μ is distributed in the high SNR as,

$$f_\mu(\mu) \doteq \begin{cases} \text{SNR}^{-\mu}, & \text{for } \mu \geq 0 \\ 0, & \text{for } \mu < 0. \end{cases} \tag{1}$$

B. Diversity-Multiplexing Tradeoff (DMT)

The trade-off between diversity and multiplexing was formally defined and studied in the context of point-to-point coherent communications in [13]. A family of codes $\mathcal{C}(\text{SNR})$ of block length T, with one code for each SNR level, is said to have a diversity gain of d and spatial multiplexing gain of r if

$$r = \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\log\text{SNR}}, \quad d = -\lim_{\text{SNR}\to\infty} \frac{\log P_e(\text{SNR})}{\log\text{SNR}}$$

where $R(\text{SNR})$ is the data rate measured in bits per channel use (BPCU) and $P_e(\text{SNR})$ is the average error probability using the maximum-likelihood (ML) decoder. For a coherent MIMO channel with M transmit antenna and N receive antenna, and for any multiplexing gain $r \leq \min\{M,N\}$ the optimal diversity gain $d^*(r)$ is given by the piecewise linear function joining the points $(K,(M-K)(N-K))$ for $K = 0,\ldots,\min\{M,N\}$. $d^*(r)$ is achieved by the random Gaussian i.i.d code ensemble for all block lengths $T \geq M+N-1$.

III. System Model and Setting

In this work, we consider communication over a relay network with one relay node (R) assisting the transmission of a source (S)-destination (D) pair as described in Fig. 1. Each link has circularly symmetric complex Gaussian zero mean unit variance channel gain h_{sd}, h_{sr}, h_{rd} corresponding to Rayleigh-fading channel, and the channel gains are mutually independent. The additive noises at the relay and the destination are mutually independent circularly symmetric white complex Gaussian. Nodes are operating in half-duplex mode. Moreover, we assume that each decoder has perfect knowledge of the channel gain. Perfect channel state information at the receivers implies that the S-R channel is known to the relay node, while the individual S-D, R-D channels are known to the destination node. The channel state information (CSI) is assumed to be absent at the node which is transmitting. Because of the ARQ protocol, limited feedback is received by the transmitting nodes. Moreover, perfect synchronization is assumed between nodes. We investigate two scenarios for the channel gains: 1) long-term static channel, where the fading is constant for all the channels over all retransmission (ARQ) rounds, and changes independently when the transmission of the current information message is stopped; 2) long-term static channel where the fading for all the channels is con-

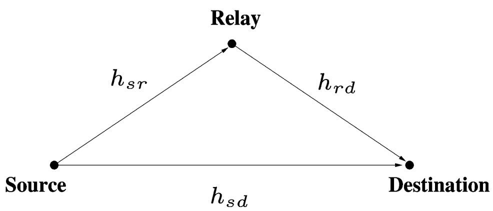

Fig. 1. System Model.

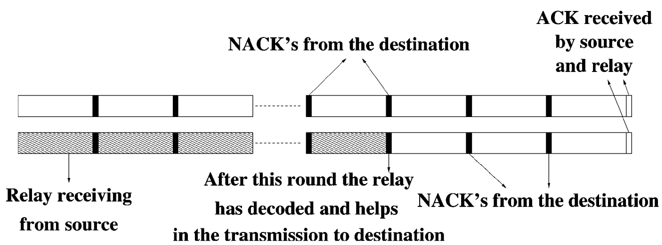

Fig. 2. Message as seen by the destination.

stant over each transmission round (or block) of the ARQ protocol and is an independent and identically distributed (i.i.d.) process across successive rounds. For the long-term static channel, only an upper bound on the diversity multiplexing delay tradeoff is derived. The ARQ protocol considered in this work is a form of incremental redundancy as studied in [16] and [17]. The transmission queue at the source is assumed to be infinite (not concerned by stability issues). The information message of b bits is encoded using a space–time code with code book $\mathcal{C} \subset \mathbb{C}^{2\times LT}$, where T is the number of channel uses taken to transmit one round and L is the maximum number of rounds that can be used to transmit the b information bits. We let $\mathcal{C}_l$ for $l = 1,\ldots,L$ denote the punctured space–time code of length lT obtained from $\mathcal{C}$ by deleting the last $(L-l)T$ columns of the space–time code.

The protocol utilizes the ARQ as follows. The destination feeds back a one bit success/failure indication to both the relay and the source. If the relay decodes before the destination then knowing the codebook $\mathcal{C}$ it begins transmitting the second row of the codebook $\mathcal{C}$ to the destination. Thus, effectively it becomes a multiple-input single-output (MISO) channel increasing the diversity (see Fig. 2). If the destination decodes before the relay, it just sends the feedback to the source and relay and the source moves on to transmitting the next message. We assume that the relay informs the destination of the starting of its transmission. The source moves on to the next information message in the transmission queue either if L rounds have been exhausted for the message or if the destination sends success feedback. If successful decoding occurs at the lth transmission, the effective coding rate for the current codeword is R/l bit/dim where $R = b/T$. In incremental redundancy, the receiver has memory of the past signals since it accumulates mutual information.

As defined above, the information message is encoded by a space–time encoder, and mapped in a sequence of L blocks, $\{\mathbf{x}_l \in \mathbb{C}^{2\times T} : l = 1,\ldots,L\}$, and the transmission is as in a MIMO system, where the rows of $\mathbf{x}_l = [\mathbf{x}_{sd;l} \quad \mathbf{x}_{rd;l}]^T$ are transmitted in parallel by the source and the relay. Each symbol of the transmitted codeword has unit power constraint. Let us call $\mathcal{T}_r$ a random variable denoting the block in which the relay was able to decode the source information message. Then, the signal model of our channel is given by

$$\mathbf{y}_l^d = \sqrt{\frac{\text{SNR}}{2}}\mathbf{h}_l\mathbf{x}_l + \mathbf{n}_l^d \tag{2}$$

where l stands for the retransmission round, $\{\mathbf{y}_l^d \in \mathbb{C}^{1\times T}\}$ is the received signal block by the destination, and $\{\mathbf{n}_l^d \in \mathbb{C}^{1\times T}\}$ is the channel noise assumed to be temporally and spatially white with i.i.d entries $\sim \mathcal{N}\mathbb{C}(0,1)$. The channel of the l-th round is characterized by the matrix $\{\mathbf{h}_l \in \mathbb{C}^{1\times 2}\}$ as follows:

$$\mathbf{h}_l = \begin{cases} [h_{sd} \quad 0], & \text{if } l \in [1, \mathcal{T}_r] \\ [h_{sd} \quad h_{rd}], & \text{if } l \in [\mathcal{T}_r+1, L]. \end{cases} \tag{3}$$

The received signal at the relay for $l = 1,\ldots,\mathcal{T}_r$ is given by

$$\mathbf{y}_l^r = \sqrt{\frac{\text{SNR}}{2}} h_{sr;l}\mathbf{x}_{sd;l}^T + \mathbf{n}_l^r \tag{4}$$

IV. Tradeoff Curves

In this section we derive the tradeoff curves for the case of the long term quasi-static and short-term quasistatic channels. Since we are in the high SNR regime, we ignore the factor 2 and use $\text{SNR} \doteq \frac{\text{SNR}}{2}$ for the remaining sections.

A. ARQ Protocol

The destination accumulates information on successive rounds and successful decoding is performed by soft-combining all the received rounds. We define the effective rate in a different manner as follows. Let $\mathcal{T}_d$ be a random variable denoting the stopping time of the transmission of the current message at the destination. Let $\overline{\mathcal{O}_l}$ be the event that the mutual information per channel use at a particular decoder exceeds the transmission rate R, i.e., $\overline{\mathcal{O}}_l = \{\sum_{i=1}^{l} I_i > R\}$ for $l = 1,\ldots,L-1$, with I_i being the mutual information of a single ARQ round as defined in (6), (7), (8). Then, we have

$$\begin{aligned} \Pr(\mathcal{T}_d = l) &= \Pr(\mathcal{O}_{d,1},\ldots,\mathcal{O}_{d,l-1},\overline{\mathcal{O}}_{d,l}) \\ &= \Pr(\mathcal{O}_{d,1},\ldots,\mathcal{O}_{d,l-1}) - \Pr(\mathcal{O}_{d,1},\ldots,\mathcal{O}_{d,l}) \\ &= \Pr(\mathcal{O}_{l-1}) - \Pr(\mathcal{O}_l) \end{aligned} \tag{5}$$

where we used the fact that the random sequence I_l is nondecreasing, and $\mathcal{O}_l \subseteq \mathcal{O}_m$ for $l \le m$ leading to $\Pr(\mathcal{O}_1,\ldots,\mathcal{O}_l) = \Pr(\mathcal{O}_l)$. We have also $\Pr(\mathcal{O}_0) = 1$, and $\Pr(\mathcal{T}_d = L) = \Pr(\mathcal{O}_{d,L-1})$. In our relay channel scenario, the instantaneous average mutual informations per channel use for the jth blocks are given by

$$I_{s;d}^j = I^j\left(\mathbf{x}_{sd,j};\mathbf{y}_j^d|h_{sd,j}\right) = \log(1+\text{SNR}\gamma_{sd,j}) \tag{6}$$

$$\begin{aligned} I_{s,r;d}^j &= I\left(\mathbf{x}_{sd,j},\mathbf{x}_{rd,j};\mathbf{y}_j^d|h_{sd,j},h_{rd,j}\right) \\ &= \log(1+\text{SNR}(\gamma_{sd,j}+\gamma_{rd,j})) \\ &\doteq \log(\text{SNR}^{(1-\min(\mu_{sd,j},\mu_{rd,j}))^+}) \end{aligned} \tag{7}$$

$$I_{s;r}^j = I(\mathbf{x}_{sr,j};\mathbf{y}_r^d|h_{sr,j}) = \log(1+\text{SNR}\gamma_{sr,j}). \tag{8}$$

The throughput of an incremental redundancy ARQ-based protocol is determined by the number of rounds needed for successful decoding, and it is defined as $\eta = R/\tau$, where τ is the average number of rounds needed for successful decoding. The effective multiplexing rate is then defined as

$$\begin{aligned} r_e &= \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\left(\sum_{l=0}^{L-1}\Pr(\mathcal{O}_l)\right)\log\text{SNR}} \\ &= \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\left(1+\sum_{l=1}^{L-1}\Pr(\mathcal{O}_l)\right)\log\text{SNR}} \\ &\doteq \frac{r}{\left(1+\sum_{l=1}^{L-1}\Pr(\mathcal{O}_l)\right)}. \end{aligned} \tag{9}$$

B. Long-Term Static Channel

Theorem 1: For a long-term static channel the outage probability at the lth round for the proposed protocol is given by

$$\Pr_{\text{out}}(l) \doteq \text{SNR}^{-d_{\text{out}}^{lt}(r,l)} \tag{10}$$

where

$$d_{\text{out}}^{lt}(r,l) = \begin{cases} (1-r), & \text{for } l=1 \\ \left(1-\frac{r}{l}\right)+\left(1-\frac{r}{l-1}\right), & \text{for } l \neq 1,3 \\ 2-5r/6, & \text{for } l=3, r<\frac{6}{7} \\ 3-2r, & \text{for } l=3, r\geq\frac{6}{7}. \end{cases} \tag{11}$$

Proof Outline 1: We used the fact that $\Pr_{\text{out}}(l) = \Pr(\mathcal{O}_l)$. For a long-term static channel, the instantaneous average mutual informations per channel use do not vary from one round to another. Denote their common values as $I_{s;d}, I_{s,r;d}$, and $I_{s;r}$. At round l, the outage probability for this cooperative channel depends on the fact that the relay was able to decode the message from the source. Suppose that the relay decodes at time k with probability given by (from (5))

$$\begin{aligned}\Pr(\mathcal{T}_r = k) &= \Pr((k-1)I_{s;r} < r\log(\text{SNR})) \\ &\quad - \Pr(kI_{s;r} < r\log(\text{SNR})) \\ &\doteq \text{SNR}^{-(1-r/(k-1))} - \text{SNR}^{-(1-r/k)}.\end{aligned} \tag{12}$$

If $k < l$, the mutual information is then the sum of the contribution of the source during k rounds single-input–single-output (SISO) and the contribution from the source and the relay during $l-k$ rounds (MISO). However if $k \geq l$, the relay has no contribution to the information conveyed to the destination. The outage probability for the ARQ relay long-term static channel is

$$\begin{aligned}\Pr_{\text{out}}(l) &= \sum_{k=1}^{L} \Pr_{\text{out}\,|\,\mathcal{T}_r=k}(l)p_k \qquad (13)\\ &= \sum_{k=1}^{l-1} \Pr(kI_{s;d} + (l-k)I_{s,r;d} < r\log(\text{SNR}))p_k \\ &\quad + \sum_{k=l}^{L} \Pr(lI_{s;d} < r\log(\text{SNR}))p_k \\ &\doteq \text{SNR}^{-2\left(1-\frac{r}{l}\right)} \sum_{k=1}^{\lfloor\frac{l}{2}\rfloor} p_k + \sum_{k=\lfloor\frac{l}{2}\rfloor+1}^{l-1} \text{SNR}^{-\left(2-\frac{r}{l-k}\right)} p_k \\ &\quad + \sum_{k=l}^{L} \text{SNR}^{-\left(1-\frac{r}{l}\right)} p_k \\ &\doteq \text{SNR}^{-d_{\text{out}}^{lt}(r,l)} \qquad (14)\end{aligned}$$

where $d_{\text{out}}^{lt}(r,l)$ is as given in (11) and $p_k = \Pr(\mathcal{T}_r = k)$. We use the following result:

$$\begin{aligned}&\Pr(kI_{s;d} + (l-k)I_{s,r;d} < r\log(\text{SNR})) \\ &\qquad \doteq \begin{cases} \text{SNR}^{-2(1-r/l)}, & \text{for } k \leq \lfloor l/2 \rfloor \\ \text{SNR}^{-(2-r/(l-k))}, & \text{for } \lfloor l/2 \rfloor < k \leq l-1 \end{cases}\end{aligned} \tag{15}$$

See Appendix A for proofs of (11) and (15). □

C. Short-Term Static Channel

Theorem 2: For a short-term static channel the outage probability at the lth round for the proposed protocol is given by

$$\Pr_{\text{out}}(l) \doteq \text{SNR}^{-d_{\text{out}}^{\text{st}}(r,l)} \tag{16}$$

where

$$d_{\text{out}}^{\text{st}}(r,l) = \begin{cases} (1-r), & \text{for } l=1 \\ l\left(1-\frac{r}{l}\right)+(l-1)\left(1-\frac{r}{(l-1)}\right), & \text{for } l\neq 1. \end{cases} \tag{17}$$

Proof Outline 2: Unlike in the case of long-term static channel, the instantaneous mutual informations defined above vary from one block to the other. We have

$$\begin{aligned}\Pr(\mathcal{T}_r = k) &= \Pr\left(\sum_{i=1}^{k-1} I_{s;r}^i < r\log(\text{SNR})\right) \\ &\quad - \Pr\left(\sum_{i=1}^{k} I_{s;r}^i < r\log(\text{SNR})\right) \\ &\doteq \text{SNR}^{-(k-1)(1-r/(k-1))} - \text{SNR}^{-k(1-r/k)}.\end{aligned} \tag{18}$$

And the outage probability for the ARQ relay long-term static channel is

$$\begin{aligned}\Pr_{\text{out}}(l) &= \sum_{k=1}^{L} \Pr_{\text{out}\,|\,\mathcal{T}_r=k}(l)p_k \qquad (19)\\ &= \sum_{k=1}^{l-1} \Pr\left(\sum_{i=1}^{k} I_{s;d}^i + \sum_{i=k+1}^{l} I_{s,r;d}^i < r\log(\text{SNR})\right) p_k \\ &\quad + \sum_{k=l}^{L} \Pr\left(\sum_{i=1}^{l} I_{s;d}^i < r\log(\text{SNR})\right) p_k \\ &\dot{\geq} \sum_{k=l}^{L} \text{SNR}^{-l(1-r/l)} p_k \\ &\doteq \text{SNR}^{-d_{\text{out}}^{\text{st}}(r,l)} \qquad (20)\end{aligned}$$

where $d_{\text{out}}^{\text{st}}(r,l)$ is as given in (11) and $p_k = \Pr(\mathcal{T}_r = k)$ as defined in (18). □

$d_{\text{out}}^{\text{st}}(r,l)$ corresponds to an upper-bound on the diversity multiplexing delay tradeoff. For a 2×1 MISO ARQ systems, the optimal diversity multiplexing delay tradeoff is $2l(1-r/l)$. One can write for $l \neq 1$, $d_{\text{out}}^{\text{st}}(r,l) \geq l(1-r/l)$ and

$$d_{\text{out}}^{\text{st}}(r,l) \leq l\left(1-\frac{r}{l}\right) + (l-1)\left(1-\frac{r}{(l-1)}\right) \leq 2l(1-r/l).$$

$l(1-r/l)$ corresponds to the case when the source relay channel is physically degraded version of the source destination channel. In this case, the diversity multiplexing delay tradeoff is that of a SISO ARQ system.

D. Diversity Multiplexing Delay Tradeoff

Theorem 3: The optimal diversity-multiplexing-delay tradeoff for the ARQ relay channel for the long-term static and short-term static relay channel is

$$\begin{aligned} d^{lt}(r_e, L) &= d_{\text{out}}^{lt}(r_e, L) = d_{\text{out}}^{lt}(r, L) \\ d^{\text{st}}(r_e, L) &= d_{\text{out}}^{\text{st}}(r_e, L) = d_{\text{out}}^{\text{st}}(r, L)\end{aligned}$$

subject to the constraint that $TL \geq 2$ for the long-term static channel and $T \geq 2$ for the short-term static channel, $0 \leq r_e < 1$.

See Appendix B for proof.

Note that the way we have defined the effective rate earlier (9) and from the expressions above for both the short-term and long-term static channel, it follows that

$$r_e \doteq \frac{r}{\left(1+\sum_{l=1}^{L-1}\text{SNR}^{-d_{\text{out}}(r,l)}\right)} \Longrightarrow r_e \doteq r. \tag{21}$$

The first phenomenon one can notice is that by increasing the value of the retransmission rounds, L, the diversity-multiplexing tradeoff curve for the long-term static channel flattens out as in Fig. 3. Consider the tradeoff curve in (17) for the short-term static channel. Since the channel fades independently to a new realization in each round, transmission in each new round gives additional diversity which explains the multiplicative L and $L-1$ factors in the diversity expression. Note that the factor is $(L-1)$ (both in the multiplication and the division) in the second term as the relay has to wait for at least one round before it can start transmitting to the destination. The reason this multiplicative factor does not show up in the case of the long-term quasistatic channel is that the channel is constant over all ARQ rounds and there is no time diversity benefit. But still there is a gain in the diversity because of the relay to destination channel and because of the ARQ protocol (the factor r/l). Moreover, as one can note from Appendix B, the multiplexing gain is determined by the rate of the first block $r_e \doteq r$, which means that most packets are decoded correctly within the first round, and ARQ rounds are used to correct the remaining error events increasing the diversity order without loss in the transmission rate, thus the diversity order is determined by the rate of the code of the combined packets. Another fact established by the ARQ protocol for the half-duplex relay channel, is that the time sharing factor (the fraction of time the relay spends in receive mode or transmit mode) chosen such that it minimizes the outage probability is accounted for automatically by incremental redundancy with ACK/NACK bit feedback and by adapting to the channel conditions.

V. Power Control

We notice that $d_{\text{out}}^{lt}(0,l)=2$ for all $l \neq 1$. Thus the long-term static channel is limiting the performance at low multiplexing-gains, which motivates the use of the power control.

Power control was recently applied to the cooperative relay channels. In [18], it was shown that by exploiting the channel state information at the transmitter and an adapted power control algorithm, the outage can be substantially lowered leading to an increase in diversity. In [20], the authors demonstrated that if the entire network state is used to determine the instantaneous transmitter power, only one bit feedback suffices to double the diversity order of the AF cooperative channel. Inspired by [14], and noticing that in long-term static channels the ARQ diversity is limited at low multiplexing gains, we construct a power control algorithm for this ARQ single relay channel. For simplicity, we consider a power control in which the relay is restricted to use a constant power in each round, but the source has the ability to vary its power to meet a long-term average power constraint.

Let $P_l = \text{SNR}^{p_l}$ be the power allocated per channel use for the lth round. The power constraint for the long-term static channel is $\frac{\sum_{l=1}^{L} P_l \Pr_{\text{out}}(l-1)}{\sum_{l=0}^{L-1} \Pr_{\text{out}}(l)} \leq 1$ (where the denominator is the expected number of rounds needed for successful decoding at the receiver). It is straightforward to show that $P_l \leq \frac{L}{\Pr_{\text{out}}(l-1)}$ and $p_l \leq d_{\text{out}}(r,l-1)$ where $d_{\text{out}}(r,l-1)$ is the SNR exponent of the $l-1$-th round outage probability for the ARQ relay channel. The power control policy is optimal when $P_l = \text{SNR}^{d(r,l-1)}$, with $P_1 = 0$ (for ease of notation the index out is omitted). Then, (6), (7), and (8) become (for long-term static channels)

$$
\begin{aligned}
I_{1,pc}^{j} &= \log(1+\text{SNR}^{1-\mu_{sd}+d(r,j-1)}) \\
I_{2,pc}^{j} &\doteq \log(\text{SNR}^{(1-\min(\mu_{sd}-d(r,j-1),\mu_{rd}))^{+}}) \\
I_{3,pc}^{j} &= I(\mathbf{x}_{sr,j};\mathbf{y}_r^d|h_{sr,j}) = \log(1+\text{SNR}^{1-\mu_{sr}+d(r,j-1)}).
\end{aligned}
$$

We define for convenience

$$
\begin{aligned}
q_k &= \Pr\left(\sum_{i=1}^{k} I_{1,pc}^{i} + \sum_{i=k+1}^{l} I_{2,pc}^{i} < r\log(\text{SNR})\right) \\
&\doteq \Pr\left(\sum_{i=1}^{k}(1-\mu_{sd}+d(r,i-1))^{+}\right. \\
&\quad \left. + \sum_{i=k+1}^{l}(1-\min(\mu_{sd}-d(r,i-1);\mu_{rd}))^{+} < r\right) \qquad (22)
\end{aligned}
$$

The outage probability for the ARQ relay long-term static channel with power control is

$$
\begin{aligned}
\Pr_{\text{out}}(l) &= \sum_{k=1}^{l-1} q_k p_k \\
&\quad + \sum_{k=l}^{L} \Pr\left(\sum_{i=1}^{l} I_{1,pc}^{i} < r\log(\text{SNR})\right) p_k \\
&\doteq \sum_{k=1}^{l-1} q_k p_k \\
&\quad + \sum_{k=l}^{L} \Pr\left(\sum_{i=1}^{l}(1-\mu_{sd}+d(r,i-1))^{+} < r\right) p_k \\
&\dot{\geq} \sum_{k=l}^{L} \Pr\left(\sum_{i=1}^{l}(1-\mu_{sd}+d(r,i-1))^{+} < r\right) p_k \qquad (23)
\end{aligned}
$$

where $\Pr(\mathcal{T}_r = k) = p_k$. Now note that

$$
\begin{aligned}
&\Pr\left(\sum_{i=1}^{l}(1-\mu_{sd}+d(r,i-1))^{+} < r\right) \\
&= \Pr\left(\left[\max_{t=1,\ldots,l}\sum_{i=1}^{t} d(r,l-i)+t(1-\mu_{sd})\right]^{+} < r\right) \qquad (24) \\
&\geq \Pr\left(\left(\sum_{i=1}^{l-1} d(r,i)+l\right)\left[1-\frac{\mu_{sd}}{d(r,l-1)+1}\right]^{+} < r\right) \\
&\doteq \text{SNR}^{-d(r,l)} \qquad (25)
\end{aligned}
$$

where one can easily show that $(\sum_{i=1}^{l-1} d(r,i)+l)[1-\frac{\mu_{sd}}{d(r,l-1)+1}]^{+}$ is, for all μ_{sd}, strictly above $[\max_{t=1,\ldots,l}\sum_{i=1}^{t} d(r,l-i)+t(1-\mu_{sd})]^{+}$, leading to

$$
d(r,l) = \left(1-\frac{r}{\sum_{i=1}^{l-1} d(r,i)+l}\right)(1+d(r,l-1)) \qquad (26)
$$

From (26) we note that $d(r,l)$ does not depend on k. Also $\Pr(\mathcal{T}_r = k)$ decreases as k increases. Combining these two facts and from (24) we see that by $r_e \doteq r$

$$
\begin{aligned}
\text{d}_{\text{out},pc}(r_e,l) \dot{\leq} \left(1-\frac{r_e}{l-1}\right) + \left(1-\frac{r}{\sum_{i=1}^{l-1} d_{\text{out},pc}(r_e,i)+l}\right) \\
\times (1+d_{\text{out},pc}(r_e,l-1)). \qquad (27)
\end{aligned}
$$

In the following, the diversity-multiplexing delay tradeoff is computed through Monte Carlo simulations. The diversity gain in a particular round and hence the power allocated can be numerically computed in a recursive manner. The diversity gain obtained using power control is significant compared to the constant power case especially at low multiplexing gains as shown in Fig. 4. Moreover, one can notice that the proposed power control is deterministic in the sense that it does not depend on the knowledge of the channel state but requires only the knowledge of the outage probabilities which can be estimated. Note that over here we have no constraint on the peak to average power ratio. An interesting question would be to investigate the gain in having a peak to average constraint of a constant greater than 1 but bounded by a constant not growing with SNR.

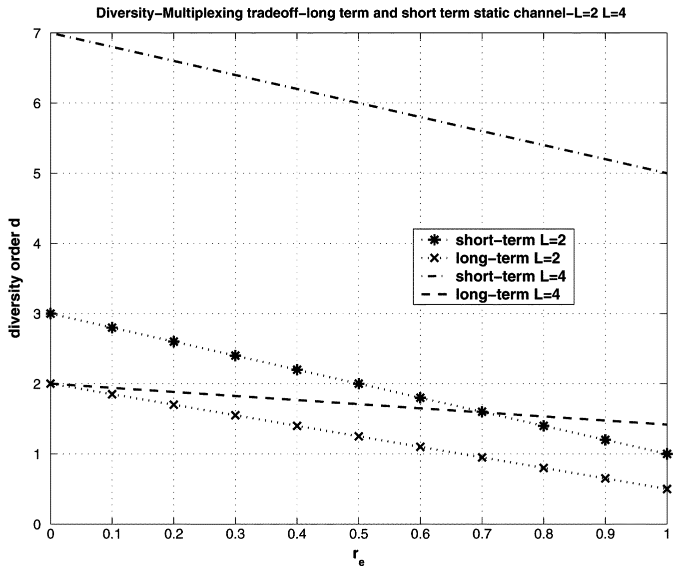

Fig. 3. DMT for different values of the maximum number of ARQ rounds for the short-term (upper-bound) and long term static channel.

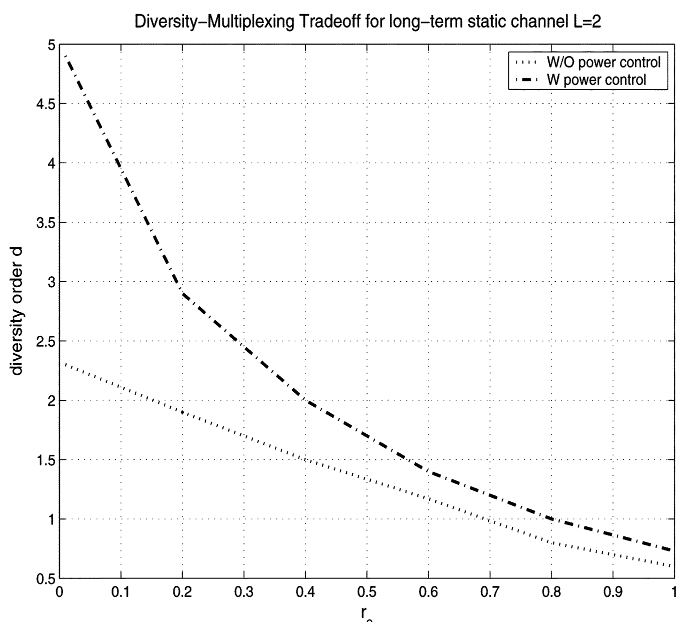

Fig. 4. The diversity-multiplexing tradeoff for $L = 2$ for the long-term static channel with and without power control.

VI. Conclusion

From the graph in Figs. 3 and 4, it can be seen that a significant gain in diversity is obtained by the proposed protocol. This is also evident from the outage probability expressions for short-term and long-term static channels. Power control is seen to be beneficial at all multiplexing gains by increasing the diversity order. An extension of this work would be the impact of multiple antennas (in particular two antennas considering the practical implications) at the receiver (base station), where the source and the relay collaborate to reach the destination. Another avenue would be to investigate the extension of these schemes to the case of multiple relays relaying the information for a single source destination pair. This protocol can then also be applied to *ad hoc* TDMA wireless networks where in each slot all the remaining nodes in the network act as relays for a particular source destination pair.

Appendix A
Proof of the Tradeoff Curve for Long-Term Static Channel

The proof of (15) is based on various cases in Fig. 5

$$
\begin{aligned}
&\Pr(kI_{s;d} + (l-k)I_{s,r;d} < r\log(\mathrm{SNR})) \\
&\doteq \Pr(k\log(\mathrm{SNR}^{(1-\mu_{sd})^+}) \\
&\quad + (l-k)\log(\mathrm{SNR}^{(1-\min(\mu_{sd},\mu_{rd}))^+}) < r\log(\mathrm{SNR})) \\
&\doteq \Pr(k(1-\mu_{sd})^+ + (l-k)(1-\min(\mu_{sd},\mu_{rd}))^+ < r) \\
&\doteq \mathrm{SNR}^{-\inf_{\underline{\mu}\in\mathcal{A}^+}(\mu_{sd}+\mu_{rd})}
\end{aligned} \tag{28}
$$

where $\mathcal{A}^+ = \{\underline{\mu} \in \mathcal{R}^{2+} : k(1-\mu_{sd})^+ + (l-k)(1-\min(\mu_{sd},\mu_{rd}))^+ < r)\}$. To solve (28), for the case where $\min(\mu_{sd},\mu_{rd}) = \mu_{sd}$, we obtain that

$$\inf_{\underline{\mu}\in\mathcal{A}^+} \mu_{sd} + \mu_{rd} = 1 - r/l + 1 - r/l = 2(1 - r/l)$$

as depicted in Fig. 5. For the case where $\min(\mu_{sd},\mu_{rd}) = \mu_{rd}$, the solution to (28) depends on k (the slot where the relay decodes). For $k \le \lfloor l/2 \rfloor$, $\inf_{\underline{\mu}\in\mathcal{A}^+} \mu_{sd} + \mu_{rd}$ corresponds to $\mu_{sd} = \mu_{rd} = 1 - r/l$ as depicted in Fig. 5(b) and for $\lfloor l/2 \rfloor < k \le l-1$, we obtain $\mu_{sd} = 1$ and $\mu_{rd} = 1 - r/(l-k)$ as depicted in Fig. 5(c) which concludes the proof of (15). To derive d_{out}^{lt} we proceed by finding the dominant term at high SNR

$$
\begin{aligned}
\Pr_{\mathrm{out}}(l) &\doteq \mathrm{SNR}^{-2(1-r/l)}\sum_{k=1}^{\lfloor l/2\rfloor}\Pr(\mathcal{T}_r = k) \\
&\quad + \sum_{k=\lfloor l/2\rfloor+1}^{l-1}\mathrm{SNR}^{-(2-r/(l-k))}\Pr(\mathcal{T}_r = k) \\
&\quad + \sum_{k=l}^{L}\mathrm{SNR}^{-(1-r/l)}\Pr(\mathcal{T}_r = k) \\
&\doteq \mathrm{SNR}^{-2(1-r/l)} + \underbrace{\mathrm{SNR}^{-(1-r/l)-(1-r/(l-1))}}_{\xi} \\
&\quad + \underbrace{\sum_{k=\lfloor l/2\rfloor+1}^{l-1}\mathrm{SNR}^{-(2-r/(l-k))}\Pr(\mathcal{T}_r = k)}_{\zeta}
\end{aligned} \tag{29}
$$

and we show

$$
\begin{aligned}
&\sum_{k=\lfloor l/2\rfloor+1}^{l-1}\mathrm{SNR}^{-\left(2-\frac{r}{l-k}\right)}\Pr(\mathcal{T}_r = k) \\
&\doteq \begin{cases} 0, & \text{for } l \in \{1,2\} \\ \mathrm{SNR}^{-(3-2r)}, & \text{for } l = 3 \\ \mathrm{SNR}^{-(2-r)-\left(1-\frac{r}{l-2}\right)}, & \text{for } l > 3. \end{cases}
\end{aligned} \tag{30}
$$

The dominant term of the outage probability is the second term of (29) ξ except for $l = 3$ where it depends on the value of r. Indeed for $l = 3$ and $r < 6/7$ the dominant term of the outage probability is the second term of (29) ξ where for $l = 3$, $r \ge 6/7$ the dominant term is the last one in (29) ζ as given in (30).

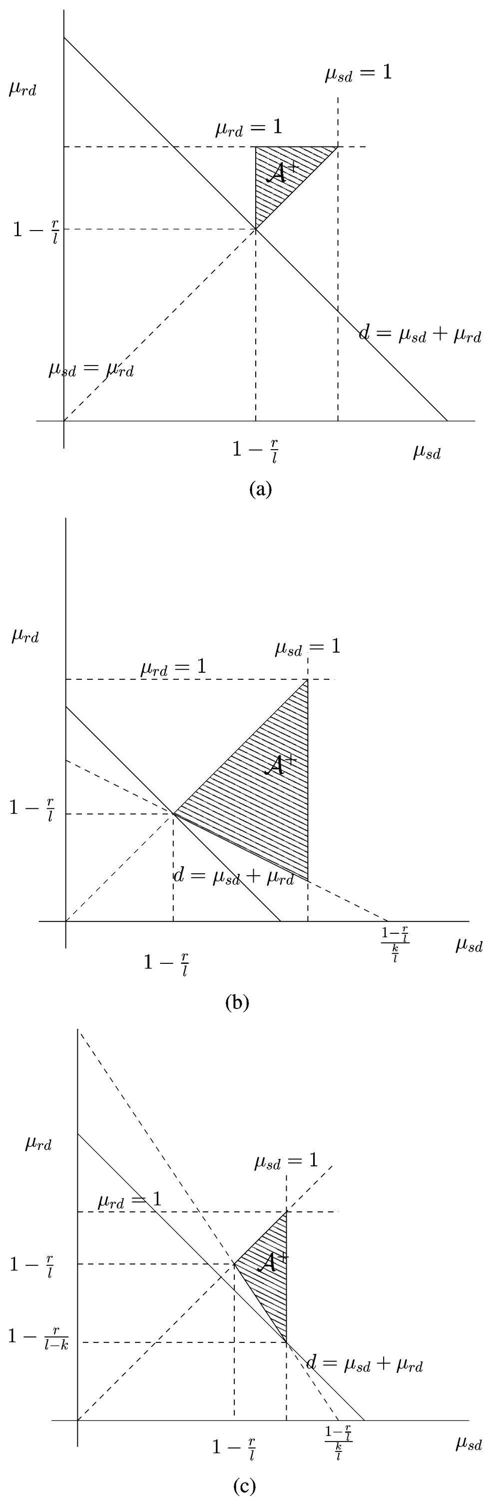

Fig. 5. The region $\mathcal{A}^+$ for the long-term static channel for various cases. (a) $\min(\mu_{sd},\mu_{rd}) = \mu_{sd}$. (b) $\min(\mu_{sd},\mu_{rd}) = \mu_{rd}$ and $k \le \lfloor l/2 \rfloor$. (c) $\min(\mu_{sd},\mu_{rd}) = \mu_{rd}$ and $\lfloor l/2 \rfloor < k \le l-1$.

Appendix B
Proof of the Diversity Multiplexing Delay Tradeoff

In the following, all the computations are done for the long-term static channel, the results are easily extended to the long-term static channel. For simplicity of notations in the proofs, we will use the following channel model:

$$\underline{\mathbf{y}}_l^d = \underline{\mathbf{h}}_l \underline{\mathbf{x}} + \underline{\mathbf{n}}_l^d \tag{31}$$

where $\underline{\mathbf{y}}_l^d \in \mathcal{C}^{Tl}$ represents the signal received over all transmitted block from 1 to l, $\underline{\mathbf{n}}_l^d$ the noise over the l rounds, $\underline{\mathbf{x}} = (\mathbf{x}_1^T,\ldots,\mathbf{x}_L^T)^T$ where $\mathbf{x}_l^T$ is defined in (2) and $\underline{\mathbf{h}}_L = \sqrt{\text{SNR}}(\mathbf{h}_1,\ldots,\mathbf{h}_L)$, $\mathbf{h}_l$ as defined in (3). $\underline{\mathbf{h}}_l$ is obtained from $\underline{\mathbf{h}}_L$ by replacing the last $2T(L-l)$ rows by zero, which corresponds to the fact that the blocks $(\mathbf{x}_{l+1},\ldots,\mathbf{x}_L)$ have not been transmitted yet and they appear multiplied by a zero channel matrix.

We wish first to show that d_{out} as defined in (11) is an upper bound to the SNR exponent of the ARQ relay system. Consider a system with codebook $\mathcal{C}$ (SNR), first-block rate $r\log(\text{SNR})$ and some decoding rule $\phi = (\phi_1,\ldots,\phi_L)$. We call k the time slot at which the relay decodes and $\overline{\mathcal{E}_k}$ the event that the relay decodes correctly, $\mathcal{E}_k$ the event for the incorrect decoding. Then, the probability of error conditioned on the vector channel: $\underline{\mathbf{h}} = [\underline{h}_{sd},\underline{h}_{rd},\underline{h}_{sr}]$, that the relay decodes at slot k, and on the particular decoder and codebook is

$$\begin{aligned} &P_{e,k}(\text{SNR}\,|\,\underline{\mathbf{h}},\mathcal{C}(\text{SNR}),\phi)\\ &= \Pr(e,\mathcal{E}_k) + \Pr(e,\overline{\mathcal{E}_k})\\ &\ge \Pr(e\,|\,\overline{\mathcal{E}}_k)\Pr(\overline{\mathcal{E}}_k)\\ &\ge \Pr(e\,|\,\overline{\mathcal{E}}_k). \end{aligned} \tag{32}$$

From the channel coding theorem, one can show that $\Pr(\overline{\mathcal{E}}_k) \ge 1-\epsilon$.

Let E_l be the event that the decoding outcome at the destination is not correct with l received rounds and $\mathcal{O}_l$ the event that the destination sends a NACK at round l and $\overline{\mathcal{O}}_l$ being the event for sending an ACK. Based on the results of [14], one can define the probability of error as (knowing that w is the transmitted message, $\hat{w} = \phi_l(\underline{\mathbf{y}}_l)$ and $\phi_l(\underline{\mathbf{y}}_l) = 0$ means that an error is detected and a NACK is sent back to the transmitter).

Clearly $\Pr(E_l,\mathcal{O}_1,\ldots,\mathcal{O}_{l-1},\overline{\mathcal{O}}_l)$ is the probability of undetected error. Indeed if an event $\mathcal{E}$ is detected a NACK is sent and a retransmission occurs, then this event does not count in the error event. $\Pr(E_L,\mathcal{O}_1,\ldots,\mathcal{O}_{L-1})$ is the probability of decoding error with L (maximum number) received blocks (at round L, the error is due to the undetected error or the failure of decoding which explains the $\bigcup_{\hat{w}}$ in (33), shown at the bottom of the following page.

The probability of error (computed by considering a decoder working on each round) is lower-bounded by the probability of error of the optimal ML decoder ϕ_{ml} that operates on the whole received signal vector $\underline{\mathbf{y}} = \underline{\mathbf{y}}_L$. By using the Fano's inequality (T is the number of channel uses per round)

$$\begin{aligned} &\Pr_{e,k}(\text{SNR}\,|\,\underline{\mathbf{h}},\mathcal{C}(\text{SNR}),\phi)\\ &\quad\ge \Pr_{e,k}(\text{SNR}\,|\,\underline{\mathbf{h}},\mathcal{C}(\text{SNR}),\phi_{ml})\\ &\quad\ge 1 - \frac{I(\underline{\mathbf{x}};\underline{\mathbf{y}}\,|\,\underline{\mathbf{h}},k)}{Tr\log\text{SNR}} - \frac{1}{Tr\log\text{SNR}} \end{aligned}$$

This leads to (based on the results in [13])

$$P_{e,k} \dot{\ge} \Pr(I(\underline{\mathbf{x}};\underline{\mathbf{y}}\,|\,\underline{\mathbf{h}},k) \le Tr\log\text{SNR}) \tag{34}$$

In our case, we obtain $P_e = E_k[P_{e,k}] \dot{\ge} \text{SNR}^{-d_{\text{out}}(r,L)}$. This is because the mutual information we use in the Fano's inequality is exactly the mutual information knowing that the relay cooperate at a particular round k and then we average on k which corresponds exactly to the outage probability computed in (13). Since $r_e \le r$ and d^{lt} is a decreasing function in r, we have $d^{lt}(r_e,L) \le d_{\text{out}}(r,L) \le d_{\text{out}}(r_e,L)$.

The achievability of the exponent upper-bound is shown based on a bounded distance decoder [14]. Let $\mathcal{C}(\text{SNR})$ denote a random code generated with i.i.d $\sim \mathcal{N}_c(0,1)$ components, block length LT and rate $r \log SNR$. Remember that in our case the source and the relay cooperate fully and we assume that both nodes are using the same codebook. The probability of error is given by (recalling that $\overline{\mathcal{E}}_k$ the event that the relay decodes correctly)

$$\begin{aligned} P_{e,k}(\text{SNR}) &= \Pr(e, \mathcal{E}_k) + \Pr(e, \overline{\mathcal{E}_k}) \\ &\leq \Pr(e \,|\, \overline{\mathcal{E}}_k)\Pr(\overline{\mathcal{E}}_k) \end{aligned} \tag{35}$$

where we used the fact that $\Pr(\mathcal{E}_k) \leq \epsilon$ (based on results in [13]), $\Pr(\overline{\mathcal{E}}_k) \leq 1-\epsilon$ and $\Pr(e \,|\, \mathcal{E}_k) \leq 1$.

We define the following bounded distance decoder ϕ at each round $l \leq L-1$ and the signal model at round l is given by (31), and $\underline{\mathbf{x}}(w)$ takes into account the signature of the source and the relay, w is the word message transmitted:

- $\phi_l(\underline{\mathbf{y}}_l) = \hat{w}$ if the channel is not in outage and the codeword $\underline{\hat{\mathbf{x}}}$ corresponding to $\hat{w}$ is the unique codeword in $\mathcal{C}(\text{SNR})$ such that $|\,\underline{\mathbf{y}}_l - \underline{\mathbf{h}}\underline{\hat{\mathbf{x}}}\,| \leq Tl(1+\delta)$
- $\phi_l(\underline{\mathbf{y}}_l) = 0$ in any other case.
- At round L, the decoder outputs the index of the minimum distance codeword, i.e., $\phi_L(\underline{\mathbf{y}}_L) = \phi_{ml}(\underline{\mathbf{y}}_L)$

Let us first bound the probability of undetected error $\Pr(E_l, \mathcal{O}_1, \ldots, \mathcal{O}_{l-1}, \overline{\mathcal{O}}_l \,|\, \overline{\mathcal{E}}_k) \leq \Pr_k(E_l, \overline{\mathcal{O}}_l)$. An error is undetected if the unique codeword $\underline{\hat{\mathbf{x}}}$ such that $|\,\underline{\mathbf{y}}_l - \underline{\mathbf{h}}\underline{\hat{\mathbf{x}}}\,| \leq Tl(1+\delta)$ does not correspond to the transmitted message w. This means that if we draw a sphere centered around the true codeword corresponding to the message transmitted w of radius $Tl(1+\delta)$, an undetected error occurs if the received signal $\underline{\mathbf{y}}_l$ belongs to the other sphere centered around other codewords. This event is included in the event that the received signal belongs to the region corresponding to the complement of the sphere corresponding to the true message transmitted; which is the event that the magnitude of the noise is bigger than the radius of the sphere

$$\begin{aligned} \Pr_k(E_l, \overline{\mathcal{O}}_l) &\leq \Pr(\,|\,\underline{\mathbf{n}}_l\,|^2 \geq Tl(1+\delta)) \\ &\leq (1+\delta)^{Tl(1+\delta)} \exp(-Tl\delta) \end{aligned} \tag{36}$$

Using the Chernoff bound and for some $\beta > 0$, and by letting $\delta = \beta \log \text{SNR}$. This leads to $\Pr_k(E_l, \overline{\mathcal{O}}_l) \dot{\leq} \text{SNR}^{-Tl\beta}$.

Assuming ML decoder, the probability of error at round L is $\Pr(E_L \,|\, \overline{\mathcal{E}}_k) = \Pr_k(E_L)$, and using the results in [13], one can show that $\Pr(E_L) \doteq \text{SNR}^{-d^{lt}_{\text{out}}(r,L)}$ for $LT \geq 2$ (corresponding to the case where the relay decodes in the first round). Using the following:

$$\Pr_{e,k}(\text{SNR}) \leq \sum_{l=1}^{L-1} \Pr(E_l, \overline{\mathcal{O}}_l \,|\, \overline{\mathcal{E}}_k) + \Pr(E_L \,|\, \overline{\mathcal{E}}_k)$$

we obtain

$$\begin{aligned} P_e(\text{SNR}) &\dot{\leq} E_k[\sum_{l=1}^{L-1} \Pr_k(E_l, \overline{\mathcal{O}}_l)] + \text{SNR}^{-d^{lt}_{\text{out}}(r,L)} \\ &\dot{\leq} \text{SNR}^{-T\beta} + \text{SNR}^{-d^{lt}_{\text{out}}(r,L)}. \end{aligned}$$

By choosing $LT \geq 2$ and a large β, one can ensure that $T\beta \geq d^{lt}_{\text{out}}(r,L)$ leading to $P_e \dot{\leq} \text{SNR}^{-d^{lt}_{\text{out}}(r,L)}$.

The next step is to prove that the effective multiplexing rate $r_e \doteq r$. Note that the effective multiplexing rate is given by

$$r_e \doteq \frac{r}{1 + \sum_{l=1}^{L-1} \Pr(\mathcal{O}_l)}$$

The condition $r_e \doteq r$ translates to the fact that $\Pr(\mathcal{O}_l)$ are $o(1)$.

Let us look at the region formed at the channel output by all possible received vectors $\underline{\mathbf{y}}_l$ and channel matrices $\underline{\mathbf{h}}_l$. We define $\mathcal{A}(\text{SNR}, l)$ as the outage space, $\mathcal{R}_\emptyset$ as the region of channel outputs not included in any sphere of radius $Tl(1+\delta)$ and centered around the codewords, and finally $\mathcal{R}$ is the region of channel outputs included in more than one of such spheres. $\mathcal{R}$ is partitioned into $\mathcal{R}_w$ (the region centered around the true codeword $\underline{\mathbf{x}}$ corresponding to the transmitted message w), and $\mathcal{R}_{\overline{w}}$

$$\begin{aligned} \Pr(\mathcal{O}_l) &= \Pr(\mathcal{A}(\text{SNR}, l) \cup \mathcal{R}_\emptyset \cup \mathcal{R}) \\ &\leq \Pr(\mathcal{A}(\text{SNR}, l)) \\ &+ \Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap (\mathcal{R}_\emptyset \cup \mathcal{R}_{\overline{w}})) \\ &+ \Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap \mathcal{R}_w) \end{aligned} \tag{37}$$

The event $\overline{\mathcal{A}}(\text{SNR}, l) \cap (\mathcal{R}_\emptyset \cup \mathcal{R}_{\overline{w}})$ corresponds to the union of the region of channel outputs not included in any sphere and the intersection of all spheres excluding the one corresponding to the transmitted codeword. One can show that: $\Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap (\mathcal{R}_\emptyset \cup \mathcal{R}_{\overline{w}})) \leq \Pr(\,|\,\underline{\mathbf{n}}_l\,|^2 \geq Tl(1+\delta))$. We then have

$$\Pr(\mathcal{O}_l) \dot{\leq} \text{SNR}^{-d^{lt}_{\text{out}}(r,l)} + \text{SNR}^{-T\beta} + \Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap \mathcal{R}_w) \tag{38}$$

Noting that $\mathcal{R}_{\overline{w}}$ is the region of the sphere centered around the true codeword and included in other spheres

$$\mathcal{R}_{\overline{w}} : \bigcup_{\substack{\hat{w} \neq w \\ \hat{w} > 0}} \{|\underline{\mathbf{y}}_l - \underline{\mathbf{h}}_l \underline{\mathbf{x}}| \leq Tl(1+\delta), |\underline{\mathbf{y}}_l - \underline{\mathbf{h}}_l \underline{\hat{\mathbf{x}}}| \leq Tl(1+\delta)\} \tag{39}$$

$$\begin{aligned} \Pr_{e,k}(\text{SNR} \,|\, \underline{\mathbf{h}}, \mathcal{C}(\text{SNR}), \phi) &\geq \sum_{l=1}^{L-1} \Pr(E_l, \mathcal{O}_1, \ldots, \mathcal{O}_{l-1}, \overline{\mathcal{O}}_l \,|\, \overline{\mathcal{E}}_k) + \Pr(E_L, \mathcal{O}_1, \ldots, \mathcal{O}_{L-1} \,|\, \overline{\mathcal{E}}_k) \\ &= \sum_{l=1}^{L-1} \Pr(\{\phi_l(\underline{\mathbf{y}}_1) = 0\}, \ldots, \{\phi_l(\underline{\mathbf{y}}_{l-1}) = 0\}, \bigcup_{\substack{\hat{w} \neq w \\ \hat{w} > 0}} \{\phi_l(\underline{\mathbf{y}}_l) = \hat{w}\} \,|\, \underline{\mathbf{h}}, \overline{\mathcal{E}}_k) \\ &+ \Pr(\{\phi_l(\underline{\mathbf{y}}_1) = 0\}, \ldots, \{\phi_l(\underline{\mathbf{y}}_{L-1}) = 0\}, \bigcup_{\hat{w} \neq w} \{\phi_l(\underline{\mathbf{y}}_L) = \hat{w}\} \,|\, \underline{\mathbf{h}}, \overline{\mathcal{E}}_k) \end{aligned} \tag{33}$$

This leads to (40)

$$\Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap \mathcal{R}_w) = \Pr(\overline{\mathcal{A}}(\text{SNR}, l), \bigcup_{\substack{\hat{w} \neq w \\ \hat{w} > 0}} |\underline{\mathbf{y}}_l - \underline{\mathbf{h}}_l \hat{\underline{\mathbf{x}}}| \le Tl(1+\delta), |\underline{\mathbf{n}}_l|^2 \le Tl(1+\delta)). \quad (40)$$

Let $a = \underline{\mathbf{h}}_l(\underline{\mathbf{x}} - \hat{\underline{\mathbf{x}}})$, $b = \underline{\mathbf{n}}_l$ and $\Delta = Tl(1+\delta)$, we have

$$\begin{aligned} \{ |a+b|^2 \le \Delta, |b|^2 \le \Delta \} &= \{|a+b|^2 \le \Delta, |b|^2 \le \Delta, |a|^2 \le 4\Delta\} \\ &\quad \cup \{|a+b|^2 \le \Delta, |b|^2 \le \Delta, |a|^2 > 4\Delta\} \quad (41) \\ &= \{|a+b|^2 \le \Delta, |b|^2 \le \Delta, |a|^2 \le 4\Delta\} \\ &\subseteq \{|a|^2 \le 4\Delta\} \quad (42) \end{aligned}$$

We obtain

$$\Pr\left(\bigcup_{\substack{\hat{w} \neq w \\ \hat{w} > 0}} |\underline{\mathbf{y}}_l - \underline{\mathbf{h}}_l \hat{\underline{\mathbf{x}}}| \le Tl(1+\delta), |\underline{\mathbf{n}}_l|^2 \le Tl(1+\delta) \right) \le \sum_{\substack{\hat{w} \neq w \\ \hat{w} > 0}} \Pr\left(\left| \frac{\underline{\mathbf{h}}_l(\underline{\mathbf{x}} - \hat{\underline{\mathbf{x}}})}{2} \right|^2 \le Tl(1+\delta) \right). \quad (43)$$

This leads to an upper-bound on the pairwise error probability summed over all distinct messages pairs and conditioned with respect to the channel. Results from [13] yield that for $Tl \ge 2$:

$$\Pr(\overline{\mathcal{A}}(\text{SNR}, l) \cap \mathcal{R}_w) \dot{\le} \text{SNR}^{-d_{\text{out}}^{lt}(r,l)} \quad (44)$$

where $d_{\text{out}}^{lt}(r,l)$ is the maximum possible SNR exponent for codes with length lT and rate r/l. Finally, one can state

$$\Pr(\mathcal{O}_l) \dot{\le} \text{SNR}^{-d_{\text{out}}^{lt}(r,l)}, \quad \text{for } 0 < l < L \quad (45)$$

and recalling the definition of the effective multiplexing rate, we obtain that $r_e \doteq r$.

Using random coding arguments, one can find codebooks which are good depending on the instant $\mathcal{T}_r$ at which the relay decodes. But we need codebooks which are simultaneously optimal irrespective of the instant $\mathcal{T}_r$ when the relay decodes. In the same spirit, we have to show that not only all the exponents ((45)) can be achieved by averaging over the code ensemble, but that there exist codes that achieve them simultaneously. The existence of codes which simultaneously achieve the error probability exponents follows from expurgation (along the lines of [14, Lemma 11]).

REFERENCES

[1] P. Gupta and P. R. Kumar, “The capacity of wireless networks,” *IEEE Trans. Inf. Theory*, vol. 46, no. 11, pp. 388–404, Mar. 2000.

[2] E. C. van der Meulen, “Three-terminal communication channels,” *Adv. Appl. Prob.*, vol. 3, pp. 120–154, 1971.

[3] T. M. Cover and A. A. El Gamal, “Capacity theorems for the relay channel,” *IEEE Trans. Inf. Theory*, vol. 25, pp. 572–584, Sep. 1979.

[4] G. Kramer, M. Gastpar, and P. Gupta, “Cooperative strategies and capacity theorems for relay networks,” *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sept. 2005.

[5] A. Host-Madsen, “Capacity bounds for cooperative diversity,” *IEEE Trans. Inf. Theory*, submitted to.

[6] M. Khojastepour, A. Sabharwal, and B. Aazhang, “On the capacity of ‘Cheap’ relay networks,” in *Proc. Conf. Inf. Sci. Syst.*, Apr. 2003.

[7] A. Sendonaris, E. Erkip, and B. Aazhang, “User cooperation diversity. Part I. System description,” *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1927–1938, Nov. 2003.

[8] A. Sendonaris, E. Erkip, and B. Aazhang, “User cooperation diversity—Part II: Implementation aspects and performance analysis,” *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1939–1948, Nov. 2003.

[9] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, “Cooperative diversity in wireless networks: Efficient protocols and outage behavior,” *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[10] K. Azarian, H. El Gamal, and P. Schniter, “On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels,” *IEEE Trans. Inf. Theory*, Jul. 2004, submitted for publication.

[11] T. Tabet, S. Dusad, and R. Knopp, “ARQ based half-duplex cooperative diversity protocol,” in *Canadian Workshop on Information Theory*, Montreal, QC, Canada, Jun. 2005.

[12] T. Tabet, S. Dusad, and R. Knopp, “Achievable diversity-multiplexing-delay tradeoff in half-duplex ARQ relay channels,” in *Proc. IEEE Int. Symp. Inf. Theory, ISIT*, Adelaide, Australia, Sep. 2005.

[13] L. Zheng and D. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple antenna channels,” *IEEE Trans. Inf. Theory*, vol. 49, pp. 1073–1096, May 2003.

[14] H. El Gamal, G. Caire, and M. O. Damen, “The MIMO ARQ channel: Diversity-multiplexing-delay tradeoff,” *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3601–3621, Aug. 2006.

[15] Y. Nam, K. Azarian, and H. El Gamal, “Cooperation through ARQ,” in *Proc. IEEE 6th Workshop on Signal Processing Adv. Wireless Commun. (SPAWC05)*, New York, Jun. 5–8, 2005, pp. 1023–1027.

[16] T. Tabet and R. Knopp, “Spatial throughput of multi-hop wireless networks under different retransmission protocols,” in *Proc. Allerton Conf. Commun., Contr., Comput.*, Monticello, IL, Oct. 2004.

[17] G. Caire and D. Tuninetti, “The throughput of hybrid-ARQ protocols for the Gaussian collision channel,” *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 1971–1988, Jul. 2001.

[18] N. Ahmed, M. Khojastepour, and B. Aazhang, “Outage minimization and optimal power control for the fading relay channel,” in *Proc. IEEE Inf. Theory Workshop*, Houston, TX, May 2004, submitted for publication.

[19] G. Caire, G. Taricco, and E. Biglieri, “Optimum power control over fading channels,” *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1468–1489, July 1999.

[20] N. Ahmed, M. Khojastepour, A. Sabharwal, and B. Aazhang, “On power control with finite rate feedback for cooperative relay networks,” in *Proc. Int. Symp. Inf. Theory Applicat.*, Mar. 2004, submitted for publication.