Lower Bound on Time-Delay Estimation Error of UWB Signals

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***Abstract*— In this work, we study fundamental limitation to the performance of time-delay estimation schemes of IR-UWB signals. We are specially interested in deriving lower bound on the mean square error (MSE). As UWB systems are expected to operate at low signal-to-noise ratio (SNR) (80211.15.4 low-power low rate standard), the improved Ziv-Zakai lower bound (IZZLB) [1] is then more suited to characterize the lower bound on the MSE, than the Cramer-Rao lower bound (CRLB). We express then the lower bound on MSE of maximum likelihood estimator based on perfect knowledge of the 2nd order statistics of the received signal by mean of the IZZLB.**

I. Introduction

An ultra-wideband (UWB) signaling scheme is defined as any wireless technology that occupies a bandwidth of more than 500 MHZ and/or has fractional bandwidth greater than 20%. The fractional bandwidth is defined as $f_{frac} = 2(f_H - f_L)/(f_H + f_L)$, where f_H and f_L are respectively the upper and lower frequency at $-10dB$.

UWB technology employs pulses of very short durations ($\leq ns$) with very low spectral densities. It is resistant to channel multipath and has very good time-domain resolution allowing for location and tracking applications, and is relatively low-complexity and low-cost. Due to low power density, duty cycle transmission, and dense UWB multipath channel [2], very fine synchronization is required for reliable transmission.

In this paper, we address the performance limitation of time-delay estimation schemes of UWB signals. Its well known that the CRLB applies only to unbiased estimates and yields tight bounds only for large signal-to-noise ratios (SNR). As UWB systems are expected to operate at low SNR, CRLB represents a very optimistic (from engineering point of view) performance limit. This motivates our choice to use the IZZLB[1] rather than the CRLB to address the time-delay estimation performance for UWB signals.

The first application of the IZZLB in time-delay estimation for UWB signals has been introduced in [3] for the case of independent channel paths. In this work, we extend this study to the case of correlated channel paths.

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In [4], a simplified Ziv-Zakai bound for the time-delay estimation problem, without taking into account the impact of the radio channel on the received signal, was introduced as extension of the work in [5].

The paper is organized as follows, in section II we introduce the system model, in section II we derive the IZZLB for the case of single and multiple frame observation. Numerical results are given in section IV and concluding remarks in section V.

II. System Model

Let $s(t) = \sqrt{\frac{E_p}{T_p}}p(t)$ be the transmitted IR-UWB single-pulse one-shot signal, with E_p been the pulse energy, $p(t)$ is the transmitted pulse of duration T_p with $\int_0^{T_p} p(t)^2 dt = 1$, and $W_b = 1/T_p$ the signal bandwidth. Propagation studies for IR-UWB signals have shown that they undergo dense multipath environment producing large number of resolvable paths [2].

A typical model for the impulse response of a multipath channel is given by

$$h(t) = \sum_{i=1}^{L} h_i \delta(t - \tau_i) \quad (1)$$

Where τ_i is the $i-th$ path delay and h_i is random variable modeling signal attenuation at τ_i, $\sum_{i=1}^{L} E[|h_i|^2] = 1$.

The received signal during an observation period of duration T_f can then be written as

$$r(t) = \begin{cases} y(t-\theta_0) + n(t) t \in [\theta_0,\ \theta_0 + T_d], \\ n(t) t \in [0,\ \theta_0] \cup [\theta_0 + T_d,\ T_f] \end{cases} \quad (2)$$

Where θ_0 is the time delay parameter to be estimated, T_d the channel delay spread, $n(t)$ is complex Gaussian noise process with zero mean and power spectral density N_o, and

$$y(t) = s(t) * h(t) = \sqrt{\frac{E_p}{T_p}} \sum_{i=1}^{L} h_i p(t - \tau_i) \quad (3)$$

Since each random variable h_i modeling the signal attenuation is a combination of many significant random variables, we model y as non-stationary circular complex Gaussian process. The autocorrelation function of y is given as

$$K_y(t,u) = \frac{E_p}{T_p} \sum_{i=1}^{L} \sum_{j=1}^{L} E[h_i h_j^{\dagger}] p(t-\tau_i) p(u-\tau_j) \quad (4)$$

† denotes complex conjugate.

III. Lower Bound on Mean Square Estimation Error

A. The Improved Ziv-Zakai Lower Bound

The Ziv-Zakai formulation of the lower bound is based on the probability of deciding correctly between two possible values (θ) and $(\theta + x)$ of the signal delay θ_0. The derivation of this bound relies on result from detection theory [6]. An optimal detection scheme which minimizes the probability of error performs a likelihood ratio test between the two hypothesized delays. On the other hand, a suboptimal procedure will be to apply, first, some estimation procedure to estimate the delay $\hat{\theta_0}$ of the received signal then decide between the two hypothesis by comparing $\hat{\theta}$ with $\theta + x/2$ (the arithmetic mean of θ and $\theta + x$). By comparing the performance the two schemes, one obtains the improved Ziv-Zakai lower bound (IZZLB) [1] on mean square error of the delay estimate

$$E[(\hat{\theta_0} - \theta_0)^2] \geq \int_0^{T_f - T_d} x dx \int_0^{T_f - T_d - x} \frac{P_d(\theta, \theta + x)}{T_f - T_d} d\theta \quad (5)$$

Where $P_d(\theta, \theta + x)$ denotes the probability of error of the likelihood ratio test when deciding between θ and $\theta + x$.
In cases where $P_d(\theta, \theta+x)$ is independent of θ_0, the expression in (5) becomes

$$E[(\hat{\theta_0} - \theta_0)^2] \geq \int_0^{T_f - T_d} x \frac{T_f - T_d - x}{T_f - T_d} P_d(x) dx \quad (6)$$

B. Single-frame observation

The received signal during the observation interval $[0,\ T_f]$ ($T_f >> T_d$) is given by

$$r(t) = \begin{cases} y(t - \theta_0) + n(t) & t \in [\theta_0,\ \theta_0 + T_d] \\ n(t) & t \in [0,\ \theta_0] \cup [\theta_0 + T_d,\ T_f] \end{cases} \quad (7)$$

We assume that the receiver has perfect information about a degenerate kernel of the second order statistics $K_y(t, u)$ characterized by a finite number of eigenmodes. Let $\theta_1 = 0$ and $\theta_2 = x$ denote the two hypothesized delays[1], $x \in]0,\ T_f - T_d]$. When the distance between the two hypothesized delays is greater than the duration of the signal, the two hypotheses can be rewritten to obtain a symmetric-hypothesis detection problem.

$$r(t) = \left\{ \begin{array}{ll} y(t) + n(t) & t \in [0,\ T_d] \\ n(t) & t \in [x,\ x + T_d] \end{array} \right| H_1 \quad (8)$$

$$r(t) = \left\{ \begin{array}{ll} y(t - x) + n(t) & t \in [x,\ x + T_d] \\ n(t) & t \in [0,\ T_d] \end{array} \right| H_2 \quad (9)$$

For $x \leq T_d$, the two hypothesis are no more symmetric. The observation interval could be reduced to cover only the region where the signal is present under at least one of the two hypotheses. The two hypotheses becomes then

$$r(t) = \left\{ \begin{array}{ll} y(t) + n(t) & t \in [0,\ T_d] \\ n(t) & t \in [T_d,\ x + T_d] \end{array} \right| H_1 \quad (10)$$

$$r(t) = \left\{ \begin{array}{ll} y(t - x) + n(t) & t \in [x,\ x + T_d] \\ n(t) & t \in [0,\ x] \end{array} \right| H_2 \quad (11)$$

In this cases, projecting the signal in term of its covariance matrix on each observation interval ($[0,\ T_d]$ and $[x,\ x + T_d]$) leads to singularities in the log-likelihood ratio function. For this reason, we will use Fourier basis, common to the two hypotheses, to obtain our sufficient statistics.

1) Case of $x > T_d$: External Bound: The two observation intervals are disjoints, the received signal over each observation interval is projected on a corresponding Fourier basis as follows

$$r(t) = \sum_{i=1}^{N} R_i^1 \psi_i^1(t) \quad ; R_i^1 = \int_0^{T_d} r(t) \psi_i^1(t) dt \quad (12)$$

$$\psi_i^1(t) = \frac{1}{\sqrt{T_d}} e^{-\frac{j2\pi it}{T_d}} \quad \text{for } t \in [0,\ T_d] \quad (13)$$

$$r(t) = \sum_{i=1}^{N} R_i^2 \psi_i^2(t) \quad ; R_i^2 = \int_x^{x+T_d} r(t) \psi_i^2(t) dt \quad (14)$$

$$\psi_i^2(t) = \frac{1}{\sqrt{T_d}} e^{-\frac{j2\pi i(t-x)}{T_d}} \quad \text{for } t \in [x,\ x + T_d] \quad (15)$$

Where $N = 2W_b T_d$, and $j = \sqrt{-1}$.
Let $R^1 = [R_1^1, \ldots, R_N^1]^T$, $R^2 = [R_1^2, \ldots, R_N^2]^T$ and $R = \left[R^{1T}\ R^{2T}\right]^T$.
Under H_1, R is zero mean circular complex Gaussian process with covariance matrix ${}_1K_x$ whose elements are given by

$$\begin{aligned} & E[R_i^1 R_k^{1\dagger} | H_1] \\ & = E\left[\frac{1}{T_d} \int_0^{T_d} \int_0^{T_d} r(t) r(u)^\dagger \psi_i^1(t) \psi_k^{1\dagger}(u) dt du\right] \\ & = \frac{N_0}{2} \delta_{i,k} + \frac{1}{T_d} \int_0^{T_d} \int_0^{T_d} K_y(t, u) e^{-\frac{j2\pi(it - ku)}{T_d}} dt du \quad (16) \end{aligned}$$

$$E[R_i^2 R_k^{2\dagger} | H_1] = \frac{N_0}{2} \delta_{i,k} \quad (17)$$

$$E[R_i^1 R_k^{2\dagger} | H_1] = 0 \quad (18)$$

Similarly, under H_2, R is zero mean circular complex Gaussian process with covariance matrix ${}_2K_x$ whose elements are given by

$$\begin{aligned} & E[R_i^2 R_k^{2\dagger} | H_2] \\ & = E\left[\frac{1}{T_d} \int_x^{x+T_d} \int_x^{x+T_d} r(t-x) r(u-x)^\dagger \psi_i^2(t) \psi_k^{2\dagger}(u) dt du\right] \\ & = \frac{N_0}{2} \delta_{i,k} + \frac{1}{T_d} \int_0^{T_d} \int_0^{T_d} K_y(t, u) e^{-\frac{j2\pi(it - ku)}{T_d}} dt du \quad (19) \end{aligned}$$

$$E[R_i^1 R_k^{1\dagger} | H_2] = \frac{N_0}{2} \delta_{i,k} \quad (20)$$

$$E[R_i^1 R_k^{2\dagger} | H_2] = 0 \quad (21)$$

[1] We assume that the detection error probability is independent of θ_0 which is the case for $T_f >> T_d$

The log-likelihood ratio function is defined as

$$L(x) = \ln\left\{\frac{P(R|H1)}{P(R|H2)}\right\} = R^\dagger Q_x R \quad (22)$$

$$\text{Where} \quad Q_x = {}_1K_x^{-1} - {}_2K_x^{-1} \quad (23)$$

The two hypothesis are then compared according to the decision rule: $L(x) \underset{H_2}{\overset{H_1}{\gtrless}} 0$. As the two hypotheses are symmetric, the resulting binary detection error probability is

$$P_d(x) = P(Z > 0|H_1 \text{ is correct}) \quad (24)$$

Where $Z = R^\dagger Q_x R$.
Z is an indefinite quadratic form on complex Gaussian random variables. In appendix A, we give a general technique to decompose Z and derive its distribution.

2) Case of $x \leq T_d$: Internal Bound: The observation interval is reduced to cover only the region where the signal is present under at least one of the two hypotheses; So the received signal during the interval $[0,\ x+T_d]$ is projected on a Fourier basis, common to the two hypotheses, as follows

$$r(t) = \sum_{i=0}^{N} R_i\psi_i(t) \quad (25)$$

$$R_i = \int_0^{x+T_d} r(t)\psi_i(t)dt \quad (26)$$

$\psi_i(t) = \frac{1}{\sqrt{x+T_d}}e^{-\frac{j2\pi it}{x+T_d}}$ are elements of the Fourier basis defined in the interval $[0,\ x+T_d]$, $N = 2W_b(x+T_d)$.
R_i are circular complex Gaussian variables with zero mean. Under H_1 their covariance matrix is given as

$$\begin{aligned} &{}_1K_x(i,k) = E[R_iR_k^\dagger|H_1] \\ &= E\left[\frac{1}{x+T_d}\int_0^{x+T_d}\int_0^{x+T_d} r(t)r(u)^\dagger\psi_i(t)\psi_k^\dagger(u)dtdu\right] \\ &= \frac{N_0}{2}\delta_{i,k} + \frac{1}{x+T_d}\int_0^{T_d}\int_0^{T_d} K_y(t,u)e^{-\frac{j2\pi(it-ku)}{x+T_d}}dtdu \end{aligned} \quad (27)$$

Similarly, under hypothesis H_2, R_i have a covariance matrix given as

$$\begin{aligned} &{}_2K_x(i,k) = E[R_iR_k^\dagger|H_2] = \frac{N_0}{2}\delta_{i,k} + \\ &\frac{1}{x+T_d}\int_x^{x+T_d}\int_x^{x+T_d} K_y(t-x,u-x)e^{-\frac{j2\pi(it-ku)}{x+T_d}}dtdu \\ &= \frac{N_0}{2}\delta_{i,k} + \frac{e^{-\frac{j2\pi x(i-k)}{x+T_d}}}{x+T_d}\int_0^{T_d}\int_0^{T_d} K_y(t,u)e^{-\frac{j2\pi(it-ku)}{x+T_d}}dtdu \\ &= e^{-\frac{j2\pi x(i-k)}{x+T_d}}\ {}_1K_x(i,k) \end{aligned} \quad (28)$$

The resultant log-likelihood ratio function is then

$$L(x) = \ln\left\{\frac{P(R|H1)}{P(R|H2)}\right\} = R^\dagger Q_x R - D_x \quad (29)$$

$$\text{Where} \quad Q_x = {}_1K_x^{-1} - {}_2K_x^{-1} \quad (30)$$

$$D_x = [\ln det({}_1K_x) - \ln det({}_2K_x)] = 0 \quad (31)$$

The two hypothesis are then compared according to the decision rule

$$L(x) \underset{H_1}{\overset{H_2}{\gtrless}} 0 \quad (32)$$

The resulting probability of detection error is

$$P_d(x) = \frac{1}{2}[P(Z>0|H_1 \text{ is correct}) + P(Z<0|H_2 \text{ is correct})] \quad (33)$$

Where $Z = R^\dagger Q_x R$.

C. Multiframe observation

In this case, the transmitted signal is repeated over N_f frames. In order to compare fairly with the single frame scheme, the transmitted signal energy over each frame is divided over N_f. The receiver signal during the observation interval $[0,\ N_fT_f]$ is then

$$r(t) = \begin{cases} y_m(t-\theta_0)+n(t) & ,\ t\in[mT_f,\ mT_f+T_d] \\ n(t) & ,\ \text{elsewhere} \end{cases} \quad (34)$$

Where

$$y_m(t) = \sqrt{\frac{E_p}{N_fT_p}}\sum_{i=1}^{L} h_ip(t-\tau_i),\ m=0\ldots N_f-1 \quad (35)$$

1) Internal Bound: The received signal during the observation interval
$\cup_{m=0}^{N_f-1}[mN_f,\ mN_f+x+T_d]$ is now projected on N_f Fourier basis as follows

$$r(t) = \sum_{m=0}^{N_f-1}\sum_{i=1}^{N} R_i^m\psi_i^m(t) \quad (36)$$

$$R_i^m = \int_{mN_f}^{mN_f+x+T_d} r(t)\psi_i^m(t)dt \quad (37)$$

$$\psi_i^m(t) = \frac{1}{\sqrt{x+T_d}}e^{-\frac{j2\pi it}{x+T_d}} \quad ,\ t\in[mT_f,\ mT_f+x+T_d] \quad (38)$$

Following the same processing as in sec.(III-B.2, Internal Bound), the detection error probability is

$$P_d(x) = \frac{1}{2}[P(Z_f>0|H_1 \text{ is correct}) + P(Z_f<0|H_2 \text{ is correct})] \quad (39)$$

Where now

$$Z_f = \sum_{m=0}^{N_f} R^{m\dagger}Q_xR^m \quad (40)$$

$R^m = (R_1^m,\ldots,R_N^m)^T$ and Q_x is defined as in sec.(III-B, Internal Bound).

2) External Bound: The received signal during the observation intervals $[mN_f,\ mN_f+T_d]$ and $[mN_f+x,\ mN_f+x+T_d]$, $m = 0\ldots N_f$, is now projected on the corresponding $2N_f$ Fourier basis. The same processing as in sec.(III-B.1, External

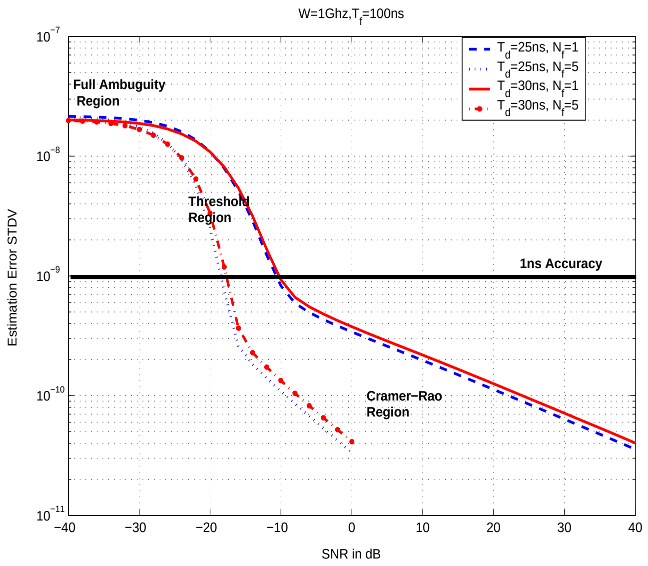

Fig. 1. IZZLB on RMSE Vs. average SNR, for $T_d = 25ns,\ 30ns$

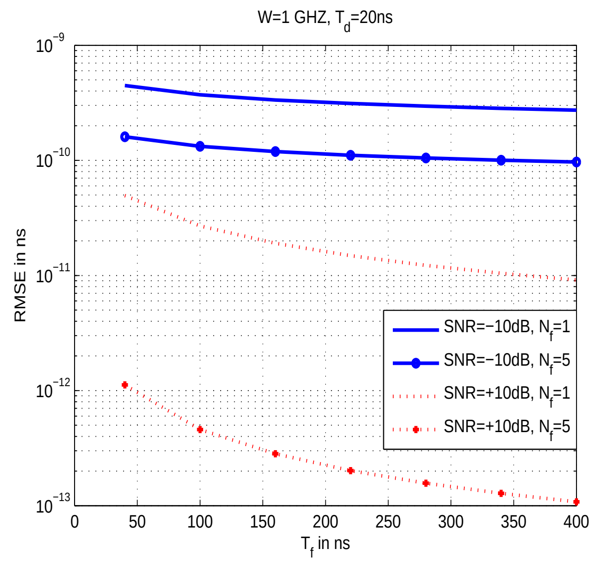

Fig. 2. IZZLB on RMSE Vs. Frame length , for $T_d = 20ns$

Bound) can be done to obtain the resulting detection error probability.

IV. Numerical Results

For numerical purpose, we take a equi-spaced multipath channel with power delay profile $E[|h_i|^2] = \exp^{-\alpha\frac{\tau_i}{T_d}}$, with α defined as the power decay factor (PDF). In Fig. IV, we plot the obtained IZZLB on the root mean square error (RMSE) for different delay spread durations T_d vs. Average transmitted SNR. The pulse is of duration $T_p = 1ns$, the observation period is of length $T_f = 100ns$, and the PDF is 2. The average SNR is defined as $SNR = \frac{E_p T_p}{T_f N_0}$. We observe then three different operating regions:

1) The full ambiguity region corresponding to a very small SNR, in this region the receiver see the signal as noise and the error in this case is uniformly distributed over the a priori interval $[0,\ T_f - T_d]$.
2) The Cramer-Rao region corresponds to a high SNR, in this case the receiver success to match well the signal with very small uncertainty. We observe also that for increasing delay spread T_d, and for the same pulse energy, the increase in error variance is small even if the energy is more spread.
3) The threshold region is located just between the two regions cited above. The estimation error in this case exceeds the CRLB by a large factor and describes more precisely the limit of the estimation error. It is then more realistic bound, especially for UWB systems that are supposed to operate on this range of SNR.

We observe also that the multiframe scheme outperforms greatly the single frame one even with a small number of repetitions.

In order to study the impact of the observation interval length on estimation performance, we plot in Fig. IV the RMSE achieved for signal of delay spread $T_d = 20ns$ Vs. varying observation interval length T_f, and different SNRs. To fairly compare, the transmitted power is adequately adapted for each value of T_f in order to obtain unique average SNR. We observe then that the RMSE decreases with increasing T_f, which means that the ML estimator performs better with increasing signal power even if the search interval is larger.

V. Conclusion

In this paper, we give fundamental limitation to the performance of time-delay estimation of UWB signals by the mean of the improved Ziv-zakai lower bound. The bound is more tight than the Cramer-rao lower bound in the operating low SNR region of UWB systems, is simple to derive, and was applied in the case of maximum likelihood estimator having perfect knowledge of the 2nd. order statistics of the propagation channel. The obtained results show that 1ns accuracy range is achievable even at low average SNR in one shot attempt. This performance can be dramatically improved with multiframe repetition.

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APPENDIX

A. Derivation of the detection error probability

We begin by making a Karhunen-Loeve decomposition of R in the basis of its covariance matrix, $K_0 = U\Lambda U^\dagger$, where Λ is a diagonal matrix with eigenvalues of K_0 as diagonal elements, and U is a unitary matrix formed by the corresponding eigenvectors. R can be written then as

$$R = U\Lambda^{\frac{1}{2}}\dot{R} \quad \text{where} \quad \dot{R} = \Lambda^{-\frac{1}{2}}U^\dagger R \quad , \quad K_{\dot{R}} = I \tag{A-1}$$

So we get

$$Z = \dot{R}^\dagger \dot{Q}_x \dot{R} \quad \text{with} \quad \dot{Q}_x = \Lambda^{\frac{1}{2}}U^\dagger Q_x U \Lambda^{\frac{1}{2}} \tag{A-2}$$

As ${}_1K_{\theta_1,x}$ and ${}_2K_{\theta_1,x}$ are Hermitian, $\dot{Q}_x$ is also Hermitian and can be decomposed also as $VMV^\dagger$ where V is an orthonormal matrix of eigenvectors of $\dot{Q}_x$ and M is a diagonal matrix of corresponding eigenvalues μ_x^i. We can thus write

$$Z = (V^\dagger \dot{R})^\dagger M (V^\dagger \dot{R}) = \sum \mu_x^i |\ddot{R}_i|^2 \tag{A-3}$$

Where $\quad \ddot{R} = V^\dagger \dot{R} = V^\dagger \Lambda^{-\frac{1}{2}} U^\dagger R \, , \quad K_{\ddot{R}} = V K_{\dot{R}} V^\dagger = I$

As R_i are circular complex Gaussian random variables $CN(0,1)$, the random variables U_i defined as $U_i = 2|\ddot{R}_i|^2$ are independent chi-square random variables with two degrees of freedom $\chi(2)$. We have thus expressed Z as weighted sum of N independent Chi-square random variables.

The eigenvalues $\{\mu_x^i\}$ are not necessary equals nor distinct, so the closed form expression of the distribution of Z is not tractable. However, a linear combination of chi-square variables can be well approximated by a Gamma distributed variable [7], [8] that have the same first and second moments.

We split the set of eigenvalues as $a_x^i = \{\mu_x^i \ , \ \mu_x^i \geq 0\}$ and $b_x^i = \{|\mu_x^i| \ , \ \mu_x^i < 0\}$. Z can then be given as

$$Z = Z_1 - Z_2 \quad \text{Where} \quad Z_1 = \sum \frac{a_x^i}{2} U_i \quad Z_2 = \sum \frac{b_x^i}{2} U_i \tag{A-4}$$

And the probability of decision error becomes

$$P_d(x) = \frac{1}{2}[P_{H_1}(Z1 - Z2 > 0) + P_{H_2}(Z1 - Z2 < 0)] \tag{A-5}$$

We approximate then Z_1 and Z_2 as a gamma distributed variables $G_1(\alpha_1, \beta_1)$ and $G_2(\alpha_2, \beta_2)$. By equating the first moments we obtain

$$\alpha_1 = \frac{(\sum a_x^i)^2}{\sum (a_x^i)^2} \quad , \quad \beta_1 = \frac{\sum (a_x^i)^2}{\sum a_x^i} \tag{A-6}$$

$$\alpha_2 = \frac{(\sum b_x^i)^2}{\sum (b_x^i)^2} \quad , \quad \beta_2 = \frac{\sum (b_x^i)^2}{\sum b_x^i} \tag{A-7}$$

From [9], we have

$$F(Z_1 - Z_2 \leq 0) = \frac{\beta_1^{\alpha_2}\beta_2^{\alpha_1} \, {}_2F_1\left(1, \alpha, \alpha_1 + 1, \frac{\beta_2}{\beta}\right)}{\alpha_1 B(\alpha_1, \alpha_2)(\beta)^\alpha} \tag{A-8}$$

$$\text{with} \qquad \alpha = \alpha_1 + \alpha_2 \, , \quad \beta = \beta_1 + \beta_2$$

Where ${}_2F_1$ is the Gauss hypergeometric function.