Now using Little's formula, since the arrival rate is $2/3n$, we conclude that

$$E[D_4] = E[A]E[\tilde{Q}] = \frac{3n}{2}O(\log n) = O(n \log n). \qquad \square$$

VI. Conclusion

In this correspondence, we studied the maximal throughput scaling and the corresponding delay scaling in a random mobile network with restricted node mobility. In [2], it was shown that a particular mobility restriction does not affect the throughput scaling. In this correspondence, we showed that it does not affect delay scaling either. In particular, we show that delay scales as $D(n) = \Theta(n \log n)$ for a network of n nodes, which is the same as the delay scaling without any mobility restriction. This was understood to be a consequence of the fact that in spite of an apparent restriction, essentially node mobility remains unchanged in the sense that: i) each node meets every other node for $\Theta(1/n)$ fraction of the time with only $\Theta(1)$ other neighboring nodes; and ii) the intermeeting time of nodes has mean of $\Theta(n)$ and variance of $O(n^2 \log n)$.

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CDMA Systems With Correlated Spatial Diversity: A Generalized Resource Pooling Result

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***Abstract*—This correspondence analyzes the behavior of code-division multiple-access (CDMA) systems with correlated spatial diversity. The users transmit to one or more antenna arrays. The centralized receiver employs a linear multiuser detector. We derive the performance of a large system with random spreading sequences and weak assumptions on the flat-fading channel gains—the fading may be correlated and contain line-of-sight components. We show that, as the number of users and the spreading factor grow large with fixed ratio, the performance of the system is fully characterized by a square matrix with size equal to the number of receiving antennas and multiuser efficiencies are not identical for all users. Our general result includes the analysis of CDMA systems with spatial diversity discussed by Hanly and Tse ('01) for independent channel gains in case of both micro-diversity and macro-diversity and provides a rigorous proof for the macro-diversity case missing in their work. We also show that to any scenario with correlated Rayleigh fading, there exists a macro-diversity scenario with independent Rayleigh fading which is characterized by the same signal-to-interference-and-noise ratio (SINR). Furthermore, sufficient conditions are given which force the multiuser efficiencies of all users to become identical also in case of statistically dependent channel gains.**

***Index Terms*—Antenna array, code-division multiple access (CDMA), correlated channels, large system analysis, line-of-sight components, multiuser detection, random spreading, resource pooling, spatial diversity.**

I. Introduction

Modeling of spreading matrices in code-division multiple-access (CDMA) systems by random matrices has been extremely fruitful from both the theoretical perspective of system analysis, see the seminal works of [2], [3], and [4], and from the practical point of view of receiver design, e.g., [5]. In the large system limit, as both the transmitted signals K and the spreading factor N tend to infinity with a fixed ratio, certain functions of random matrices show self-averaging properties. This allows for the description of the system in terms of few macroscopic system parameters and provides deep insights into the system behavior. Modeling the spreading matrices as random matrices, Hanly and Tse [1] analyzed a CDMA system consisting of users transmitting to a multiuser receiver with spatial diversity. The spatial diversity can be obtained by multiple antenna elements at a single base station, or by combining of signals received at multiple base stations. These two cases of spatial diversity are referred to as micro-diversity and macro-diversity, respectively. This celebrated

Manuscript received July 24, 2004; revised November 14, 2006. This work was supported in part by the Kplus Programme of the Austrian Government and the European Network of Excellence NEWCOM.. The material in this correspondence was presented in part at the 2004 International Symposium on Information Theory and Its Applications (ISITA2004), Parma, Italy, October 2004.

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Communicated by X. Wang, Associate Editor for Detection and Estimation.
Digital Object Identifier 10.1109/TIT.2006.890784

TABLE I
ASYMPTOTIC CONSTANTS CHARACTERIZING CDMA SYSTEMS WITH CORRELATED AND INDEPENDENT SPATIAL DIVERSITY

Micro-diversity [1] • Channel gains independent for all users and antennas. • For a given user, the channel gains at all receiving antennas are identically distributed.	$\boldsymbol{A} = a\boldsymbol{I}_L$
Macro-diversity [1] • Channel gains independent for all users and antennas.	$\boldsymbol{A} = \begin{pmatrix} a_1 & 0 & \ddots & \ddots \\ 0 & a_2 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & 0 & a_L \end{pmatrix}$
General case • Correlated channel gains.	$\boldsymbol{A} = \boldsymbol{A}^H$

work covered many interesting aspects of CDMA systems with spatial diversity.

- There is a simple relation between the degrees of freedom introduced by spatial diversity (L receiving antennas) and the degrees of freedom in frequency given by spread-spectrum techniques (spreading factor N): the multiple-antenna system behaves like a system with a single receive antenna but with spreading factor multiplied by the number of receiving antennas, and with the received power of each user being the sum of the received powers at the individual antennas. This behavior is known as the resource pooling effect. It shows the possibility to trade bandwidth (spreading factor) for the number of antennas and *vice versa* according to the peculiarity of the communication system.
- The effect of a single interferer on the user of interest is captured by the concept of effective interference.
- A power control algorithm, the power-limited capacity region for finite number of classes of users, and the interference-limited user capacity are defined.

The work in [1] is based on the performance analysis of linear multiuser receivers under the assumption that the spreading sequences are Gaussian and the random channel gains are circular symmetric and independent for all users and antennas, and for any user the gains to all antennas are identically distributed. The analysis does not span cases of practical interest like multiple-antenna element systems with correlated channels and/or line-of-sight components.

The pioneering works in [6] and [7] on antenna arrays at the transmitter and the receiver promise huge increases in the throughput of wireless communication systems. Therefore, they motivated the flurry of a rich output of works that study the capacities of such systems in more realistic situations. In this stream are works that analyze the effects of channel correlation [8]–[14], line-of-sight components [15]–[17], multiple scattering [18], and keyholes [11] (this list does not claim to be comprehensive). Fading correlation and line-of-sight components were found to affect channel capacity severely. It is natural and of practical interest to consider their effects also in CDMA systems with spatial diversity.

In this correspondence, we consider a general framework with one or more antenna arrays at the receive side including combined micro- and macro-diversity scenarios. The transmitting users may use multiple-element antennas, but need not do so. The channel gains may be correlated and contain line-of-sight components, i.e., their mean may be different from zero. The analysis is based on the assumption of independent random spreading. Our general result includes the results in [1] as special cases. Additionally, we provide a rigorous proof of the results for the macro-diversity case, only conjectured in [1].

In the micro-diversity case with independent channel gains analyzed in [1], the system behavior is captured by the multiuser efficiency, a normalized signal-to-interference-and-noise ratio (SINR) defined in [19], conditioned on the fading state of the user of interest. The multiuser efficiency is shown to converge to a deterministic constant in the large system limit. In the macro-diversity case with L receiving antennas, L constants, $a_1, a_2, \ldots, a_L$, characterize the system. With correlated channel gains, we show that the large system behavior is captured by a deterministic positive definite Hermitian matrix $\boldsymbol{A}$ with size equal to the number of receive antennas.

Table I compares the scenarios investigated in [1] with the general case considered in this work. The results in [1] are revisited in the light of the general results so that all scenarios are represented by a matrix $\boldsymbol{A}$.

- In the micro-diversity scenario with independent channels, $\boldsymbol{A}$ is the identity matrix multiplied by the constant multiuser efficiency a.
- In the macro-diversity case with independent channels, $\boldsymbol{A}$ is a diagonal matrix.

In this contribution, we consider three linear receivers corresponding to different levels of knowledge of the interference structure and noise at the receiver.

- Linear minimum mean-square error (MMSE) receiver, which requires a complete knowledge of the spreading sequences and the channel gains of the interferers.
- Single-user Bayesian receiver, which assumes only a statistical knowledge of the spreading sequences and the channel gains of the interferers;
- Single-user matched filter (SUMF) receiver. In this case, the receiver has no information about noise and interference.

Thanks to the assumption of independence among the chips, the analysis shows that these linear receivers are not affected by channel correlation due to coupling effects among transmitting antennas and suffer only from channel correlations due to coupling effects among receiving antennas. For large CDMA systems without receive antenna diversity, the multiuser efficiency fully characterizes the system since it is identical for all users. In contrast, we show that the multiuser efficiency in CDMA systems with spatial diversity changes from user to user, in general. Additionally, we give sufficient conditions under which also a system with linear MMSE at the receiver, spatial diversity and statistically dependent channel gains is characterized by a unique multiuser efficiency.

The single-user Bayesian receiver and the SUMF receiver are shown to be asymptotically equivalent, in terms of SINR, to a finite CDMA system with i) linear MMSE detector and SUMF, respectively, at the receiver; ii) spreading factor L; and iii) spreading sequences equal to the vector of the channel gains.

II. Notations

Throughout this work, the superscripts $\cdot^T$ and $\cdot^H$ denote the transpose and the conjugate transpose of the matrix argument, respectively. $\boldsymbol{I}_n$ is the identity matrix of size $n \times n$ and $\mathbb{C}$ and $\mathbb{R}$ are the fields of complex and real numbers, respectively. $\mathrm{tr}(\cdot)$, $\|\cdot\|$, and $|\cdot|$ are the trace, the Frobenius norm, and the spectral norm of the argument, respectively, i.e., $\|\boldsymbol{A}\| = \sqrt{\mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^H)}$, $|\boldsymbol{A}| = \max_{\boldsymbol{x}^H\boldsymbol{x}\le 1} \boldsymbol{x}^H\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{x}$. $\mathrm{E}\{\cdot\}$ is the expectation operator. δ_{ij} is the Kronecker symbol and $\delta(\lambda)$ is the Dirac's delta function. $\boldsymbol{X} = (x_{ij})_{i=1,\dots,n_1}^{j=1,\dots,n_2}$ is the $n_1 \times n_2$ matrix whose (i,j)-element is the scalar x_{ij}. $\boldsymbol{X} = (\boldsymbol{X}_{ij})_{i=1,\dots,n_1}^{j=1,\dots,n_2}$ is the $n_1q_1 \times n_2q_2$ block matrix whose (i,j)-block is the $q_1 \times q_2$ matrix $\boldsymbol{X}_{ij}$. $\otimes$ and $\wedge$ denote the Kronecker product and the logical "AND," respectively. $\boldsymbol{e}_l$ is the L-dimensional unit column vector whose elements are zero except the lth that equals 1, i.e., $\boldsymbol{e}_l = (\delta_{lj})_{j=1\dots L}$. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ are the real and imaginary parts of the argument, respectively. mod denotes the modulus and $\lfloor\cdot\rfloor$ is the operator that yields the maximum integer not greater than its argument. Furthermore, $\chi(\boldsymbol{x} \in \mathcal{A})$ denotes the indicator function of the multivariate random variable $\boldsymbol{x}$ on the set $\mathcal{A}$ and $\chi(\boldsymbol{x} \in \mathcal{A}) = 1$ if $\boldsymbol{x} \in \mathcal{A}$ and it is zero otherwise.

III. System Model

We consider a CDMA system with spreading factor N and K' users. Each user employs a transmit antenna array with N_{T} elements sending independent data streams through each of the elements. Thus, we may speak of a system with $K = K'N_{\mathrm{T}}$ virtual users. The signal is received by L receive antennas. These antennas can be part of an array or can be placed at different locations, but processed jointly.

The baseband discrete-time system model, as the channel is flat fading, is given by

$$\boldsymbol{y} = \boldsymbol{\mathcal{H}}\boldsymbol{b} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{y}$ is the NL-dimensional vector of received signal samples, $\boldsymbol{b}$ is the K-dimensional vector of transmitted symbols, and $\boldsymbol{n}$ is discrete-time, circularly symmetric complex-valued additive white Gaussian noise with zero mean and variance σ^2. The influence of spreading and fading is described by the $NL \times K$ matrix

$$\boldsymbol{\mathcal{H}} = \sum_{l=1}^{L} (\boldsymbol{S}\boldsymbol{T}\boldsymbol{\Lambda}_l) \otimes \boldsymbol{e}_l \tag{2}$$

where $\boldsymbol{S}$ is the $N \times K$ spreading matrix whose kth column is the spreading sequence of the kth virtual user. Furthermore, the diagonal matrix $\boldsymbol{T} \in \mathbb{C}^{K\times K}$ contains the transmitted amplitudes of all virtual users such that its kth diagonal element t_k is the amplitude of the signal transmitted by the virtual user indexed by k. The diagonal matrices $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \dots, \boldsymbol{\Lambda}_L \in \mathbb{C}^{K\times K}$ take into account the effects of the flat-fading channels. The kth diagonal element of $\boldsymbol{\Lambda}_l$ is the channel gain between the transmitting antenna element of the kth virtual user and the lth receive antenna and will be denoted by λ_{lk} in the following. The channel gains can be, in general, correlated and contain line of sight components as in Rice channels.

In the following, the spreading matrix $\boldsymbol{S}$ is modeled as a complex random matrix whose elements are independent[1] with zero mean, variance $\frac{1}{N}$, and fourth moment such that there exists a $\gamma > 1$ for which $\mathrm{E}\{|s_{ij}|^4\} \le \frac{1}{N^\gamma}$ with $i = 1,\dots,N$ and $j = 1,\dots,K$. This condition is satisfied by all practically relevant choices of chips, like Gaussian or binary chips. Moreover, we assume the transmitted symbols to be uncorrelated and identically distributed random variables with zero mean and unit variance, i.e., $\mathrm{E}\{\boldsymbol{b}\boldsymbol{b}^H\} = \boldsymbol{I}_K$. In order to simplify notation, it will be helpful in the following to define the L-dimensional vectors $\boldsymbol{l}_k = t_k[\lambda_{1k}, \lambda_{2k}, \dots, \lambda_{Lk}]^T$, $k = 1,\dots,K$, and the corresponding L-variate random variable $\boldsymbol{l} = t[\lambda_1, \lambda_2, \dots, \lambda_L]^T$.

[1]Note that the random variables s_{nk} are not required to be identically distributed.

IV. Linear MMSE Receiver

The linear MMSE detector generates a soft decision $\widehat{b}_k = \boldsymbol{c}_k^H\boldsymbol{y}$ based on the observation $\boldsymbol{y}$. The linear MMSE detector $\boldsymbol{c}_k$ for the detection of b_k, the transmitted symbol of virtual user k, can be derived from the Wiener–Hopf theorem [20] for the estimation of zero-mean random variables. It is given by

$$\boldsymbol{c}_k = \mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\}^{-1}\mathrm{E}\{b_k^*\boldsymbol{y}\} \tag{3}$$

with the expectation taken over all variables that are unknown to the receiver, i.e., the transmitted symbols $\boldsymbol{b}$ and the noise. Specializing the Wiener–Hopf equation to the system model (1) yields

$$\begin{aligned} \boldsymbol{c}_k &= (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{h}_k && (4)\\ &= c\cdot(\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{h}_k && (5)\end{aligned}$$

for some $c \in \mathbb{R}$. Here, $\boldsymbol{h}_k$ denotes the kth column of $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}_k$ is the $NL \times (K-1)$ matrix obtained from $\boldsymbol{\mathcal{H}}$ suppressing the kth column $\boldsymbol{h}_k$. The second step follows from the matrix inversion lemma. Its performance is measured by the signal-to-interference-and-noise ratio SINR_k at its output which is well known [19] to be given by

$$\mathrm{SINR}_k = \boldsymbol{h}_k^H(\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{h}_k. \tag{6}$$

The SINR_k can be conveniently expressed in terms of the multiuser efficiency η_k

$$\mathrm{SINR}_k = \frac{\|\boldsymbol{l}_k\|^2}{\sigma^2}\eta_k. \tag{7}$$

The multiuser efficiency η_k is defined as the ratio between the SINR for the user k at the output of the multiuser detector of interest and the SINR for the same user in absence of multiuser interference. The multiuser efficiency is a useful measure, since it is identical to all users in special cases [19].

A. General Case

Let us notice that SINR_k depends on the spreading sequences and the channel parameters of all the virtual users. To get deeper insights on the linear MMSE behavior it is convenient to analyze the performance as $K, N \to \infty$ with constant ratio $\beta = \frac{K}{N}$. To this aim, we have to define how the matrices $\boldsymbol{T}, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \dots, \boldsymbol{\Lambda}_L$ behave as the system grows large. Let us consider a system with K virtual users and the K corresponding $(L+1)$-variate random variables $(t_k, \lambda_{1k}, \lambda_{2k}, \dots, \lambda_{Lk})$ for $k = 1,\dots,K$. The empirical joint distribution function for the random variables $(t_k, \lambda_{1k}, \lambda_{2k}, \dots, \lambda_{Lk})$ for $k = 1,\dots,K$ is the distribution function

$$F^{(K)}_{\boldsymbol{T},\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2,\dots,\boldsymbol{\Lambda}_L}(t,\lambda_1,\lambda_2,\dots,\lambda_L) = \frac{1}{K} \times\sum_{k=1}^{K}\chi((t\ge t_k)\wedge(\lambda_1\ge\lambda_{1k})\wedge(\lambda_2\ge\lambda_{2k})\wedge\cdots\wedge(\lambda_L\ge\lambda_{Lk})).$$

We assume that the joint empirical distribution $F^{(K)}_{\boldsymbol{T},\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2,\dots,\boldsymbol{\Lambda}_L}(t,\lambda_1,\lambda_2,\dots,\lambda_L)$ converges weakly with probability 1 to a limit distribution function $F_{\boldsymbol{T},\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2,\dots,\boldsymbol{\Lambda}_L}(t,\lambda_1,\lambda_2,\dots,\lambda_L)$

with bounded support. Let us notice that, if $(t_k, \lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{Lk})$ for all k, are independent realizations of a common cumulative distribution function (cdf), then the empirical distribution function $F^{(K)}(t, \lambda_1, \lambda_2, \ldots, \lambda_L)$ is the natural estimate of the common cdf. The Glivenko–Cantelli theorem guaranties that, if $(t_k, \lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{Lk})$ are independent and identically distributed (i.i.d.) in k, then the empirical distribution converges weakly to the common distribution function with probability 1. For example, if, for each virtual user k, $(t_k, \lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{Lk})$ is a realization of the same Gaussian distribution $F(t, \lambda_1, \lambda_2, \ldots, \lambda_L)$, then the Glivenko–Cantelli lemma guarantees that the sequence of the empirical distribution functions converges almost surely to the same distribution function $F(t, \lambda_1, \lambda_2, \ldots, \lambda_L)$.

In the following, when possible, we will use the limiting joint distribution of the received amplitudes $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ rather than the limit distribution $F_{\boldsymbol{T}, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \ldots, \boldsymbol{\Lambda}_L}(t, \lambda_1, \lambda_2, \ldots, \lambda_L)$ in order to simplify the notation. $F_{\boldsymbol{l}}$ is obtained by the projection $(l_1, l_2, \ldots, l_L) = (t\lambda_1, t\lambda_2, \ldots, t\lambda_L)$. Under these assumptions the asymptotic performance depends on a small set of parameters, as shown by the following theorem.

Theorem 1: Let $\boldsymbol{S}$ be an $N \times K$ random matrix with independent entries. Let its elements s_{ij} be zero mean, with variance $\mathrm{E}\{|s_{ij}|^2\} = \frac{1}{N}$ and forth moment $\mathrm{E}\{|s_{ij}|^4\} \leq \frac{1}{N^\gamma}$ where $\gamma > 1$. Let $\boldsymbol{l} = [l_1, l_2, \ldots, l_L]$ and let $\boldsymbol{l}_k$ be the vector of received amplitudes of the virtual user k. Let us assume that the norm of the channel gain vector $\|\boldsymbol{l}_k\|$ is uniformly bounded for all K. Furthermore, the empirical joint distribution of $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_{k-1}, \boldsymbol{l}_{k+1}, \ldots \boldsymbol{l}_K$ converges almost surely to some limiting joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ as $K \to \infty$. Then, as $N, K \to \infty$ with $\frac{K}{N} \to \beta$ and L fixed, the SINR of virtual user k, given the fading amplitude $\boldsymbol{l}_k$, converges in probability to the value

$$\lim_{K,N\to\infty} \mathrm{SINR}_k = \frac{\boldsymbol{l}_k^H \boldsymbol{A} \boldsymbol{l}_k}{\sigma^2} \tag{8}$$

where $\boldsymbol{A}$ is the unique deterministic $L \times L$ Hermitian matrix solution to the matrix-valued fixed point equation

$$\boldsymbol{A}^{-1} = \boldsymbol{I}_L + \beta \int \frac{\boldsymbol{l}\boldsymbol{l}^H}{\sigma^2 + \boldsymbol{l}^H \boldsymbol{A} \boldsymbol{l}} \, \mathrm{d}F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L) \tag{9}$$

such that $\boldsymbol{A}$ is positive definite for any positive value of the noise variance σ^2.

Proof: See Appendix II.

Theorem 1 provides the asymptotic output SINR of a linear MMSE detector for a synchronous CDMA system with correlated spatial diversity. This result holds under very general conditions on the channel gains and demonstrates interesting and useful properties of synchronous CDMA systems with correlated spatial diversity and linear MMSE detector at the receiver. The remainder of this section is devoted to the discussion of these properties. More specifically, in Section IV-B, Theorem 1 is specialized to the relevant situation of practical interest where the received amplitudes are correlated Gaussian distributed. In Section IV-C, Theorem 1 is utilized to derive sufficient conditions under which the resource pooling effect arises. General properties of CDMA systems with correlated spatial diversity evinced from Theorem 1 are presented in Section IV-D.

B. *Correlated Gaussian Received Amplitudes*

In practice, fading amplitudes are often complex Gaussian distributed and correlated. Rayleigh fading also violates the demand for uniformly bounded channel gains. However, it can be approximated arbitrary closely by a distribution with bounded support. Thus, from an engineering perspective, we need not worry about that fact.[2] Assume that the limiting joint distribution is given as

$$f_{\boldsymbol{l}}(\boldsymbol{l}) = \frac{1}{\pi^L \det \boldsymbol{C}_{\boldsymbol{l}}} \exp\left(-\boldsymbol{l}^H \boldsymbol{C}_{\boldsymbol{l}}^{-1} \boldsymbol{l}\right). \tag{10}$$

In the absence of power control, i.e., $\boldsymbol{T} = \boldsymbol{I}_K$, this implies that $\boldsymbol{C}_{\boldsymbol{l}}$ is the correlation matrix of the fading at the receive side with entries $(\boldsymbol{C}_{\boldsymbol{l}})_{ij} = \mathrm{E}\left\{\lambda_i \lambda_j^*\right\}$. Consider the eigenvalue decomposition

$$\boldsymbol{C}_l = \boldsymbol{M} \boldsymbol{\Psi} \boldsymbol{M}^H \tag{11}$$

with $\boldsymbol{\Psi} = \mathrm{diag}(\psi_1, \ldots, \psi_L)$ and the change of variables

$$\boldsymbol{g} = \boldsymbol{M}^H \boldsymbol{l} \tag{12}$$

$$\boldsymbol{g}_k = [g_{1k}, \ldots, g_{Lk}]^T = \boldsymbol{M}^H \boldsymbol{l}_k \tag{13}$$

creating statistically independent components in the random vector $\boldsymbol{g}$. Plugging (12) into (9), we see that the matrix $\boldsymbol{M}$ also diagonalizes the deterministic limit matrix $\boldsymbol{A}$, i.e., the eigenvectors of the matrix $\boldsymbol{A}$ coincide with the eigenvectors of the correlation matrix $\boldsymbol{C}_{\boldsymbol{l}}$. Thus, we obtain for correlated Rayleigh fading

$$\lim_{K,N\to\infty} \mathrm{SINR}_k = \frac{1}{\sigma^2} \sum_{\ell=1}^{L} a_\ell |g_{\ell k}|^2 \tag{14}$$

where the coefficients a_ℓ, $\ell = 1 \ldots L$, are solutions of the fixed point equations

$$a_\ell = \frac{1}{1 + \frac{\beta}{\pi^L} \int \frac{\psi_\ell |x_\ell|^2}{\sigma^2 + \sum_{n=1}^{L} a_n \psi_n |x_n|^2} \prod_{n=1}^{L} \exp(-|x_n|^2) \mathrm{d}\, x_n} \tag{15}$$

for $\ell = 1, 2 \ldots, L$. Let us notice that, in the case of correlated Rayleigh spatial diversity, the system of L^2 fixed-point equation (9) reduces to a system of L fixed-point equations as in the macro-diversity case with independent channel gains analyzed in [1] (cf. Table I in this correspondence). Furthermore, the CDMA system with correlated Rayleigh spatial diversity characterized by the limiting joint distribution of the channel gains (10) is equivalent in performance to a CDMA network with macro-diversity and with independent Rayleigh channel gains. For this equivalent CDMA network, the virtual channel gains of user k are given in (13) and the virtual limiting joint distribution of the channel gains is Gaussian with covariance matrix $\boldsymbol{\Psi}$ defined in (11). Therefore, it is apparent that to any correlated Rayleigh-fading scenario, there exists an equivalent macro-diversity scenario with independent Rayleigh fading.

C. *Uncorrelated Received Amplitudes*

It is clear from (7) and (8) that unless the matrix $\boldsymbol{A}$ is a multiple of the identity matrix, multiuser efficiency is, in general, not unique for all the virtual users. In this subsection, we analyze under which conditions on the limiting joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ or, equivalently, on the corresponding limiting pdf $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ the matrix $\boldsymbol{A}$ is diagonal or proportional to the identity matrix. In fact, for diagonal $\boldsymbol{A}$, the general result in Theorem 1 simplifies to the system of fixed-point equations in [1, Theorem 3]. The following corollary summarizes some sufficient conditions that yield a diagonal structure of $\boldsymbol{A}$.

Corollary 1: Let $\boldsymbol{S}$ and $\boldsymbol{l}_k$ be as in Theorem 1. If the joint pdf $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$, for any r, is an even function of $\mathrm{Re}(l_r)$ and $\mathrm{Im}(l_r)$ for any value of the parameters $(l_1, \ldots, l_{k-1}, l_{k+1}, \ldots, l_L)$ then as

[2] In fact, real-world channel gains are always bounded. The infinite support Rayleigh distribution is just a close approximation of the distribution for real-world bounded channel gains.

$N, K \to \infty$ with $\frac{K}{N} \to \beta$ and L fixed, the SINR of virtual user k, given the fading amplitude $\boldsymbol{l}_k$, converges in probability to the value

$$\lim_{K,N\to\infty} \mathrm{SINR}_k = \frac{1}{\sigma^2} \sum_{\ell=1}^{L} a_\ell |l_{\ell k}|^2 \tag{16}$$

where a_ℓ, $\ell = 1 \ldots L$, are the unique positive solutions to the system of fixed-point equations

$$a_\ell = \frac{1}{1 + \beta \int \frac{|l_\ell|^2}{\sigma^2 + \sum_{n=1}^{L} a_n |l_n|^2} f_{\boldsymbol{l}}(l_1, \ldots, l_L) \mathrm{d}l_1 \ldots \mathrm{d}l_L} \tag{17}$$

for $\ell = 1, \ldots, L$.

Proof: Corollary 1 is proven if the system of (9) reduces to the system of (17) under the conditions on $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ required by Corollary 1. This is verified if, for all $i, j = 1, \ldots, L$, with $i \neq j$, the off-diagonal elements of $\boldsymbol{A}$ are zero. The uniqueness of the solution for system (9) guarantees that the constants a_ℓ are the solutions we are looking for. In fact, $\forall i, j = 1, \ldots, L$ and $i \neq j$ the off-diagonal elements of $\boldsymbol{A}$ are given by

$$\int \frac{l_i l_j f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)}{\sigma^2 + \sum_{\ell=1}^{L} a_\ell |l_\ell|^2} \mathrm{d}l_1 \mathrm{d}l_2 \ldots \mathrm{d}l_L$$
$$= \int l_i \left(\int \frac{l_j f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)}{\sigma^2 + \sum_{\ell=1}^{L} a_\ell |l_\ell|^2} \mathrm{d}l_j \right) \mathrm{d}l_1 \ldots \mathrm{d}l_{j-1} \mathrm{d}l_{j+1} \ldots \mathrm{d}l_L.$$

Since the function $l_j / (\sigma^2 + \sum_{\ell=1}^{L} a_\ell |l_\ell|^2)$ is an odd function of $\mathrm{Re}(l_j)$ and $\mathrm{Im}(l_j)$, the integral with respect to l_j will be always zero if $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ is an even function in $\mathrm{Re}(l_j)$ and $\mathrm{Im}(l_j)$. Since this property is satisfied for all random variables l_j with $j = 1, \ldots, L$ then all the off-diagonal elements of $\boldsymbol{A}$ are zero and this concludes the proof of Corollary 1.

Following the same approach used for Corollary 4 in [1] and using Corollary 1, we find sufficient conditions under which the matrix $\boldsymbol{A}$ is proportional to the identity matrix, i.e., $\boldsymbol{A} = \eta \boldsymbol{I}$. If $\boldsymbol{A} = \eta \boldsymbol{I}$, then the scalar η coincides with the multiuser efficiency of a linear MMSE detector as it is apparent from (7) and (8). Let us assume that the conditions of Corollary 1 are satisfied. If we additionally assume that the joint pdf $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ is exchangeable, i.e., for any permutation π of $\{1, \ldots, L\}$

$$f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L) = f_{\boldsymbol{l}}(l_{\pi(1)}, l_{\pi(2)}, \ldots, l_{\pi(L)})$$

then the system of (17), which defines the diagonal matrix $\boldsymbol{A}$, satisfies $a_\ell = \eta$, for all $\ell = 1, \ldots, L$, $\boldsymbol{A} = \eta \boldsymbol{I}$, and the system of (17) reduces a single fixed-point equation. This result is stated in the following corollary.

Corollary 2: Let $\boldsymbol{S}$ and $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ be as in Corollary 1. If the limiting pdf $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ is exchangeable, i.e., for any permutation π of $\{1, \ldots, L\}$

$$f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L) = f_{\boldsymbol{l}}(l_{\pi(1)}, l_{\pi(2)}, \ldots, l_{\pi(L)})$$

then, as $N, K \to \infty$ with $\frac{K}{N} \to \beta$ and L fixed, $\frac{\mathrm{SINR}_k \sigma^2}{P_k}$, with $P_k = \|\boldsymbol{l}_k\|^2$ converges in probability to a deterministic constant η which is the unique scalar multiuser efficiency, solution to the fixed-point equation

$$\eta = \frac{1}{1 + \frac{\beta}{L} \int \frac{P}{\sigma^2 + \eta P} \mathrm{d}\, F_P(P)}.$$

Here, P is the random variable defined by $P = \|\boldsymbol{l}\|^2$ and $F_P(P)$ is its distribution.

The conditions of Corollaries 1 and 2 imply that $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_K$ are uncorrelated. However, the converse is not true in general, i.e., the matrix $\boldsymbol{A}$ is not typically diagonal for asymptotically uncorrelated received amplitudes. Corollaries 1 and 2 provide sufficient conditions such that the matrix $\boldsymbol{A}$ is diagonal and proportional to the identity matrix, respectively, when the received amplitudes are asymptotically uncorrelated. Under the conditions of Corollary 2, the resource pooling effect arises. In fact, the multiuser efficiency of a synchronous CDMA system with L receive antennas and spreading factor N is equal to the multiuser efficiency of a synchronous CDMA system with a single receive antenna, with spreading factor NL, and with the received power of each virtual user being the sum of the received powers at the individual antennas [1], [3].

D. Remarks

The empirical joint distributions $F_{\boldsymbol{l}}^{(K)}(l_1, l_2, \ldots, l_L)$ and the limiting joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ are not able to capture and describe the effects of the correlation due to antenna coupling at the transmitter side. Since the effects of the channel gains on the system performance are taken into account only by $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$, we can conclude that the correlations of the channel gains due to coupling effects at the transmitter side do not effect the asymptotic performance of the linear MMSE receiver. This property is intrinsically related to the assumption of the statistical independence of the spreading sequences of the transmitting antennas. It does not hold true if the condition of independence is not satisfied. In fact, in this case $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ would not be sufficient to describe the system behavior.

As a consequence of Theorem 1, the asymptotic behavior of the general system is completely described by a an $L \times L$ matrix $\boldsymbol{A}$. In contrast to the case of single receive antenna or the cases in which the resource pooling effect arises, the multiuser efficiency of the linear MMSE receiver varies from user to user in general. In particular, for virtual user k, it depends on the direction of the channel gains $\boldsymbol{l}_k$ with respect to the eigenvectors of $\boldsymbol{A}$: The SINR is maximum if $\boldsymbol{l}_k$ has the same direction as the eigenvector corresponding to the maximum eigenvalue of $\boldsymbol{A}$.

Typically, in order to determine the eigenvectors of the matrix $\boldsymbol{A}$, the solution of the matrix fixed point (9) is required. However, in the special case that the limiting pdf $f_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ is zero-mean Gaussian, the eigenvectors of $\boldsymbol{A}$ coincide with the eigenvectors of the covariance matrix $\mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\}$.

In light of Theorem 1, we can revisit the known results in [1]. Theorem 1 in [1] states that if the elements s_{ij} are i.i.d. Gaussian random variables with zero mean and variance $\frac{1}{N}$, if the received amplitudes are independent for all users and antennas, if the received amplitudes are identically distributed for a given user, and if the sequence of the empirical distributions converges asymptotically to a bounded distribution function, then the matrix $\boldsymbol{A}$ in (8) is given by

$$\boldsymbol{A} = a \boldsymbol{I}_L$$

with

$$a = \left(1 + \frac{\beta}{L} \mathrm{E}\left(\frac{P}{\sigma^2 + aP}\right)\right)^{-1}$$

where $P = \boldsymbol{l}^H \boldsymbol{l}$.

By comparing this result to the result in Corollary 1 it becomes evident that Corollary 2 implies Theorem 1 in [1].

The following result is conjectured in Theorem 3 in [1]. If the chip elements s_{ij} are independent, zero mean, Gaussian distributed, if the received signal amplitudes are independent, and if the sequence of the empirical distribution converges to a bounded distribution function, then Theorem 3 in [1] conjectures that the matrix $\boldsymbol{A}$ in (8) is given by

$$\boldsymbol{A} = \mathrm{diag}(a_1, a_2, \ldots, a_L)$$

TABLE II
SUMMARY OF COROLLARY 1 AND COROLLARY 2

Assumptions	Condition A	The joint probability density function $f(l_1, l_2, \ldots, l_L)$ is an even function of $Re(l_i)$ and $Im(l_i)$ for $i = 1, \ldots, L$. $\Downarrow$ Received amplitudes uncorrelated at the receiver.
	Condition B	$f(l_1, l_2, \ldots, l_L)$ is exchangeable, i.e. for any permutation π of $\{1, 2, \ldots, L\}$ $f(l_1, l_2, \ldots, l_L) = f(l_{\pi(1)}, l_{\pi(2)}, \ldots, l_{\pi(L)})$
Implications	Condition A Conditions A $\wedge$ B	$\boldsymbol{A}$ is diagonal $\boldsymbol{A} = a\boldsymbol{I}$
Conclusions	The resource pooling effect arises if Condition A $\wedge$ B are satisfied.	

with

$$a_\ell = \left(1 + \beta \mathrm{E}\left(\frac{|l_\ell|^2}{\sigma^2 + \sum_{n=1}^{L} a_n |l_n|^2}\right)\right)^{-1}, \qquad \ell = 1, \ldots, L.$$

By comparing the previous conjecture to Corollary 1, we notice that Corollary 1 includes and proves rigorously Theorem 3 in [1].

In Table II, we recapitulate the results of Corollaries 1 and 2 and summarize the sufficient conditions under which the resource pooling effect arises.

V. SINGLE-USER BAYESIAN RECEIVER

The single-user Bayesian receiver is the linear detector, optimum in a mean-squared error sense, when the receiver is synchronized and has complete information about the virtual user of interest, i.e., spreading sequence, and received power, but it does not know the spreading sequences of the interferers and has only statistical knowledge of the interference. More specifically, we assume that the following information is known at the receiver:

- knowledge of the signature sequence, channel gains, and transmit power of virtual user k;
- knowledge of the statistics of the signature sequences, the channel gains, and transmit powers of all interferers.

This detector has been analyzed for the case of independent channel gains in [1] under the denomination of matched filter.

The single-user Bayesian detector $\boldsymbol{c}_k$ for the virtual user of interest k that minimizes the mean-squared error between its output, $\widehat{\boldsymbol{b}}_{Bf,k} = \boldsymbol{c}_k^H \boldsymbol{y}$, and the transmitted symbol is given by the Wiener–Hopf equation

$$\boldsymbol{c}_k = \mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\}^{-1}\mathrm{E}\{b_k^* \boldsymbol{y}\} \tag{18}$$

as for the linear MMSE receiver. However, in this case, the expectation operator is taken not only over the transmitted signals and the noise, as for the linear MMSE receiver, but also with respect to the signature sequences, the channel gains, and the transmit powers of all interferers. Equation (18) yields the following explicit expression for the Bayesian filter:

$$\boldsymbol{c}_k = \frac{(\boldsymbol{I}_N \otimes ((\beta - \frac{1}{N})\boldsymbol{C}_{\boldsymbol{l}} + \sigma^2 \boldsymbol{I}_L)^{-1})\boldsymbol{h}_k}{1 + (\boldsymbol{l}_k^H((\beta - \frac{1}{N})\boldsymbol{C}_{\boldsymbol{l}} + \sigma^2 \boldsymbol{I}_L)^{-1}\boldsymbol{l}_k)(\boldsymbol{s}_k^H \boldsymbol{s}_k)} \tag{19}$$

with $\boldsymbol{C}_{\boldsymbol{l}} = \mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\}$. A better insight into the Bayesian filter receiver can be obtained from (19) by performing a permutation Π of the elements of $\boldsymbol{c}_k$ and $\boldsymbol{y}_k$ such that the elements corresponding to the same antenna are relocated next to each other ($\Pi : i \rightarrow ((i-1)\mathrm{mod}L)N + \lfloor \frac{i}{L} \rfloor + 1$). Let us denote with $\boldsymbol{c}_k^\Pi$ and $\boldsymbol{y}^\Pi$ the Bayesian filter receiver and the received signal vector obtained by such a permutation. Let $\boldsymbol{\xi}_k = ((\beta - \frac{1}{N})\boldsymbol{C}_{\boldsymbol{l}} + \sigma^2 \boldsymbol{I}_L)^{-1}\boldsymbol{l}_k$ and let ξ_l be the lth element of $\boldsymbol{\xi}_k$. Then

$$\boldsymbol{c}_k^\Pi = \frac{\boldsymbol{\xi}_k \otimes \boldsymbol{s}_k}{1 + \boldsymbol{l}_k^H \boldsymbol{\xi}_k \boldsymbol{s}_k^H \boldsymbol{s}_k}. \tag{20}$$

Equation (21) shows that, similarly to the case of completely independent channel gains, the Bayesian filter despreads the received signal at each antenna using the spreading sequence $\boldsymbol{s}_k$ and, then, it performs a maximal ratio combining of the despread signals using as weight the coefficients

$$\frac{\xi_l}{1 + \boldsymbol{l}_k^H \boldsymbol{\xi}_k \boldsymbol{s}_k^H \boldsymbol{s}_k}, \qquad l = 1, \ldots, L. \tag{21}$$

The coefficients for maximal ratio combining depend on the correlation matrix of the channel gains of the interferers.

The following theorem provides the performance of the Bayesian filter in terms of its limiting SINR as the system dimensions grow large with constant ratio.

Theorem 2: Let $\boldsymbol{l}_k$ be the vector of received amplitudes of virtual user k. Let us assume that, almost surely, the empirical joint distribution of $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_{k-1}, \boldsymbol{l}_{k+1}, \ldots, \boldsymbol{l}_K$ converges to some limiting joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \ldots, l_L)$ as $K \rightarrow \infty$. Additionally, the elements of the spreading vector $\boldsymbol{s}_k$ are assumed to be i.i.d. with zero mean and variance $\mathrm{E}|s_{jk}|^2 = \frac{1}{N}$. Then, if $N, K \rightarrow \infty$ with $\frac{K}{N} \rightarrow \beta$ and L fixed, SINR_k of the Bayesian filter for the transmitted signal k, conditioned on the vector of received amplitudes $\boldsymbol{l}_k$, converges almost surely to a constant value

$$\lim_{\substack{K,N \rightarrow \infty \\ \frac{K}{N} \rightarrow \beta}} \mathrm{SINR}_k \stackrel{a.s.}{=} \boldsymbol{l}_k^H (\beta \mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\} + \sigma^2 \boldsymbol{I}_L)^{-1} \boldsymbol{l}_k \tag{22}$$

where $\boldsymbol{l}$ is the L-variate random variable with joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \ldots l_L)$.

Proof: See Appendix III.

The asymptotic analysis provides a result of simple interpretation: the SINR of the virtual user k is equivalent to the SINR at the output of a linear MMSE detector for a CDMA system with the following characteristics.

- Spreading factor equal to the number of receiving antennas.
- Spreading sequence of the virtual user of interest equal to the vector $\boldsymbol{l}_k$ of channel gains.
- Spreading sequences of the interferers equal to the vectors of the channel gains attenuated by a factor $\sqrt{\beta}$. This takes into account the beneficial effects of the spreading in the original CDMA system.

In contrast to the case of independent channel gains in [1], the performance depends on the direction of the vector of the channel gains. For a given received power, the SINR is maximized as $\boldsymbol{l}_k$ has the direction of the eigenvector corresponding to the minimum eigenvalue of the correlation matrix $\mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\}$.

VI. Matched Filter

The single-user matched filter requires only the knowledge of the spreading sequence of the virtual user of interest. Its output is given by

$$\widehat{b}_{mf,k} = \boldsymbol{h}_k^H \boldsymbol{y}.$$

As in the case of the single-user Bayesian receiver, the matched filter despreads the received signals at each antenna and then it combines the despread signals using as weighting coefficients the received energy at each antenna.

The asymptotic performance of the single-user matched filter is given by the following theorem.

Theorem 3: Let $\boldsymbol{l}_k$, $\boldsymbol{s}_k$, and $F_{\boldsymbol{l}}(l_1, l_2, \dots l_L)$ be as in Theorem 2. Then, if $N, K \to \infty$ with $\frac{K}{N} \to \beta$ and L fixed, SINR_k of the matched filter for the transmitted signal k, conditioned on the vector of received amplitudes $\boldsymbol{l}_k$, converges almost surely to a constant value

$$\lim_{\substack{K,N\to\infty \\ \frac{K}{N}\to\beta}} \mathrm{SINR}_k \overset{a.s.}{=} \frac{(\boldsymbol{l}_k^H \boldsymbol{l}_k)^2}{\boldsymbol{l}_k^H (\beta \mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\} + \sigma^2 \boldsymbol{I}_L)\boldsymbol{l}_k} \tag{23}$$

where $\boldsymbol{l}$ is the L-variate random variable with joint distribution $F_{\boldsymbol{l}}(l_1, l_2, \dots .l_L)$.

Theorem 3 is proven in Appendix IV.

The SINR of the matched filter is equivalent to the SINR of a matched filter for a CDMA system with spreading factor L, spreading sequence of the virtual user of interest equal to the vector of the channel gains, and the spreading sequence of the interferers equal to their channel gains attenuated by a factor $\sqrt{\beta}$.

The multiuser efficiency depends on the direction of $\boldsymbol{l}_k$. It is maximized when $\boldsymbol{l}_k$ has the direction of the eigenvector corresponding to the minimum eigenvalue of the correlation matrix of the interferers $\mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\}$.

VII. Conclusion

In this contribution, we determined the asymptotic performance of linear MMSE detector, the single-user Bayesian receiver, and the single-user matched filter receiver in CDMA systems with random spreading and spatial diversity. We consider the general case where the channel gains are correlated and there are line-of-sight components.

When a linear MMSE detector is used at the receiver, our general Theorem 1 shows that the system is asymptotically described by an $L \times L$ matrix $\boldsymbol{A}$ that characterizes completely the effects of channel correlation and line-of-sight components. Our result includes as special cases the results in [1] that were derived under the constraints of independence of the channel gains, uniformly distributed phases, and Gaussian spreading. Deriving the results in [1] from the general equations (8) and (9) we could prove the results for the macro-diversity case, which was only conjectured in [1]. The performance analysis shows that the efficiency of the system in recovering the symbol transmitted by the virtual user k strongly depends on the direction of the channel gain vector $\boldsymbol{l}_k$ with respect to the eigenvectors of $\boldsymbol{A}$. However, the system performance is asymptotically independent of coupling effects at the transmitting antennas.

The single-user Bayesian filter and the single-user matched filter in a large CDMA scenario with correlated spatial diversity were shown to be equivalent, in terms of performance, to a linear MMSE detector and a matched filter, respectively, in a CDMA system with spreading factor L and spreading sequences equal to the channel gains.

Appendix I
Mathematical Tools

The following mathematical tools are developed along the lines of the REFORM method proposed by Girko in [21] and [22]. The complete derivation of the results utilized in this work was omitted in [21] and it is presented here. We will make large use of the following well-known inequalities:

$$|\boldsymbol{A}+\boldsymbol{B}| \le |\boldsymbol{A}| + |\boldsymbol{B}| \quad \text{Triangular Inequality for Spectral Norm} \tag{24}$$

$$|\boldsymbol{A}\boldsymbol{B}| \le |\boldsymbol{A}||\boldsymbol{B}| \quad \text{Sub-multiplicative Inequality for Spectral Norm} \tag{25}$$

$$\|\boldsymbol{A}+\boldsymbol{B}\|^2 \le 2\|\boldsymbol{A}\|^2 + 2\|\boldsymbol{B}\|^2 \tag{26}$$

$$\|\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}\|^2 \le 3\|\boldsymbol{A}\|^2 + 3\|\boldsymbol{B}\|^2 + 3\|\boldsymbol{C}\|^2 \tag{27}$$

$$\|\boldsymbol{A}\boldsymbol{B}\| \le |\boldsymbol{A}|\|\boldsymbol{B}\| \tag{28}$$

$$\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^H + \boldsymbol{B}\boldsymbol{A}^H) \le \|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\|^2 \tag{29}$$

where $\boldsymbol{A} \in \mathbb{C}^{a_1 \times a_2}$ and $\boldsymbol{B} \in \mathbb{C}^{b_1 \times b_2}$ are matrices with consistent dimensions, i.e., $a_1 = b_1$ and $a_2 = b_2$ for $\boldsymbol{A}+\boldsymbol{B}$ and $a_2 = b_1$ for $\boldsymbol{A}\boldsymbol{B}$.

In this section we adopt the following notation:

(i) $n \in \mathbb{Z}^+$ is a parameter.

(ii) $n_1, n_2, p_1, p_2, q_1, q_2 \in \mathbb{Z}^+$, with $n_1 = n_1(n)$, $n_2 = n_2(n)$, $p_1 = p_1(n)$, $p_2 = p_2(n)$, and $n_1 = p_1 q_1$, $n_2 = p_2 q_2$. n_1, n_2, p_1, p_2 are increasing functions of n and $n_1, n_2, p_1, p_2 \to \infty$ as $n \to \infty$.

(iii) $\Xi^{(n)} = (\xi_{ij}^{(n)})_{i=1,\dots n_1}^{j=1,\dots n_2}$ is an $n_1 \times n_2$ matrix with complex random elements ξ_{ij}. The superscript (n) is omitted as not necessary, i.e., $\Xi = \Xi^{(n)}$.

(iv) $\widetilde{\Xi}^{(n)}$ is obtained from $\Xi^{(n)}$ by structuring $\Xi^{(n)}$ in $p_1 p_2$ blocks of size $q_1 \times q_2$, Ξ_{ij}, i.e.,
$$\widetilde{\Xi}^{(n)} = \Xi^{(n)} = (\Xi_{ij})_{i=1,\dots p_1}^{j=1,\dots p_2}.$$
In the following, we use $\widetilde{\Xi}^{(n)}$ instead of $\Xi^{(n)}$ to stress the fact that the matrix is block-structured. As the parameter n varies, we obtain the sequence of random matrices $\widetilde{\Xi} = \{\widetilde{\Xi}^{(n)}\}$. The superscript (n) is omitted when not necessary, i.e., $\widetilde{\Xi} = \widetilde{\Xi}^{(n)}$.

(v) $\widetilde{\boldsymbol{S}} = \widetilde{\Xi}\widetilde{\Xi}^H$ is a $p_1 \times p_1$ matrix of complex $q_1 \times q_1$ blocks $\boldsymbol{S}_{ij}$, i.e., $\widetilde{\boldsymbol{S}} = (\boldsymbol{S})_{i,j=1,\dots,p_1}$.

(vi) $\widetilde{\boldsymbol{P}} = \widetilde{\Xi}^H\widetilde{\Xi}$ is a $p_1 \times p_1$ matrix of complex $q_2 \times q_2$ blocks $\boldsymbol{P}_{ij}$, i.e., $\widetilde{\boldsymbol{P}} = (\boldsymbol{P}_{ij})_{i,j=1,\dots,p_2}$.

(vii) $\alpha = t + is \in \mathbb{C}$ with $s \neq 0$ and $t \ge 0$.

(viii) $\widetilde{\boldsymbol{Q}} = (\boldsymbol{Q}_{ij})_{ij=1,\dots,p_1} = [\widetilde{\boldsymbol{S}} + \alpha \boldsymbol{I}]^{-1}$, where $\boldsymbol{Q}_{ij}$ are complex blocks of size $q_1 \times q_1$.

(ix) $\widetilde{\boldsymbol{G}} = (\boldsymbol{G}_{ij})_{ij=1,\dots,p_2} = [\widetilde{\boldsymbol{P}} + \alpha \boldsymbol{I}]^{-1}$, where $\boldsymbol{G}_{ij}$ are complex blocks of size $q_2 \times q_2$.

(x) $\vec{\Xi}_k$ is the kth block row of $\widetilde{\Xi}$.

(xi) $\vec{\Xi}(k)$ is a $q_1 \times (p_2 - 1)q_2$ matrix obtained by suppressing the kth block from $\vec{\Xi}_k$, i.e.,
$$\vec{\Xi}(k) = \{\widehat{\Xi}_{k\ell}, \ell = 1, \dots, p_2 - 1\} = \{\Xi_{k\ell}, \ell \neq k, \ell = 1, \dots, p_2\}.$$
The previous relation defines implicitly the $q_1 \times q_2$ blocks $\widehat{\Xi}_{k\ell}$ as $\widehat{\Xi}_{k\ell} = \widetilde{\Xi}_{k,\ell-\chi(\ell>k)}$.

(xii) $\widetilde{\Xi}\begin{pmatrix} j_1, j_2, \dots, j_r \\ k_1, k_2, \dots, k_s \end{pmatrix}$ is a matrix obtained from $\widetilde{\Xi}$ suppressing the s block rows $k_1, k_2, \dots, k_s$ and the r block columns $j_1, j_2, \dots, j_r$.

(xiii) $\widetilde{\boldsymbol{S}}\begin{pmatrix} j_1, j_2, \dots, j_r \\ k_1, k_2, \dots, k_s \end{pmatrix} = \widetilde{\Xi}\begin{pmatrix} j_1, j_2, \dots, j_r \\ k_1, k_2, \dots, k_s \end{pmatrix} \widetilde{\Xi}^H \begin{pmatrix} j_1, j_2, \dots, j_r \\ k_1, k_2, \dots, k_s \end{pmatrix}$.

(xiv) $\widetilde{\boldsymbol{P}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = \widetilde{\boldsymbol{\Xi}}^H\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\widetilde{\boldsymbol{\Xi}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$.

(xv) $\widetilde{\boldsymbol{Q}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = \left(\widetilde{\boldsymbol{S}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) + \alpha\boldsymbol{I}\right)^{-1}$.

(xvi) $\widetilde{\boldsymbol{G}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = \left(\widetilde{\boldsymbol{P}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) + \alpha\boldsymbol{I}\right)^{-1}$.

(xvii) $\vec{\boldsymbol{S}}_k$ is a $q_1 \times (p_1-1)q_1$ matrix obtained by suppressing the kth $q_1 \times q_1$ block from the kth block row of $\widetilde{\boldsymbol{S}}$, i.e.,

$$\vec{\boldsymbol{S}}_k = \{\widehat{\boldsymbol{S}}_{ku}, u=1,\dots p_1-1\} = \{\widetilde{\boldsymbol{S}}_{ku}, u\neq k, u=1,\dots,p_1\}.$$

The $p_1 \times p_1$ blocks $\widehat{\boldsymbol{S}}_{ku}$ are implicitly defined as $\widehat{\boldsymbol{S}}_{ku} = \widetilde{\boldsymbol{S}}_{k,u+\chi(u>k)}$.

(xviii) $\vec{\boldsymbol{S}}_k\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is a $q_1 \times (p_1-s-1)q_1$ matrix obtained by suppressing the kth $q_1 \times q_1$ block from the kth block row of $\widetilde{\boldsymbol{S}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$, i.e.,

$$\begin{aligned}\vec{\boldsymbol{S}}_k&\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\\ &= \left\{\widehat{\boldsymbol{S}}_{ks}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right), s=1,\dots p_1-s-1\right\}\\ &= \left\{\left(\widetilde{\boldsymbol{S}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\right)_{kj}, j\neq k, j=1,\dots p_1-s\right\}.\end{aligned}$$

The $q_1 \times q_1$ blocks $\widehat{\boldsymbol{S}}_{kj}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ are implicitly defined as

$$\widehat{\boldsymbol{S}}_{kj}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = \left(\widetilde{\boldsymbol{S}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\right)_{k,j+\chi(j>k)}.$$

(xix) $\widetilde{\boldsymbol{H}}$ is the $p_1q_1 \times p_2q_2$ block matrix given by $\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{\Xi}} - \mathrm{E}(\widetilde{\boldsymbol{\Xi}})$. $\boldsymbol{H}_{k\ell}$ is the block (k,ℓ) of $\widetilde{\boldsymbol{H}}$ of dimensions $q_1 \times q_2$.

(xx) $\widetilde{\boldsymbol{H}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is a matrix with (p_1-s) block rows and p_2-r block columns obtained from $\widetilde{\boldsymbol{H}}$ suppressing the s block rows $k_1,k_2,\dots,k_s$ and the r block columns $j_1,j_2,\dots,j_r$.

(xxi) $\vec{\boldsymbol{H}}_k$ is the kth block row of the matrix $\widetilde{\boldsymbol{H}}$.

(xxii) $\vec{\boldsymbol{H}}(k)$ is a $q_1 \times (p_2-1)q_2$ matrix obtained by suppressing the kth block from $\vec{\boldsymbol{H}}_k$, i.e.,

$$\vec{\boldsymbol{H}}(k) = \{\widehat{\boldsymbol{H}}_{k\ell}, \ell=1,\dots p_2-1\} = \{\boldsymbol{H}_{k\ell}, \ell\neq k, \ell=1,\dots,p_2\}.$$

The previous relation defines implicitly the $q_1 \times q_2$ blocks $\widehat{\boldsymbol{H}}_{k\ell}$ as $\widehat{\boldsymbol{H}}_{k\ell} = \widetilde{\boldsymbol{H}}_{k,\ell-\chi(\ell>k)}$.

(xxiii) $\widetilde{\boldsymbol{H}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is a matrix obtained from $\widetilde{\boldsymbol{H}}$ suppressing the s block rows $k_1,k_2,\dots,k_s$ and the r block columns $j_1,j_2,\dots,j_r$.

(xxiv) $\boldsymbol{A} = (\boldsymbol{A}_{kj})_{k=1,\dots,p_1}^{j=1,\dots,p_2}$ is the $p_1 \times p_2$ matrix of $q_1 \times q_2$ blocks given by $\widetilde{\boldsymbol{A}} = \mathrm{E}(\widetilde{\boldsymbol{\Xi}})$.

(xxv) $\widetilde{\boldsymbol{A}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is a matrix obtained from $\widetilde{\boldsymbol{A}}$ suppressing the s block rows $k_1,k_2,\dots,k_s$ and the r block columns $j_1,j_2,\dots,j_r$.

(xxvi) $\vec{\boldsymbol{A}}_k$ is the kth block row of the matrix $\widetilde{\boldsymbol{A}}$.

(xxvii) $\vec{\boldsymbol{A}}(k)$ is a $q_1 \times (p_2-1)q_2$ matrix obtained by suppressing the kth block from $\vec{\boldsymbol{A}}_k$, i.e.,

$$\vec{\boldsymbol{A}}(k) = \{\widehat{\boldsymbol{A}}_{k\ell}, \ell=1,\dots p_2-1\} = \{\boldsymbol{A}_{k\ell}, \ell\neq k, \ell=1,\dots,p_2\}.$$

The previous relation defines implicitly the $q_1 \times q_2$ blocks $\widehat{\boldsymbol{A}}_{k\ell}$ as $\widehat{\boldsymbol{A}}_{k\ell} = \widetilde{\boldsymbol{A}}_{k,\ell-\chi(\ell>k)}$.

(xxviii) $\widetilde{\boldsymbol{D}}^{(1)} = \mathrm{diag}(\boldsymbol{D}_{ii}^{(1)})_{i=1,\dots,p_1}$ is a $p_1q_1 \times p_1q_1$ block-diagonal Hermitian matrix with p_1 blocks of dimensions $q_1 \times q_1$. $\widetilde{\boldsymbol{D}}^{(1)}(k_1,k_2,\dots,k_s)$ is a $(p_1-s)q_1 \times (p_1-s)q_1$ matrix obtained from $\widetilde{\boldsymbol{D}}^{(1)}$ suppressing the k_1th, k_2th, $\dots,k_s$th block rows and columns.

(xxix) $\widetilde{\boldsymbol{D}}^{(2)} = \mathrm{diag}(\boldsymbol{D}_{ii}^{(2)})_{i=1,\dots,p_2}$ is a $p_2q_2 \times p_2q_2$ block-diagonal Hermitian matrix with p_2 blocks of dimensions $q_2 \times q_2$. $\widetilde{\boldsymbol{D}}^{(2)}(j_1,j_2,\dots,j_r)$ is a $(p_2-r)q_2 \times (p_2-r)q_2$ matrix obtained from $\widetilde{\boldsymbol{D}}^{(2)}$ suppressing the j_1th, j_2th, $\dots,j_r$th block rows and columns.

(xxx) $\widetilde{\boldsymbol{M}} = (\boldsymbol{M})_{i=1,\dots,p_1}^{j=1,\dots,p_2}$ is a $p_1q_1 \times p_2q_2$ complex matrix structured in p_1p_2 blocks of size $q_1 \times q_2$.

(xxxi) $\vec{\boldsymbol{M}}_k$ is the kth block row of $\widetilde{\boldsymbol{M}}$.

(xxxii) $\widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is the matrix with (p_1-s) block rows and p_2-r block columns obtained from $\widetilde{\boldsymbol{M}}$ suppressing the s block rows $k_1,k_2,\dots,k_s$ and the r block columns $j_1,j_2,\dots,j_r$.

(xxxiii) $\widetilde{\boldsymbol{U}} = (\boldsymbol{U}_{ij})_{i,j=1,\dots,p_1} = (\widetilde{\boldsymbol{D}}^{(1)} + \widetilde{\boldsymbol{M}}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^H)^{-1}$, where $\boldsymbol{U}_{ij}$ are complex blocks of size $q_1 \times q_1$. Similarly as in (xxxi), $\widetilde{\boldsymbol{U}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is defined in the first equation at the bottom of the page.

(xxxiv) $\widetilde{\boldsymbol{V}} = (\boldsymbol{V}_{ij})_{i,j=1,\dots,p_1} = (\widetilde{\boldsymbol{D}}^{(2)} + \widetilde{\boldsymbol{M}}^H(\widetilde{\boldsymbol{D}}^{(1)})^{-1}\widetilde{\boldsymbol{M}})^{-1}$, where $\boldsymbol{V}_{ij}$ are complex blocks of size $q_2 \times q_2$. Similarly as in (xxxi) $\widetilde{\boldsymbol{V}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is defined in the second equation at the bottom of the page.

(xxxv) $\widetilde{\boldsymbol{Z}} = (\boldsymbol{Z}_{i,j})_{i,j=1,\dots,p_2} = \widetilde{\boldsymbol{M}}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^H$. $\vec{\boldsymbol{Z}}_k$ is a $q_1 \times (p_1-1)q_1$ matrix obtained by suppressing the kth block from the kth block row of $\widetilde{\boldsymbol{Z}}$, i.e.,

$$\begin{aligned}\vec{\boldsymbol{Z}}_k &= \{\widehat{\boldsymbol{Z}}_{ks}, s=1,\dots,p_1-1\}\\ &= \{\widetilde{\boldsymbol{Z}}_{ks}, s\neq k, s=1,\dots,p_1\}\\ &= \vec{\boldsymbol{M}}_k(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^H\left(\begin{smallmatrix} - \\ k \end{smallmatrix}\right).\end{aligned}$$

The $q_1 \times q_1$ blocks $\widehat{\boldsymbol{Z}}_{ks}$ are implicitly defined as $\widehat{\boldsymbol{Z}}_{ks} = \widetilde{\boldsymbol{Z}}_{k,s+\chi(s>k)}$.

(xxxvi) $\widetilde{\boldsymbol{Z}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)$ is a $q_1(p_1-s) \times q_1(p_1-s)$ matrix obtained as

$$\begin{aligned}\widetilde{\boldsymbol{Z}}&\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\\ &= \widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)\left(\widetilde{\boldsymbol{D}}^{(2)}(j_1,j_2,\dots,j_r)\right)^{-1}\widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)^H.\end{aligned}$$

Lemma 1: Let $\widetilde{\boldsymbol{C}}$ be a block matrix of size $p_1q_1 \times p_1q_1$. $\widetilde{\boldsymbol{A}}$ is a nonsingular block matrix of size $p_1q_1 \times p_1q_1$ Then

$$\frac{\partial}{\partial\gamma}\ln\det[\widetilde{\boldsymbol{A}} + \gamma\widetilde{\boldsymbol{C}}]\Big|_{\gamma=0} = \mathrm{Tr}(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{-1}).$$

$$\widetilde{\boldsymbol{U}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = (\widetilde{\boldsymbol{D}}^{(1)}(k_1,k_2,\dots,k_s) + \widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)(\widetilde{\boldsymbol{D}}^{(2)}(j_1,j_2,\dots,j_r))^{-1}\widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)^H)^{-1}.$$

$$\widetilde{\boldsymbol{V}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right) = (\widetilde{\boldsymbol{D}}^{(2)}(j_1,j_2,\dots,j_r) + \widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right)^H(\widetilde{\boldsymbol{D}}^{(1)}(k_1,k_2,\dots,k_s))^{-1}\widetilde{\boldsymbol{M}}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r \\ k_1,k_2,\dots,k_s \end{smallmatrix}\right))^{-1}.$$

$$\mathrm{Tr}(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{U}}) = \frac{\partial}{\partial\gamma}\ln\det\left(\begin{array}{c|c}\widetilde{\boldsymbol{Z}}\binom{-}{k} + \widetilde{\boldsymbol{D}}^{(1)}(k) & \vec{\boldsymbol{Z}}_k^H \\ \hline \vec{\boldsymbol{Z}}_k + \gamma\vec{\boldsymbol{C}}_{k\ell} & \boldsymbol{Z}_{kk} + \boldsymbol{D}_{kk}^{(1)} + \gamma\chi(k=\ell)\boldsymbol{C}\end{array}\right). \tag{32}$$

$$\mathrm{Tr}(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{U}}) = \frac{\partial}{\partial\gamma}\ln\left(\det\left(\widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)\right)\det(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}+\gamma\chi(k=\ell)\boldsymbol{C}-(\vec{\boldsymbol{Z}}_k+\gamma\vec{\boldsymbol{C}}_{k\ell})\times\left(\widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)\right)^{-1}\vec{\boldsymbol{Z}}_k^H)\right) \tag{33}$$

$$= \frac{\partial}{\partial\gamma}\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H+\gamma\chi(\ell=k)\boldsymbol{C}-\gamma\boldsymbol{C}\left(\chi(\ell<k)\sum_{j=1}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\ell j}\widehat{\boldsymbol{Z}}_{kj}^H+\chi(\ell>k)\sum_{j=1}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\ell-1,j}\widehat{\boldsymbol{Z}}_{kj}^H\right)\right) \tag{34}$$

$$= \begin{cases}\mathrm{Tr}\left(\boldsymbol{C}(\boldsymbol{Z}_{kk}+\widetilde{\boldsymbol{D}}^{(1)}(k)-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H)^{-1}\right), & k=\ell\\ -\mathrm{Tr}\left(\boldsymbol{C}\left(\sum_{j=1}^{p_1-1}(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\ell j}\widehat{\boldsymbol{Z}}_{kj}^H\right)\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\right)\vec{\boldsymbol{Z}}_k^H\right)^{-1}, & \ell<k\\ -\mathrm{Tr}\left(\boldsymbol{C}\sum_{j=1}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\ell-1,j}\widehat{\boldsymbol{Z}}_{kj}^H\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\right)\vec{\boldsymbol{Z}}_k^H\right)^{-1}, & \ell>k\end{cases} \tag{35}$$

Proof: Let $\lambda_i(\cdot)$ denote the ith eigenvalue of the matrix argument in general not real for non-Hermitian matrices. Lemma 1 can be derived as follows:

$$\begin{aligned}\frac{\partial}{\partial\gamma}\ln\det(\widetilde{\boldsymbol{A}}+\gamma\widetilde{\boldsymbol{C}})\Big|_{\gamma=0} &= \frac{\partial}{\partial\gamma}\ln\det((\boldsymbol{I}+\gamma\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{-1})\widetilde{\boldsymbol{A}})\Big|_{\gamma=0}\\ &= \frac{\partial}{\partial\gamma}\ln\left(\det(\boldsymbol{I}+\gamma\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}})\det(\widetilde{\boldsymbol{A}}^{-1})\right)\Big|_{\gamma=0}\\ &= \frac{\partial}{\partial\gamma}\ln\prod_{i=1}^{p_1q_1}(1+\gamma\lambda_i(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{-1}))|_{\gamma=0}\\ &= \sum_{i=1}^{p_1q_1}\lambda_i(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{-1})\\ &= \mathrm{Tr}(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{A}}^{-1}).\end{aligned}$$

Lemma 2: Definitions (xxviii), (xxix), (xxx), (xxxi), (xxxii), (xxxiii), (xxxiv) hold. Then

$$\boldsymbol{U}_{\ell k} = -\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{U}}\binom{-}{k}_{\ell-\chi(\ell>k),j}\widehat{\boldsymbol{Z}}_{kj}^H\right)\boldsymbol{U}_{kk}, \qquad k\neq\ell, k,\ell=1,\dots p_1 \tag{30}$$

$$\boldsymbol{U}_{kk} = \left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right)^{-1}. \tag{31}$$

Proof: Let $\widetilde{\boldsymbol{C}}$ be a block matrix of size $p_1q_1 \times p_1q_1$ having all $q_1 \times q_1$ blocks equal to zero except $\boldsymbol{C}_{k\ell} = \boldsymbol{C}$. $\vec{\boldsymbol{C}}_{k\ell}$ is a $q_1 \times q_1(p_1-1)$ row vector block with the $(\ell-1)$th or the ℓth $q_1 \times q_1$ block equal to $\boldsymbol{C}$ if $k<\ell$ or $k>\ell$, respectively, and zero elsewhere. Then, the application of Lemma 1 to the matrices $\widetilde{\boldsymbol{C}}$ and $\widetilde{\boldsymbol{U}}$ yields (32) at the top of the page. The rule of determinant of block matrices[3] applied to (32) yields (33)–(35) at the top of the page, and (35) follows from Lemma 1. By choosing $\boldsymbol{C}$ such that $(\boldsymbol{C})_{ij}=1$ and zero elsewhere, we obtain an expression for $(\boldsymbol{U}_{kk})_{ij}$, the (i,j) element of the block matrix $\boldsymbol{U}_{kk}$

$$(\boldsymbol{U}_{kk})_{ij} = \left(\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right)^{-1}\right)_{ij}.$$

[3]Let $\boldsymbol{A}\in\mathbb{C}^{m\times m}$ be non singular, $\boldsymbol{D}\in\mathbb{C}^{n\times n}$, $\boldsymbol{B}\in\mathbb{C}^{m\times n}$ and $\boldsymbol{C}\in\mathbb{C}^{n\times m}$. Then

$$\det\left(\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{B}\\ \hline \boldsymbol{C} & \boldsymbol{D}\end{array}\right) = \det(\boldsymbol{A})\det(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}).$$

Then, the relation holds also for the full block

$$\boldsymbol{U}_{kk} = \left(\boldsymbol{Z}_{kk}+\boldsymbol{D}_{kk}^{(1)}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right)^{-1}. \tag{36}$$

Using (36) in (35) for $\ell<k$ and $\ell>k$ we obtain

$$(\boldsymbol{U}_{\ell k})_{ij} = \begin{cases}-\left(\sum_{j=1}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\ell j}\widehat{\boldsymbol{Z}}_{kj}^H\boldsymbol{U}_{kk}\right)_{ij}, & \ell<k\\ -\left(\sum_{j=1}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\ell-1,j}\widehat{\boldsymbol{Z}}_{kj}^H\boldsymbol{U}_{kk}\right)_{ij}, & \ell>k.\end{cases}$$

This concludes the proof of Lemma 2.

Lemma 3: Definitions (xxviii), (xxix), (xxx), (xxxi), (xxxii), (xxxiii), (xxxiv) hold and $\widetilde{\ell}=\ell-\chi(\ell>k)$. Then

$$\boldsymbol{U}_{\ell\ell}-\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell}\widetilde{\ell}} = -\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{U}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}_{kj}^H\right)\boldsymbol{U}_{kk}\left\{\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{U}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}_{kj}^H\right\}^H$$

for $\ell\neq k$ and

$$\mathrm{tr}(\widetilde{\boldsymbol{U}})-\mathrm{tr}(\widetilde{\boldsymbol{U}}\binom{-}{k}) = -\mathrm{tr}\left(\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{U}}\binom{-}{k})^2\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk}\right)+\mathrm{tr}(\boldsymbol{U}_{kk}).$$

Proof: Let $\widetilde{\boldsymbol{C}}$ be a block matrix of size $p_1q_1\times p_1q_1$ having all blocks equal to zero except $\boldsymbol{C}_{\ell\ell}=\boldsymbol{C}$ where $\boldsymbol{C}$ is an arbitrary $q_1\times q_1$ matrix. $\widetilde{\boldsymbol{C}}(k)$ is a $(p_1-1)q_1\times(p_1-1)q_1$ block matrix obtained from $\widetilde{\boldsymbol{C}}$ by suppressing the kth block row and the kth block column. $\boldsymbol{C}_k$ and $\boldsymbol{E}_k$ denote two block vectors of dimensions $(p_1-1)q_1\times q_1$ with all $q_1\times q_1$ blocks equal to zero except the kth equal to $\boldsymbol{C}$ and to a $q_1\times q_1$ identity matrix, respectively. Then, $\widetilde{\boldsymbol{C}}(k)=\boldsymbol{C}_{\widetilde{\ell}}\boldsymbol{E}_{\widetilde{\ell}}^H$.

Here, $\lambda_i(\cdot)$ denotes the ith eigenvalue of the matrix argument.

Let $v=\mathrm{tr}(\boldsymbol{C}(\boldsymbol{U}_{\ell\ell}-(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\widetilde{\ell}\widetilde{\ell}}))$. By applying Lemma 1 we obtain

$$\begin{aligned}v &= \mathrm{tr}(\widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{U}})-\mathrm{tr}(\widetilde{\boldsymbol{C}}(k)\widetilde{\boldsymbol{U}}\binom{-}{k})\\ &= \frac{\partial}{\partial\gamma}\left(\ln\det\left(\widetilde{\boldsymbol{Z}}+\widetilde{\boldsymbol{D}}^{(1)}+\gamma\widetilde{\boldsymbol{C}}\right)\right)\Big|_{\gamma=0}\\ &\quad-\frac{\partial}{\partial\gamma}\left(\ln\det\left(\widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)+\gamma\widetilde{\boldsymbol{C}}(k)\right)\right)\Big|_{\gamma=0}.\end{aligned} \tag{37}$$

$$
\begin{aligned}
\upsilon &= \frac{\partial}{\partial\gamma}\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k\left(\widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)+\gamma\widetilde{\boldsymbol{C}}(k)\right)^{-1}\vec{\boldsymbol{Z}}_k^H\right)\Bigg|_{\gamma=0}\\
&= \lim_{\Delta\gamma\to0+}\frac{1}{\Delta\gamma}\left(\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)+\Delta\gamma\widetilde{\boldsymbol{C}}(k))^{-1}\vec{\boldsymbol{Z}}_k^H\right)-\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{Z}}\binom{-}{k}\right.\right.\\
&\quad\left.\left.+\widetilde{\boldsymbol{D}}^{(1)}(k))^{-1}\vec{\boldsymbol{Z}}_k^H\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\upsilon &= \lim_{\Delta\gamma\to0+}\frac{1}{\Delta\gamma}\left(\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right.\right.\\
&\quad\left.\left.-\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{U}}\binom{-}{k}\Delta\gamma\boldsymbol{C}_{\widetilde{\ell}}(\boldsymbol{I}+\Delta\gamma\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k}\boldsymbol{C}_{\widetilde{\ell}})^{-1}\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k})\vec{\boldsymbol{Z}}_k^H\right)-\ln\det\left(\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right)\right)\\
&= \lim_{\Delta\gamma\to0+}\frac{1}{\Delta\gamma}\left(\ln\det\left(\boldsymbol{I}-\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{U}}\binom{-}{k}\Delta\gamma\boldsymbol{C}_{\widetilde{\ell}}(\boldsymbol{I}+\Delta\gamma\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k}\boldsymbol{C}_{\widetilde{\ell}})^{-1}\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k})\vec{\boldsymbol{Z}}_k^H\left(\widetilde{\boldsymbol{Z}}_{kk}+\boldsymbol{D}^{(1)}_{kk}-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\right)^{-1}\right)\right)\\
&= \lim_{\Delta\gamma\to0+}\ln\prod_{i=1}^{p_1-1}\left(1+\Delta\gamma\lambda_i\left(-\vec{\boldsymbol{Z}}_k\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\boldsymbol{C}_{\widetilde{\ell}}(\boldsymbol{I}+\Delta\gamma\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k}\boldsymbol{C}_{\widetilde{\ell}})^{-1}\boldsymbol{E}^H_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k}\right)\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk}\right)\right)^{\frac{1}{\Delta\gamma}}.
\end{aligned}
$$

$$
\begin{aligned}
\upsilon &= \sum_{i=1}^{p_1-1}\lambda_i\left(-\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\widetilde{\boldsymbol{C}}(k)\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk}\right)\\
&= -\operatorname{tr}\left(\vec{\boldsymbol{Z}}_k\widetilde{\boldsymbol{U}}\binom{-}{k}\boldsymbol{C}(k)\widetilde{\boldsymbol{U}}\binom{-}{k}\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk}\right)\\
&= -\operatorname{tr}\left(\left(\sum_{j=1}^{p_1-1}(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}^H_{kj}\right)\boldsymbol{U}_{kk}\left(\sum_{j=1}^{p_1-1}(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}^H_{kj}\right)^H\boldsymbol{C}\right).
\end{aligned}
$$

The application of the rule of determinant for block matrices to the matrix

$$
\widetilde{\boldsymbol{Z}}+\widetilde{\boldsymbol{D}}^{(1)}=\left(\begin{array}{c|c}\boldsymbol{Z}_{kk}+\boldsymbol{D}^{(1)}_{kk} & \vec{\boldsymbol{Z}}_k\\ \hline \vec{\boldsymbol{Z}}_k^H & \widetilde{\boldsymbol{Z}}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)+\gamma\widetilde{\boldsymbol{C}}(k)\end{array}\right)
$$

and (37) yield the first equation at the top of the page.

By applying the Woodbury formula[4] we obtain the second equation at the top of the page.

By applying (31) of Lemma 2 and the limit $\mathrm{e}^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ we obtain the third equation at the top of the page.

Setting a single element of the matrix $\boldsymbol{C}$ equal to 1 and keeping the others equal to zero we can establish the identity between the elements of the matrix $\boldsymbol{U}_{\ell\ell}-\widetilde{\boldsymbol{U}}\binom{-}{k}_{\ell\ell}$ and the elements of the argument of the trace. Therefore

$$
\boldsymbol{U}_{\ell\ell}-\widetilde{\boldsymbol{U}}\binom{-}{k}_{\widetilde{\ell},\widetilde{\ell}}=-\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{U}}_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}^H_{kj}\right)\boldsymbol{U}_{kk}\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{U}}_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}^H_{kj}\right)^H.
$$

Thus

$$
\sum_{\substack{\ell=1\\ \ell\neq k}}^{p_1}\operatorname{tr}(\boldsymbol{U}_{\ell\ell}-(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\widetilde{\ell\ell}})=-\operatorname{tr}(\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{U}}\binom{-}{k})^2\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk})
$$

and

$$
\operatorname{tr}(\widetilde{\boldsymbol{U}})-\operatorname{tr}(\widetilde{\boldsymbol{U}}\binom{-}{k})=-\operatorname{tr}(\vec{\boldsymbol{Z}}_k(\widetilde{\boldsymbol{U}}\binom{-}{k})^2\vec{\boldsymbol{Z}}_k^H\boldsymbol{U}_{kk})+\operatorname{tr}(\boldsymbol{U}_{kk}).
$$

[4]Let $\boldsymbol{A}$ be an $N\times N$ matrix and $\boldsymbol{X},\boldsymbol{Y}$ be $N\times M$ matrices. Then, $(\boldsymbol{A}+\boldsymbol{X}\boldsymbol{Y}^H)^{-1}=\boldsymbol{A}^{-1}-(\boldsymbol{A}^{-1}\boldsymbol{X}(\boldsymbol{I}+\boldsymbol{Y}^H\boldsymbol{A}^{-1}\boldsymbol{X})^{-1}\boldsymbol{Y}^H\boldsymbol{A}^{-1})$.

This concludes the proof of Lemma 3.

Lemma 4: Definitions (i), (ii), (xxviii), (xxix), (xxx), (xxxiii), (xxxiv), (xxxv), (xxxvi) hold. Let $\widetilde{\ell}=\ell-\chi(\ell>k)$. Then

$$
\begin{aligned}
\boldsymbol{U}_{\ell k} &= -\boldsymbol{U}_{\widetilde{\ell\ell}}\binom{-}{k}(\vec{\boldsymbol{M}}_\ell\widetilde{\boldsymbol{V}}\left(\begin{smallmatrix}-\\ k,\ell\end{smallmatrix}\right)\vec{\boldsymbol{M}}_k^H)\boldsymbol{U}_{kk}\\
&\qquad k\neq\ell,\ \text{and } k,\ell=1,\ldots,p_1 \qquad (38)\\
\boldsymbol{U}_{kk} &= (\boldsymbol{D}^{(1)}_{kk}+\vec{\boldsymbol{M}}_k\widetilde{\boldsymbol{V}}\binom{-}{k}\vec{\boldsymbol{M}}_k^H)^{-1}\\
&\qquad k=\ell,\ \text{and } k,\ell=1,\ldots p_1. \qquad (39)
\end{aligned}
$$

Proof: Equation (30) with $\ell\neq k$ can be rewritten as

$$
\boldsymbol{U}_{\ell k}=-\left(\sum_{\substack{j=1\\ j\neq\widetilde{\ell}}}^{p_1-1}\widetilde{\boldsymbol{U}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{Z}}^H_{kj}\right)\boldsymbol{U}_{kk}-\widetilde{\boldsymbol{U}}_{\widetilde{\ell},\widetilde{\ell}}\binom{-}{k}\widehat{\boldsymbol{Z}}^H_{k\ell}\boldsymbol{U}_{kk}. \qquad (40)
$$

Applying (30) to the matrix $\widetilde{\boldsymbol{U}}\binom{-}{k}$ instead of $\widetilde{\boldsymbol{U}}$ we obtain

$$
\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{j\ell}=-\sum_{i=1}^{p_1-2}\left(\widetilde{\boldsymbol{U}}\left(\begin{smallmatrix}-\\ k,\ell\end{smallmatrix}\right)\right)_{\widetilde{j},i}\widehat{\boldsymbol{Z}}^H_{\ell i}\binom{-}{k}\widetilde{\boldsymbol{U}}_{\ell\ell}\binom{-}{k},
$$
$$
j\neq\ell,\ j,\ell=1,\ldots,p_1-1
$$

where $\widetilde{\boldsymbol{U}}_{\ell\ell}\binom{-}{k}$ denotes the (ℓ,ℓ) block of the matrix $\widetilde{\boldsymbol{U}}\binom{-}{k}$ and $\widetilde{j}=j-\chi(j>\ell)$. Since $\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{U}}\binom{-}{k}$, and $\widetilde{\boldsymbol{Z}}\binom{-}{k}$ are Hermitian

$$
(\widetilde{\boldsymbol{U}}\binom{-}{k})_{\ell j}=(\widetilde{\boldsymbol{U}}\binom{-}{k})^H_{j\ell}=-\widetilde{\boldsymbol{U}}_{\ell\ell}\binom{-}{k}\sum_{i=1}^{p_1-2}\widehat{\boldsymbol{Z}}_{\ell i}\binom{-}{k}(\widetilde{\boldsymbol{U}}\left(\begin{smallmatrix}-\\ k,\ell\end{smallmatrix}\right))^H_{\widetilde{j},i},
$$
$$
j\neq\ell,\ j,\ell=1,\ldots,p_1-1. \qquad (41)
$$

Substituting (41) in (40) we obtain

$$\begin{aligned}\boldsymbol{U}_{\ell k} &= \sum_{i=1}^{p_1-2}\sum_{\substack{j=1\\ j\neq\widetilde{\ell}}}^{p_1-1}\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell},\widetilde{\ell}}\widehat{\boldsymbol{Z}}_{\widetilde{\ell},i}\binom{-}{k}\widetilde{\boldsymbol{U}}\binom{-}{k,\ell}_{\widetilde{j},i}^{H}\widehat{\boldsymbol{Z}}_{kj}^{H}\boldsymbol{U}_{kk}\\ &\quad-\left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell},\widetilde{\ell}}\widetilde{\boldsymbol{Z}}_{k,\widetilde{\ell}}^{H}\boldsymbol{U}_{kk}\\ &= \left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell},\widetilde{\ell}}\left(\left(\vec{\boldsymbol{Z}}\binom{-}{k}\right)_{\widetilde{\ell}}\widetilde{\boldsymbol{U}}\binom{-}{k,\ell}\vec{\boldsymbol{Z}}_{k}^{H}\binom{-}{\widetilde{\ell}}-\boldsymbol{Z}_{k\ell}^{H}\right)\boldsymbol{U}_{kk}.\end{aligned}$$

By applying definitions (xxxv) and (xxxiii) and using the following identities:

$$\begin{aligned}\vec{\boldsymbol{Z}}_{\widetilde{\ell}}\binom{-}{k} &= \vec{\boldsymbol{M}}_{\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}\\ \vec{\boldsymbol{Z}}_{k}(\ell-\chi(\ell>k)) &= \vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}\\ \boldsymbol{Z}_{k\ell} &= \vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\vec{\boldsymbol{M}}_{\ell}^{H}\end{aligned}$$

we obtain (42) at the bottom of the page.

The inversion lemma yields the following identity:

$$\begin{aligned}&-(\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(1)}(k,\ell))^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}+\widetilde{\boldsymbol{D}}^{(2)})^{-1}\\ &\quad=(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}(\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}\\ &\qquad+\widetilde{\boldsymbol{D}}^{(1)}(k,\ell))^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}-(\widetilde{\boldsymbol{D}}^{(2)})^{-1}. \quad (43)\end{aligned}$$

Here, (42) and (43) are used to derive

$$\begin{aligned}\boldsymbol{U}_{\ell k} &= -\widetilde{\boldsymbol{U}}_{\widetilde{\ell},\widetilde{\ell}}\vec{\boldsymbol{M}}_{\ell}\left(\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(1)}(k,\ell))^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}+\widetilde{\boldsymbol{D}}^{(2)}\right)^{-1}\\ &\quad\times\vec{\boldsymbol{M}}_{k}^{H}\boldsymbol{U}_{kk}\\ &= -\widetilde{\boldsymbol{U}}_{\widetilde{\ell},\widetilde{\ell}}\vec{\boldsymbol{M}}_{\ell}\widetilde{\boldsymbol{V}}\binom{-}{k,\ell}\vec{\boldsymbol{M}}_{k}^{H}\boldsymbol{U}_{kk}. \quad (44)\end{aligned}$$

Definition (xxxiv) is applied to obtain (44).

By using definitions (xxxv) and (xxxiii) and the identities

$$\boldsymbol{Z}_{kk}=\vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\vec{\boldsymbol{M}}_{k}^{H},\quad \vec{\boldsymbol{Z}}_{k}=\vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k}$$

(31) can be rewritten as follows:

$$\begin{aligned}\boldsymbol{U}_{kk} &= \Bigg(\vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\vec{\boldsymbol{M}}_{k}^{H}+\widetilde{\boldsymbol{D}}_{kk}^{(1)}-\vec{\boldsymbol{M}}_{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k}\\ &\qquad\left(\widetilde{\boldsymbol{M}}\binom{-}{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k}+\widetilde{\boldsymbol{D}}^{(1)}(k)\right)^{-1}\\ &\qquad\widetilde{\boldsymbol{M}}\binom{-}{k}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\vec{\boldsymbol{M}}_{k}^{H}\Bigg)^{-1}\\ &= \left(\vec{\boldsymbol{M}}_{k}\left(\widetilde{\boldsymbol{M}}^{H}\binom{-}{k}(\widetilde{\boldsymbol{D}}^{(1)}(k))^{-1}\widetilde{\boldsymbol{M}}(k)+(\widetilde{\boldsymbol{D}}^{(2)})\right)^{-1}\vec{\boldsymbol{M}}_{k}^{H}+\widetilde{\boldsymbol{D}}_{kk}^{(1)}\right)^{-1} \quad (45)\\ &= \left(\vec{\boldsymbol{M}}_{k}\widetilde{\boldsymbol{V}}\binom{-}{k}\vec{\boldsymbol{M}}_{k}^{H}+\widetilde{\boldsymbol{D}}_{kk}^{(1)}\right)^{-1}. \quad (46)\end{aligned}$$

Equation (45) is derived applying the inversion lemma as in (43). In (46), we make use of definition (xxxiv).

This concludes the proof of Lemma 4.

Specializing Lemmas 2–4 to the matrix $\widetilde{\boldsymbol{Q}}$ by setting $\widetilde{\boldsymbol{D}}^{(1)}=\alpha\boldsymbol{I}$, $\widetilde{\boldsymbol{D}}^{(2)}=\boldsymbol{I}$, $\widetilde{\boldsymbol{M}}=\widetilde{\Xi}$, and $\widetilde{\boldsymbol{Z}}=\widetilde{\boldsymbol{S}}$, we obtain Corollary 3.

Corollary 3: Definitions (i), (ii), (iv), (xii) (v), (vi), (vii), (ix), (x), (viii), (xv), (xiv), (xvi), and (xvii) hold and $\widetilde{\ell}=\ell-\chi(\ell>k)$. Then

$$\boldsymbol{Q}_{\ell k}=-\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{Q}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{S}}_{kj}^{H}\right)\boldsymbol{Q}_{kk},\qquad k\neq\ell,\ k,\ell=1,\dots p_1 \quad (47)$$

$$\boldsymbol{Q}_{kk}=\left(\boldsymbol{S}_{kk}+\alpha\boldsymbol{I}-\vec{\boldsymbol{S}}_{k}\widetilde{\boldsymbol{Q}}\binom{-}{k}\vec{\boldsymbol{S}}_{k}^{H}\right)^{-1} \quad (48)$$

$$\boldsymbol{Q}_{\ell\ell}-\widetilde{\boldsymbol{Q}}\binom{-}{k}_{\ell\ell}=-\left(\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{Q}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{S}}_{kj}^{H}\right)\boldsymbol{Q}_{kk}\left\{\sum_{j=1}^{p_1-1}\widetilde{\boldsymbol{Q}}\binom{-}{k}_{\widetilde{\ell},j}\widehat{\boldsymbol{S}}_{kj}^{H}\right\}^{H},\qquad \ell\neq k \quad (49)$$

$$\mathrm{tr}\widetilde{\boldsymbol{Q}}-\mathrm{tr}\widetilde{\boldsymbol{Q}}\binom{-}{k}=-\,\mathrm{tr}\left(\vec{\boldsymbol{S}}_{k}(\widetilde{\boldsymbol{Q}}\binom{-}{k})^{2}\vec{\boldsymbol{S}}_{k}^{H}\boldsymbol{Q}_{kk}\right)+\mathrm{tr}(\boldsymbol{Q}_{kk}) \quad (50)$$

$$\boldsymbol{Q}_{\ell k}=-\alpha\boldsymbol{Q}_{\ell\ell}\binom{-}{k}(\vec{\Xi}_{\ell}\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\Xi}_{k}^{H})\boldsymbol{Q}_{kk},\qquad k\neq\ell \text{ and } k,\ell=1,\dots,p_1 \quad (51)$$

$$\boldsymbol{Q}_{kk}=\alpha^{-1}(\boldsymbol{I}+\vec{\Xi}_{k}\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\Xi}_{k}^{H})^{-1},\qquad k=\ell \text{ and } k,\ell=1,\dots p_1. \quad (52)$$

Analogous results hold for the blocks of the matrix $\widetilde{\boldsymbol{G}}$ defined in (ix).

Lemma 5: Let $\widetilde{\boldsymbol{A}}$ be a Gram $p_1q_1\times p_1q_1$ block matrix and α be defined as in (vii). Then, the spectral radius of the matrix $\widetilde{\boldsymbol{B}}=(\widetilde{\boldsymbol{A}}+\alpha\boldsymbol{I})^{-1}$ and the spectral radius of each block $\boldsymbol{B}_{ij}$ of $\widetilde{\boldsymbol{B}}$ are upper-bounded as follows:

$$|\widetilde{\boldsymbol{B}}|<|s|^{-1}\quad\text{and}\quad|\boldsymbol{B}_{ij}|<|s|^{-1}. \quad (53)$$

Proof: Let $\lambda_i(\widetilde{\boldsymbol{A}})$ be the eigenvalues of the matrix $\widetilde{\boldsymbol{A}}$. Then, given $\alpha=t+is$ with $t\geq 0$

$$|\widetilde{\boldsymbol{B}}|=\max_{\lambda_i(\widetilde{\boldsymbol{A}})}\left(|\lambda_i(\widetilde{\boldsymbol{A}})+t+is|^{-1}\right)<|\alpha|^{-1}<|s|^{-1}.$$

The upper bound is obtained for $\lambda_i(\widetilde{\boldsymbol{A}})=0$. Let $\boldsymbol{E}_k$ be a $q_1\times p_1q_1$ block matrix with all blocks equal to the null block matrix except the ith block equal to the identity matrix. Then, the submultiplicative inequality for spectral norms (25) yields

$$|\boldsymbol{B}_{ij}|=|\boldsymbol{E}_i\widetilde{\boldsymbol{B}}\boldsymbol{E}_j^{H}|\leq|\boldsymbol{E}_i||\widetilde{\boldsymbol{B}}||\boldsymbol{E}_j^{H}|\leq|s|^{-1}.$$

Lemma 6: Let Definition (iii), (iv), (x), (xii), (viii), (ix), (xix), (xxi), (xxii), (xi), (xxiv), (xxv), (xxvi), (xxvii), (xxv) hold.

$$\begin{aligned}\boldsymbol{U}_{\ell k} &= \left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell},\widetilde{\ell}}\Bigg(\vec{\boldsymbol{M}}_{\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}(\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}^{H}\\ &\qquad+\widetilde{\boldsymbol{D}}^{(1)}\binom{-}{k,\ell})^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}(\boldsymbol{D}^{(2)})^{-1}\vec{\boldsymbol{M}}_{k}^{H}-\vec{\boldsymbol{M}}_{\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\vec{\boldsymbol{M}}_{k}^{H}\Bigg)\boldsymbol{U}_{kk}\\ &= \left(\widetilde{\boldsymbol{U}}\binom{-}{k}\right)_{\widetilde{\ell},\widetilde{\ell}}\vec{\boldsymbol{M}}_{\ell}\left((\widetilde{\boldsymbol{D}}^{(2)})^{-1}\widetilde{\boldsymbol{M}}^{H}\binom{-}{k,\ell}\left(\widetilde{\boldsymbol{U}}\binom{-}{k,\ell}+\widetilde{\boldsymbol{D}}^{(1)}\binom{-}{k,\ell}\right)^{-1}\widetilde{\boldsymbol{M}}\binom{-}{k,\ell}(\widetilde{\boldsymbol{D}}^{(2)})^{-1}-(\widetilde{\boldsymbol{D}}^{(2)})^{-1}\right)\vec{\boldsymbol{M}}_{k}^{H}\boldsymbol{U}_{kk}. \quad (42)\end{aligned}$$

$$\zeta_1 = \mathrm{E}\|\alpha \sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)} \boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H + \vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H + \vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H\|.$$

$$\begin{aligned}
\zeta_1^2 \leq & 9\mathrm{Etr}\left(\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2}\alpha^2|\boldsymbol{H}_{k\ell}|^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}^H\boldsymbol{H}_{ki}^H\right) + 6\mathrm{Etr}\left(\sum_{i,\ell,r=1}^{p_2}\alpha^2|\boldsymbol{H}_{k\ell}|^2\boldsymbol{A}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{r\ell}\binom{-}{k}^H\boldsymbol{A}_{kr}^H\right)\\
\leq & \max_{\ell=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{k\ell}|^2)\left[9\mathrm{Etr}\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2}\alpha^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}^H\boldsymbol{H}_{ki}^H + 6\mathrm{Etr}\left(\sum_{i,\ell,r=1}^{p_2}\alpha^2\boldsymbol{A}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{r\ell}\binom{-}{k}^H\boldsymbol{A}_{kr}^H\right)\right]\\
\leq & \max_{\ell=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{k\ell}|^2)\left[9\mathrm{Etr}\left(\sum_{i,\ell=1}^{p_2}\alpha^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}^H\boldsymbol{H}_{ki}^H - \sum_{i=1}^{p_2}\alpha^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{ii}\binom{-}{k}\widetilde{\boldsymbol{G}}_{ii}\binom{-}{k}^H\boldsymbol{H}_{ki}^H\right) + 6\mathrm{E}\|\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\|^2\right]\\
\leq & 9\max_{\ell=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{k\ell}|^2)\mathrm{Etr}\sum_{i=1}^{p_2}\alpha^2\boldsymbol{H}_{ki}(\widetilde{\boldsymbol{G}}\binom{-}{k}\widetilde{\boldsymbol{G}}^H\binom{-}{k})\boldsymbol{H}_{ki}^H + 6\max_{\ell=1,\dots,p_2}\mathrm{E}|\boldsymbol{H}_{k\ell}|^2\mathrm{E}\|\vec{\boldsymbol{A}}_k\|^2
\end{aligned} \tag{55}$$

Additionally, assume

H-1
$$\sup_n \max_{i=1,\dots,p_1(n)}\sum_{j=1}^{p_2(n)}\mathrm{E}\|\boldsymbol{H}_{ij}\|^2 + \sup_n \max_{j=1,\dots,p_2(n)}\sum_{i=1}^{p_1(n)}\mathrm{E}\|\boldsymbol{H}_{ij}\|^2 < +\infty,$$

H-2
$$\sup_n \max_{i=1,\dots,p_1(n)}\sum_{j=1}^{p_2(n)}|\boldsymbol{A}_{ij}| + \sup_n \max_{j=1,\dots,p_2(n)}\sum_{i=1}^{p_1(n)}|\boldsymbol{A}_{ij}| < +\infty,$$

H-3 Lindeberg condition: $\forall\tau > 0$

$$\lim_{n\to\infty}\left(\max_{i=1,\dots,p_1(n)}\sum_{j=1}^{p_2(n)}\mathrm{E}\left(\|\boldsymbol{H}_{ij}\|^2\chi\{\|\boldsymbol{H}_{ij}\|>\tau\}\right) + \max_{j=1,\dots,p_2(n)}\sum_{i=1}^{p_1(n)}\mathrm{E}\left(\|\boldsymbol{H}_{ij}\|^2\chi\{\|\boldsymbol{H}_{ij}\|>\tau\}\right)\right) = 0.$$

H-4 $\boldsymbol{\Xi}_{ks}$, $k = 1,\dots,p_1$, $s = 1,\dots,p_2$, the random blocks of the matrix $\widetilde{\boldsymbol{\Xi}}$ are independent for every n.

and set ζ_1 as shown in the first equation at the top of the page. Then

$$\lim_{n\to\infty}\zeta_1 = 0. \tag{54}$$

Proof: Applying the Liapunov inequality[5] and inequality (27) we obtain

$$\begin{aligned}
\zeta_1^2 \leq & \mathrm{E}\left\|\alpha\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H + \vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H + \vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H\right\|^2\\
\leq & 3\mathrm{E}\left\|(\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\alpha\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H)\right\|^2 + 3\mathrm{E}\|\alpha\vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H\|^2\\
& + 3\mathrm{E}\|\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H\|^2.
\end{aligned}$$

[5]Liapunov inequality: Given a random variable x, $(\mathrm{E}(|x|^{k-1}))^{\frac{1}{k-1}} \leq (\mathrm{E}(|x|^k))^{\frac{1}{k}}$ for $k > 1$. In particular, $(\mathrm{E}(|x|))^2 \leq \mathrm{E}(|x|^2)$.

Considering that from the definition of $\widetilde{\boldsymbol{H}}$ $\mathrm{E}(\boldsymbol{H}_{ki}) = \mathbf{0}$, where $\mathbf{0}$ is the null matrix, and applying inequality (29), yields

$$\begin{aligned}
& \mathrm{E}\left\|(\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\alpha\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H)\right\|^2\\
\leq & \mathrm{E}\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\left\|\alpha\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H\right\|^2\\
& + \sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\alpha^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H\boldsymbol{H}_{k\ell}\widetilde{\boldsymbol{G}}_{i\ell}^H\binom{-}{k}\boldsymbol{H}_{ki}^H\\
\leq & \mathrm{E}\sum_{\substack{i,\ell=1\\ i\neq\ell}}^{p_2(n)}\left\|\alpha\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H\right\|^2\\
& + 2\mathrm{E}\sum_{\substack{i,\ell=1\\ i<\ell}}^{p_2(n)}\left\|\alpha\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H\right\|^2
\end{aligned}$$

and

$$\begin{aligned}
& \mathrm{E}\|\alpha\vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H\|^2\\
= & \mathrm{E}\|\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H\|^2\\
= & \mathrm{Etr}\sum_{i,\ell,r=1}^{p_2}\alpha^2\boldsymbol{A}_{ki}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\boldsymbol{H}_{k\ell}^H\boldsymbol{H}_{k\ell}\widetilde{\boldsymbol{G}}_{r\ell}^H\binom{-}{k}\boldsymbol{A}_{kr}^H.
\end{aligned}$$

Therefore, applying inequality (28), we get (55) at the top of the page. To obtain (55) we make use of the fact that $\sum_{\ell=1}^{p_2}\widetilde{\boldsymbol{G}}_{i\ell}\binom{-}{k}\widetilde{\boldsymbol{G}}_{i\ell}^H\binom{-}{k}$ is the ith diagonal block of the matrix $\widetilde{\boldsymbol{G}}\binom{-}{k}\widetilde{\boldsymbol{G}}^H\binom{-}{k}$ and we neglect the term

$$\sum_{i=1}^{p_2}\alpha^2\boldsymbol{H}_{ki}\widetilde{\boldsymbol{G}}_{ii}\binom{-}{k}\widetilde{\boldsymbol{G}}_{ii}^H\binom{-}{k}\boldsymbol{H}_{ki}^H$$

since it is always positive. Considering that $\left|\alpha\widetilde{\boldsymbol{G}}\binom{-}{k}\right|^2 \leq 1$ and applying the interlacing theorem, we obtain that also the maximum eigenvalue of $\left(\alpha^2\widetilde{\boldsymbol{G}}\binom{-}{k}\widetilde{\boldsymbol{G}}(k)^H\right)_{ii}$ is upper-bounded by 1. Therefore
$$\zeta_1^2 \leq 9\max_{\ell=1,\dots p_2}\mathrm{E}|\boldsymbol{H}_{k\ell}|^2\mathrm{E}\sum_{i=1}^{p_2}\|\boldsymbol{H}_{ki}\|^2 + 6\max_{\ell=1,\dots,p_2}\mathrm{E}|\boldsymbol{H}_{k\ell}|^2\mathrm{E}\sum_{i=1}^{p_2}\|\boldsymbol{A}_{ki}\|^2.$$

Assumptions H-1 and H-2 imply that $\sum_{\ell=1}^{p_2} \mathrm{E}\|\boldsymbol{H}_{ki}\|^2 < +\infty$ and $\sum_{\ell=1}^{p_2} |\boldsymbol{A}_{ki}|^2 < \infty$. Additionally, from H-3, the Lindeberg condition, it follows that

$$\begin{aligned}
&\lim_{\tau\to 0^+} \lim_{n\to+\infty} \max_{\ell=1,\ldots,p_2} \mathrm{E}\|\boldsymbol{H}_{k\ell}\|^2 \\
&= \lim_{\tau\to 0^+} \lim_{n\to+\infty} \max_{\ell=1,\ldots,p_2} \mathrm{E}\|\boldsymbol{H}_{k\ell}\|^2 \chi\{\|\boldsymbol{H}_{k\ell}\|^2 > \tau\} \\
&\quad + \mathrm{E}\|\boldsymbol{H}_{k\ell}\|^2 \chi\{\|\boldsymbol{H}_{k\ell}\|^2 \le \tau\} \\
&\le \lim_{\tau\to 0^+} \lim_{n\to+\infty} \max_{k=1,\ldots,p_2} \mathrm{E}\|\boldsymbol{H}_{k\ell}\|^2 \chi\{\|\boldsymbol{H}_{k\ell}\|^2 > \tau\} + \tau \\
&= 0.
\end{aligned} \tag{56}$$

Then, $\lim_{n\to+\infty} \zeta_1^2 = 0$. Since $\zeta_1 \ge 0$, this completes the proof of Lemma 6.

Lemma 7: Let Definitions (iii), (iv), (x), (xii), (viii), (ix), (xix), (xxi), (xxii), (xi), (xxiv), (xxv), (xxvi), (xxvii), (xxv) hold.

Assume that Conditions H-1, H-2, H-3, and H-4 in Lemma 6 are satisfied and define ζ_2 as shown in the first equation at the bottom of the page. Then

$$\lim_{n\to\infty} \zeta_2 = 0. \tag{57}$$

Proof: Let us define $\boldsymbol{L}_s = \alpha \boldsymbol{H}_{ks} \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right) \boldsymbol{H}_{ks}^H$. Then, ζ_2 can be rewritten as

$$\zeta_2 = \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s - \mathrm{E}\{\boldsymbol{L}_s | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\|.$$

By applying the triangular inequality of the Froboenius norm and the linearity of the expectation we derive

$$\begin{aligned}
\zeta_2 &\le \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\| \\
&\quad + \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\| \\
&= \delta_1 + \delta_2
\end{aligned}$$

where

$$\delta_1 = \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\|$$

and

$$\delta_2 = \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\|.$$

Let us focus on δ_1, the triangular inequality yields

$$\begin{aligned}
\delta_1 &\le \mathrm{E}(\sum_{s=1}^{p_2} \|\boldsymbol{L}_s\| \chi(\|\boldsymbol{H}_{ks}\| > \tau) \\
&\quad + \sum_{s=1}^{p_2} \|\mathrm{E}(\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right))\|).
\end{aligned}$$

The triangular inequality and the linearity of expectation imply
$$\begin{aligned}
&\mathrm{E}\|\mathrm{E}(\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| > \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right))\| \\
&\quad \le \mathrm{E}\left(\mathrm{E}(\|\boldsymbol{L}_s\| \chi(\|\boldsymbol{H}_{ks}\| > \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right))\right).
\end{aligned}$$

Therefore, using the bound on the spectral norm $|\alpha \boldsymbol{G}_{ss}\left(\bar{k}\right)| \le 1$

$$\delta_1 \le 2 \sum_{s=1}^{p_2} \mathrm{E}(\|\boldsymbol{H}_{ks}\|^2 \chi(\|\boldsymbol{H}_{ks}\| > \tau)).$$

Thanks to the Lindeberg condition

$$\lim_{n\to+\infty} \delta_1 = 0. \tag{58}$$

Let us consider δ_2. Applying the Lyapunov inequality we get the second equation at the bottom of the page. Since $\boldsymbol{H}_{ks}$ and $\boldsymbol{H}_{kt}$ are independent for $s \ne t$

$$\begin{aligned}
&\mathrm{E}\left(\boldsymbol{L}_s \boldsymbol{L}_t^H \chi(\|\boldsymbol{H}_{ks}\| < \tau) \chi(\|\boldsymbol{H}_{kt}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right), \boldsymbol{G}_{tt}\left(\bar{k}\right)\right) \\
&= \mathrm{E}\left(\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)\right) \mathrm{E}\left(\boldsymbol{L}_t^H \chi(\|\boldsymbol{H}_{kt}\| < \tau) | \boldsymbol{G}_{tt}\left(\bar{k}\right)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta_2^2 &\le \mathrm{E}\,\mathrm{tr} \sum_{s=1}^{p_2} \mathrm{E}(\boldsymbol{L}_s \boldsymbol{L}_s^H \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)) \\
&\quad - \mathrm{E}\,\mathrm{tr} \sum_{s=1}^{p_2} \mathrm{E}(\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)) \\
&\quad \times \mathrm{E}(\boldsymbol{L}_s^H \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}(k)) \\
&\le \mathrm{E}\,\mathrm{tr} \sum_{s=1}^{p_2} \mathrm{E}(\boldsymbol{L}_s \boldsymbol{L}_s^H \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)) \\
&= \mathrm{E} \sum_{s=1}^{p_2} \mathrm{E}(\|\boldsymbol{L}_s\|^2 \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)).
\end{aligned}$$

By applying inequality (28) we obtain
$$\delta_2^2 \le \mathrm{E} \sum_{s=1}^{p_2} \mathrm{E}(|\boldsymbol{H}_{ks}|^2 |\alpha \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)|^2 \|\boldsymbol{H}_{ks}\|^2 \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)).$$

The bounds $|\alpha \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)| < 1$ and $|\boldsymbol{H}_{ks}|^2 \chi(\|\boldsymbol{H}_{ks}\| < \tau) < \tau^2$ yield

$$\delta_2^2 \le \tau^2 \sum_{s=1}^{p_2} \mathrm{E}(\|\boldsymbol{H}_{ks}\|^2).$$

$$\zeta_2 = \mathrm{E}\left\|\sum_{s=1}^{p_2} \left(\alpha \boldsymbol{H}_{ks} \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right) \boldsymbol{H}_{ks}^H - \mathrm{E}\{\alpha \boldsymbol{H}_{ks} \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right) \boldsymbol{H}_{ks}^H | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\right)\right\|.$$

$$\begin{aligned}
\delta_2^2 &\le \mathrm{E}\|\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\|^2 \\
&= \mathrm{E}\,\mathrm{tr}\left(\sum_{s=1}^{p_2} \boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) - \mathrm{E}\{\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| \le \tau) | \widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\}\right)^2 \\
&= \mathrm{E}\,\mathrm{tr} \sum_{s,t=1}^{p_2} \left(\mathrm{E}(\boldsymbol{L}_s \boldsymbol{L}_t^H \chi(\|\boldsymbol{H}_{ks}\| < \tau) \chi(\|\boldsymbol{H}_{kt}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right), \boldsymbol{G}_{tt}\left(\bar{k}\right))\right. \\
&\quad \left. - \mathrm{E}\left(\boldsymbol{L}_s \chi(\|\boldsymbol{H}_{ks}\| < \tau) | \boldsymbol{G}_{ss}\left(\bar{k}\right)\right) \mathrm{E}\left(\boldsymbol{L}_t^H \chi(\|\boldsymbol{H}_{kt}\| < \tau) | \boldsymbol{G}_{tt}\left(\bar{k}\right)\right)\right).
\end{aligned}$$

Since $\sum_{s=1}^{p_2-1}\mathrm{E}(\|\boldsymbol{H}_{ks}\|^2)$ is upper-bounded thanks to hypothesis H-1, it results that

$$\lim_{\tau\to 0^+}\lim_{n\to+\infty}\delta_2 = 0.$$

This completes the proof of Lemma 7.

Lemma 8: Let Definition (iii), (iv), (x), (xii), (viii), (ix), (xix), (xxi), (xxii), (xi), (xxiv), (xxv), (xxvi), (xxvii), (xxv) hold.

Assume that Conditions H-1, H-2, H-3, and H-4 in Lemma 6 are satisfied and define ζ_3 as shown in the first equation at the bottom of the page. Then

$$\lim_{n\to\infty}\zeta_3 = 0. \tag{59}$$

Proof: The triangular inequality and the linearity of expectation yield

$$\begin{aligned}\zeta_3 &= \mathrm{E}\left\|\sum_{s=1}^{p_2}\mathrm{E}\left(\alpha\boldsymbol{H}_{ks}(\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{X})\boldsymbol{H}_{ks}^H|\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\right)\Big|_{\boldsymbol{X}=\boldsymbol{G}_{ss}}\right\|\\ &\le \mathrm{E}\left(\sum_{s=1}^{p_2}\mathrm{E}\left(\|\boldsymbol{H}_{ks}\alpha(\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{X})\boldsymbol{H}_{ks}^H\|\,|\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\right)\Big|_{\boldsymbol{X}=\boldsymbol{G}_{ss}}\right).\end{aligned}$$

By applying inequality (28) we obtain the second equation at the bottom of the page.

Applying the Woodbury formula, we obtain

$$\widetilde{\boldsymbol{G}}\left(\bar{k}\right)-\widetilde{\boldsymbol{G}} = (\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\vec{\boldsymbol{\Xi}}_k^H\boldsymbol{C}\vec{\boldsymbol{\Xi}}_k\widetilde{\boldsymbol{G}}\left(\bar{k}\right))$$

with $\boldsymbol{C}=(\boldsymbol{I}+\vec{\boldsymbol{\Xi}}_k\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\vec{\boldsymbol{\Xi}}_k^H)^{-1}$. Note that the spectral norm of $\boldsymbol{C}$ is bounded by $|\boldsymbol{C}|\le 1$ and

$$\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{G}_{ss} = \left(\sum_{j=1}^{p_2}\widetilde{\boldsymbol{G}}_{sj}\left(\bar{k}\right)\widetilde{\boldsymbol{\Xi}}_{kj}^H\right)\boldsymbol{C}\left(\sum_{\ell=1}^{p_2}\widetilde{\boldsymbol{G}}_{s\ell}\left(\bar{k}\right)\widetilde{\boldsymbol{\Xi}}_{kj}\right)^H.$$

By appealing to inequality (28)

$$\left\|\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{G}_{ss}\right\|\le\left\|\sum_{j=1}^{p_2}\widetilde{\boldsymbol{G}}_{sj}\left(\bar{k}\right)\vec{\boldsymbol{\Xi}}_{kj}^H\right\|^2.$$

Therefore, we have (60) at the bottom of the page. The bounds $\|\alpha\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\|\le 1$ and $\|\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\|\le|\alpha|^{-1}$ are applied to derive (60).

Hypotheses H-1 and H-2 of Lemma 6 imply that $\sum_{j=1}^{p_2}\mathrm{E}\|\widetilde{\boldsymbol{\Xi}}_{kj}\|^2$ is upper-bounded as $n\to+\infty$. Following the same line as in Lemma 6, we get

$$\lim_{n\to+\infty}\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)=0.$$

This implies (59) and completes the proof of Lemma 8.

Lemma 9: Let definitions (iii), (iv), (x), (xii), (viii), (ix), (xix), (xxi), (xxii), (xi), (xxiv), (xxv), (xxvi), (xxvii), (xxv) hold.

Assume that Conditions H-1, H-2, H-3, and H-4 in Lemma 6 are satisfied and define

$$\boldsymbol{E}_{k\ell}=\alpha\vec{\boldsymbol{\Xi}}_k\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{\Xi}}_k-\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k,\qquad k\ne\ell. \tag{61}$$

Then

$$\lim_{n\to\infty}\mathrm{E}(\|\boldsymbol{E}_{k\ell}\|)=0. \tag{62}$$

Proof: Referring to the Liapunov inequality and inequality (27) we obtain

$$\begin{aligned}(\mathrm{E}\|\boldsymbol{E}_{k\ell}\|)^2 &\le \mathrm{E}\|\boldsymbol{E}_{k\ell}\|^2\\ &\le 3\mathrm{E}(\|\alpha\vec{\boldsymbol{H}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{H}}_k^H\|^2+\|\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{H}}_k^H\|^2\\ &\quad+\|\alpha\vec{\boldsymbol{H}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H\|^2).\end{aligned}$$

Taking into account that $\vec{\boldsymbol{H}}_\ell$ and $\vec{\boldsymbol{H}}_k$ are independent for $\ell\ne k$ and $\mathrm{E}\vec{\boldsymbol{H}}_\ell=\boldsymbol{0}$, where $\boldsymbol{0}$ is the null matrix, it results in equation (63) at the

---

$$\zeta_3=\mathrm{E}\left\|\alpha\sum_{s=1}^{p_2}\mathrm{E}\left(\boldsymbol{H}_{ks}\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\boldsymbol{H}_{ks}^H|\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\right)-\mathrm{E}\left(\boldsymbol{H}_{ks}\boldsymbol{X}\boldsymbol{H}_{ks}^H\right)\Big|_{\boldsymbol{X}=\boldsymbol{G}_{ss}}\right\|.$$

$$\begin{aligned}\zeta_3&\le\mathrm{E}\left(\sum_{s=1}^{p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\|\alpha(\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{X})\|\,|\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)\right)\Big|_{\boldsymbol{X}=\boldsymbol{G}_{ss}}\right)\\ &=\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\sum_{s=1}^{p_2}\|\alpha(\widetilde{\boldsymbol{G}}_{ss}\left(\bar{k}\right)-\boldsymbol{G}_{ss})\|\right).\end{aligned}$$

$$\begin{aligned}\zeta_3&\le\max_{s=1,\ldots p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\sum_{s=1}^{p_2}\alpha\,\mathrm{tr}\left((\sum_{j=1}^{p_2}\widetilde{\boldsymbol{G}}_{sj}\left(\bar{k}\right)\widetilde{\boldsymbol{\Xi}}_{kj}^H)^H(\sum_{\ell=1}^{p_2}\widetilde{\boldsymbol{G}}_{s\ell}\left(\bar{k}\right)\widetilde{\boldsymbol{\Xi}}_{k\ell}^H)\right)\right)\\ &=\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\mathrm{tr}\alpha\sum_{j,\ell=1}^{p_2}\left(\sum_{s=1}^{p_2}(\widetilde{\boldsymbol{\Xi}}_{kj}\widetilde{\boldsymbol{G}}_{sj}\left(\bar{k}\right)^H\widetilde{\boldsymbol{G}}_{s\ell}\left(\bar{k}\right)\widetilde{\boldsymbol{\Xi}}_{k\ell}^H)\right)\right)\\ &=\max_{s=1,\ldots p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\mathrm{tr}\sum_{j,\ell=1}^{p_2}\alpha\left(\widetilde{\boldsymbol{\Xi}}_{kj}(\widetilde{\boldsymbol{G}}\left(\bar{k}\right)^H\widetilde{\boldsymbol{G}}\left(\bar{k}\right))_{jl}\widetilde{\boldsymbol{\Xi}}_{k\ell}^H)\right)\right)\\ &\le\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\|\alpha^{\frac{1}{2}}\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\vec{\boldsymbol{\Xi}}_k\|^2\right)\\ &\le|\alpha|^{-1}\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\mathrm{E}\left(\|\vec{\boldsymbol{\Xi}}_k^H\|^2\right)\\ &\le|\alpha|^{-1}\max_{s=1,\ldots,p_2}\mathrm{E}\left(|\boldsymbol{H}_{ks}|^2\right)\sum_{j=1}^{p_2}\mathrm{E}\left(\|\widetilde{\boldsymbol{\Xi}}_{kj}\|^2\right).\end{aligned}\tag{60}$$

$$
\begin{aligned}
(\mathrm{E}\|\boldsymbol{E}_{k\ell}\|)^2 \leq\, & 3\mathrm{E}\left(\mathrm{tr}\left(\alpha^2\sum_{i,j=1}^{p_2}\boldsymbol{H}_{\ell j}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\boldsymbol{H}_{kj}^H\boldsymbol{H}_{kj}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}^H\boldsymbol{H}_{\ell i}^H\right)\right)\\
&+3\mathrm{E}\left(\mathrm{tr}(\alpha^2\sum_{i,j,r=1}^{p_2}\boldsymbol{A}_{\ell i}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\boldsymbol{H}_{kj}^H\boldsymbol{H}_{kj}\widetilde{\boldsymbol{G}}_{rj}\binom{-}{k,\ell}^H\boldsymbol{A}_{\ell r})\right)+3\mathrm{E}\left(\mathrm{tr}(\alpha^2\sum_{i,j,r=1}^{p_2}\boldsymbol{H}_{\ell i}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\boldsymbol{A}_{kj}^H\boldsymbol{A}_{kr}\widetilde{\boldsymbol{G}}_{ir}\binom{-}{k,\ell}^H\boldsymbol{H}_{\ell i}^H)\right)\\
\leq\, & 3\max_{j=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{kj}|^2)\left(\mathrm{E}\left(\mathrm{tr}\sum_{i=1}^{p_2}\alpha^2\boldsymbol{H}_{\ell i}(\sum_{j=1}^{p_2}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\widetilde{\boldsymbol{G}}_{ij}^H\binom{-}{k,\ell})\boldsymbol{H}_{\ell i}^H\right)+\mathrm{E}\left(\mathrm{tr}\sum_{i,j,r=1}^{p_2}\alpha^2\boldsymbol{A}_{\ell i}\widetilde{\boldsymbol{G}}_{ij}^H\binom{-}{k,\ell}\right.\right.\\
&\left.\left.\times\widetilde{\boldsymbol{G}}_{rj}\binom{-}{k,\ell}\boldsymbol{A}_{\ell r}^H\right)\right)+3\max_{j=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{\ell j}|^2)\mathrm{E}\left(\mathrm{tr}\sum_{i,j,r=1}^{p_2}\alpha^2\boldsymbol{A}_{kr}\widetilde{\boldsymbol{G}}_{ir}^H\binom{-}{k,\ell}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\boldsymbol{A}_{kj}^H\right).
\end{aligned}\tag{63}
$$

top of the page. In (63) inequality (28) has been applied. Note that the term

$$\left(\sum_{j=1}^{p_2}\alpha^2\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}^H\right)$$

coincides with the ith diagonal block of the matrix

$$\alpha^2\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\widetilde{\boldsymbol{G}}^H\binom{-}{k,\ell}.$$

Appealing to the interlacing theorem and the bound on the eigenvalues of $\alpha\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}$ we easily recognize that

$$|\sum_{j=1}^{p_2}\alpha^2\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}\widetilde{\boldsymbol{G}}_{ij}\binom{-}{k,\ell}^H|\leq 1.$$

Additionally

$$\sum_{i,j,r=1}^{p_2}\alpha^2\boldsymbol{A}_{\ell i}\widetilde{\boldsymbol{G}}_{ij}\widetilde{\boldsymbol{G}}_{rj}^H\boldsymbol{A}_{\ell r}^H=\alpha^2\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}^H\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{A}}_\ell^H.$$

These considerations yield

$$
\begin{aligned}
(\mathrm{E}\|\boldsymbol{E}_{k\ell}\|)^2\leq\, & 3\max_{j=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{kj}|^2)\left(\mathrm{E}\mathrm{tr}(\sum_{i=1}^{p_2}\boldsymbol{H}_{\ell i}\boldsymbol{H}_{\ell i}^H)\right.\\
&\left.+\mathrm{E}\mathrm{tr}(\alpha^2\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}^H\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{A}}_\ell^H)\right)\\
&+3\max_{j=1,\dots p_2}\mathrm{E}(|\boldsymbol{H}_{\ell j}|^2)\mathrm{E}\mathrm{tr}(\alpha^2\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}^H\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{A}}_k^H)\\
\leq\, & 3\max_{j=1,\dots p_2}\mathrm{E}(|\boldsymbol{H}_{kj}|^2)\left(\mathrm{E}\sum_{i=1}^{p_2}\|\boldsymbol{H}_{\ell i}\|^2+\sum_{i=1}^{p_2}\|\boldsymbol{A}_{\ell i}\|^2\right)\\
&+3\max_{j=1,\dots p_2}\mathrm{E}(\|\boldsymbol{H}_{\ell j})\|^2\sum_{i=1}^{p_2}\|\boldsymbol{A}_{\ell i}\|^2.
\end{aligned}\tag{64}
$$

Inequality (28) and the bound on the spectral norm of $\alpha\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}$ are used to derive (64). Hypotheses H-1 and H-2 imply that the sums in (64) are upper-bounded for any n. Additionally, as in (56), the Lindeberg condition implies that

$$\lim_{n\to\infty}\max_{j=1,\dots,p_2}\mathrm{E}(|\boldsymbol{H}_{kj}|^2)=0.$$

This yields the thesis of Lemma 9.

Lemma 10: Let definitions (iii), (iv), (x), (xii), (viii), (ix), (xix), (xxi), (xxii), (xi), (xxiv), (xxv), (xxvi), (xxvii), (xxv) hold.

Assume that Conditions H-1, H-2, H-3, and H-4 in Lemma 6 are satisfied.

Then, for $\alpha\in\mathbb{C}\backslash\mathbb{R}^-$

$$\lim_{n\to\infty}\mathrm{E}|\boldsymbol{Q}_{p\ell}(\alpha)-\boldsymbol{T}_{p\ell}(\alpha)|=0,\qquad p,\ell=1,\dots,p_1$$

and

$$\lim_{n\to\infty}\mathrm{E}|\boldsymbol{G}_{p\ell}(\alpha)-\alpha^{-1}\boldsymbol{R}_{p\ell}(\alpha)|=0,\qquad p,\ell=1,\dots,p_2$$

i.e., the blocks of the matrices $\widetilde{\boldsymbol{Q}}$ and $\widetilde{\boldsymbol{G}}$ converge in the first mean to the corresponding blocks of the matrices

$$\widetilde{\boldsymbol{T}}=(\widetilde{\boldsymbol{C}}^{(1)}+\widetilde{\boldsymbol{A}}^H(\widetilde{\boldsymbol{C}}^{(2)})^{-1}\widetilde{\boldsymbol{A}})^{-1}$$

and

$$\widetilde{\boldsymbol{R}}=(\widetilde{\boldsymbol{C}}^{(2)}+\widetilde{\boldsymbol{A}}(\widetilde{\boldsymbol{C}}^{(1)})^{-1}\widetilde{\boldsymbol{A}}^H)^{-1}$$

respectively, with

$$\widetilde{\boldsymbol{C}}^{(1)}(\alpha)=\mathrm{diag}(\boldsymbol{C}_{kk}^{(1)}(\alpha))_{k=1,\dots,p_1}\tag{65}$$

and

$$\widetilde{\boldsymbol{C}}^{(2)}(\alpha)=\mathrm{diag}(\boldsymbol{C}_{kk}^{(2)}(\alpha))_{k=1,\dots,p_2}.\tag{66}$$

The matrix blocks $\boldsymbol{C}_{kk}^{(1)}(\alpha)$ of size $q_1\times q_1$ and $\boldsymbol{C}_{kk}^{(2)}(\alpha)$ of size $q_2\times q_2$ are equal to

$$\boldsymbol{C}_{kk}^{(1)}(\alpha)=\alpha\boldsymbol{I}+\sum_{j=1}^{p_2}\left(\mathrm{E}\boldsymbol{H}_{kj}(\boldsymbol{X})_{jj}\boldsymbol{H}_{kj}^H\right)_{\boldsymbol{X}=\alpha\widetilde{\boldsymbol{G}}},\quad k=1,\dots,p_1\tag{67}$$

$$\boldsymbol{C}_{\ell\ell}^{(2)}(\alpha)=\boldsymbol{I}+\sum_{j=1}^{p_1}\left(\mathrm{E}\boldsymbol{H}_{j\ell}^H(\boldsymbol{Y})_{jj}\boldsymbol{H}_{j\ell}^H\right)_{\boldsymbol{Y}=\widetilde{\boldsymbol{Q}}},\quad k=1,\dots,p_2\tag{68}$$

with $\widetilde{\boldsymbol{Q}}$ and $\widetilde{\boldsymbol{G}}$ defined in (viii) and (ix), respectively.

Proof: Let us consider the matrices $\widetilde{\boldsymbol{Q}},\widetilde{\boldsymbol{G}},\widetilde{\boldsymbol{C}}^{(1)}$, and $\widetilde{\boldsymbol{C}}^{(2)}$ defined in (viii), (ix), (65), and (66), respectively. Corollary 3, (51), and (52), applied to the matrix $\widetilde{\boldsymbol{Q}}$ yields

$$\boldsymbol{Q}_{\ell k}=-\boldsymbol{Q}_{\ell\ell}\binom{-}{k}(\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{A}}_k^H+\boldsymbol{E}_{\ell,k}\boldsymbol{Q}_{kk},\qquad \ell\neq k\tag{69}$$

$$\boldsymbol{Q}_{kk}=(\boldsymbol{C}_{kk}^{(1)}+\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H+\boldsymbol{E}_{kk})^{-1}\tag{70}$$

with $\boldsymbol{E}_{\ell k}$ defined already in (61) as

$$\boldsymbol{E}_{\ell k}=\alpha\vec{\boldsymbol{\Xi}}_\ell\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{\Xi}}_k^H-\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\binom{-}{k,\ell}\vec{\boldsymbol{A}}_k^H,\qquad \ell\neq k$$

and

$$\boldsymbol{E}_{kk}=\alpha\vec{\boldsymbol{\Xi}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{\Xi}}_k^H-\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H-\boldsymbol{C}_{kk}^{(1)}+\alpha\boldsymbol{I}.\tag{71}$$

Thanks to Lemma 9

$$\lim_{n\to+\infty}\max_{\substack{k=1,\dots,p_1\\ \ell=1,\dots,p_2}}\mathrm{E}(\|\boldsymbol{E}_{k\ell}\|)=0,\qquad \text{for } \ell\neq k.\tag{72}$$

In order to prove an similar property for $\boldsymbol{E}_{kk}$ let us rewrite $\boldsymbol{E}_{kk}$ as follows:

$$
\begin{aligned}
\boldsymbol{E}_{kk}=\,&\alpha\vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H+\alpha\vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{A}}_k^H+\alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\binom{-}{k}\vec{\boldsymbol{H}}_k^H\\
&-\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\boldsymbol{X}_{ss}\boldsymbol{H}_{ks}^H)_{\boldsymbol{X}=\alpha\widetilde{\boldsymbol{G}}}+\alpha\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\boldsymbol{G}_{ss}\binom{-}{k}\\
&\times\boldsymbol{H}_{ks}^H|\boldsymbol{G}_{ss}\binom{-}{k})-\alpha\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\boldsymbol{G}_{ss}\binom{-}{k}\boldsymbol{H}_{ks}^H|\boldsymbol{G}_{ss}\binom{-}{k})
\end{aligned}
$$

$$= \sum_{\substack{i,\ell\\ i\neq\ell}}^{p_2} \alpha \boldsymbol{H}_{ki}\boldsymbol{G}_{i\ell}\left(\bar{k}\right)\boldsymbol{H}_{k\ell}^H + \alpha\vec{\boldsymbol{H}}_k\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\vec{\boldsymbol{A}}_k^H + \alpha\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{G}}\left(\bar{k}\right)\vec{\boldsymbol{H}}_k^H + \alpha\sum_{s=1}^{p_2}\boldsymbol{H}_{ks}\boldsymbol{G}_{ss}\left(\bar{k}\right)\boldsymbol{H}_{ks}^H - \alpha\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\boldsymbol{G}_{ss}\left(\bar{k}\right)\boldsymbol{H}_{ks}^H|\boldsymbol{G}_{ss}\left(\bar{k}\right)) + \alpha\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\boldsymbol{G}_{ss}\left(\bar{k}\right)\boldsymbol{H}_{ks}^H|\boldsymbol{G}_{ss}\left(\bar{k}\right)) - \alpha\sum_{s=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{ks}\widetilde{\boldsymbol{G}}_{ss}\boldsymbol{H}_{ks}).$$

Appealing to the triangular inequality of the Frobenius norm

$$\mathrm{E}\|\boldsymbol{E}_{kk}\| \leq \zeta_1+\zeta_2+\zeta_3 \tag{73}$$

with ζ_1, ζ_2, and ζ_3 defined in Lemmas 6–8, respectively. These lemmas and (73) imply

$$\lim_{n\to\infty}\max_{k=1,\dots,p_1}\mathrm{E}\|\boldsymbol{E}_{kk}\| = 0. \tag{74}$$

Making use of Lemma 4, the block elements of the matrix $\widetilde{\boldsymbol{T}}$ can be rewritten as

$$\begin{aligned}\boldsymbol{T}_{\ell k} &= -\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)(\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{R}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H)\boldsymbol{T}_{kk},\\ &\qquad k\neq\ell, \text{ and } k,\ell=1,\dots,p_1\\ \boldsymbol{T}_{kk} &= (\boldsymbol{C}_{kk}^{(1)}+\vec{\boldsymbol{A}}_k\widetilde{\boldsymbol{R}}\left(\bar{k}\right)\vec{\boldsymbol{A}}_k^H)^{-1}, \qquad k=1,\dots,p_1\end{aligned} \tag{75}$$

with $\widetilde{\boldsymbol{R}}\left(\bar{k}\right) = (\boldsymbol{C}^{(2)}+\boldsymbol{A}\left(\bar{k}\right)^H(\boldsymbol{C}^{(1)}(k))^{-1}\boldsymbol{A}\left(\bar{k}\right))^{-1}$ and

$$\widetilde{\boldsymbol{R}}\left(\overline{k,\ell}\right) = (\boldsymbol{C}^{(2)}+\boldsymbol{A}\left(\overline{k,\ell}\right)^H(\boldsymbol{C}^{(1)}(k,\ell))^{-1}\boldsymbol{A}\left(\overline{k,\ell}\right)^H)^{-1}.$$

$\widetilde{\boldsymbol{C}}^{(1)}(k)$ is the matrix obtained from $\widetilde{\boldsymbol{C}}^{(1)}$ by suppressing the kth block row and the kth block column. Analogously, $\widetilde{\boldsymbol{C}}^{(1)}(k,\ell)$ is the block-diagonal matrix obtained from $\widetilde{\boldsymbol{C}}^{(1)}$ by suppressing the kth and ℓth block rows and the kth and ℓth block columns. For further study, we derive the following bound:

$$\begin{aligned}&\mathrm{E}|\boldsymbol{Q}_{\ell k}-\boldsymbol{T}_{\ell k}|\\ &\quad=\mathrm{E}|-\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)(\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H+\boldsymbol{E}_{k\ell})\boldsymbol{Q}_{kk}\\ &\qquad+\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{R}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H\boldsymbol{T}_{kk}|\\ &\quad=\mathrm{E}|(-\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)+\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right))\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H\boldsymbol{Q}_{kk}\\ &\qquad-\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\alpha\vec{\boldsymbol{A}}_\ell\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\vec{\boldsymbol{A}}_k^H(\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk})-\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\vec{\boldsymbol{A}}_\ell\\ &\qquad\times(\alpha\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)-\widetilde{\boldsymbol{R}}\left(\overline{k,\ell}\right))\vec{\boldsymbol{A}}_k^H\boldsymbol{T}_{kk}-\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)\boldsymbol{E}_{k\ell}\boldsymbol{Q}_{kk}|\\ &\quad\leq\mathrm{E}(|-\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)+\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\|\alpha\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\|\vec{\boldsymbol{A}}_\ell\|\vec{\boldsymbol{A}}_k\|\boldsymbol{Q}_{kk}|)\\ &\qquad+\mathrm{E}(|\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\|\alpha\widetilde{\boldsymbol{G}}\left(\overline{k,\ell}\right)\|\vec{\boldsymbol{A}}_\ell\|\vec{\boldsymbol{A}}_k\|\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}|)\\ &\qquad+\mathrm{E}(|\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)\|\boldsymbol{T}_{kk}\|\sum_{s,t=1}^{p_2}\boldsymbol{A}_{\ell s}(\alpha\widetilde{\boldsymbol{G}}_{st}\left(\overline{k,\ell}\right)\\ &\qquad-\boldsymbol{R}_{st}\left(\overline{k,\ell}\right))\boldsymbol{A}_{kt}^H|)+\mathrm{E}(|\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)\|\boldsymbol{E}_{k\ell}\|\boldsymbol{Q}_{kk}|).\end{aligned} \tag{76}$$

Here, (76) derives from the triangular inequality (24) and the submultiplicative inequality (25). Taking into account that

$$|\boldsymbol{Q}_{kk}|,|\boldsymbol{T}_{kk}|,|\boldsymbol{T}_{\ell\ell}(k)|<|\alpha|^{-1}\leq|s|^{-1}\quad\text{and}\quad|\alpha\widetilde{\boldsymbol{C}}\left(\overline{k,\ell}\right)|<1$$

and

$$|\vec{\boldsymbol{A}}_\ell|\leq\sqrt{\sum_{i=1}^{p_1}\|\boldsymbol{A}_{\ell i}\|^2}\leq\sqrt{q_1\sum_{i=1}^{p_1}|\boldsymbol{A}_{\ell i}|^2}$$

we obtain

$$\begin{aligned}&\mathrm{E}|\boldsymbol{Q}_{\ell k}-\boldsymbol{T}_{\ell k}|\\ &\leq\sqrt{q_1\sum_{i=1}^{p_1}|\boldsymbol{A}_{\ell i}|^2}\sqrt{q_1\sum_{i=1}^{p_1}|\boldsymbol{A}_{ki}|^2|s|^{-1}}\\ &\quad\times\Big(\mathrm{E}|\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)-\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)|+\mathrm{E}|\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}|\\ &\quad+\max_{s,t=1,\dots p_2}\mathrm{E}|\widetilde{\boldsymbol{G}}_{st}-\alpha^{-1}\boldsymbol{R}_{st}\left(\overline{k,\ell}\right)|\Big)+|s|^{-2}\mathrm{E}|\boldsymbol{E}_{k\ell}|\\ &\leq c'|\alpha|^{-1}(\max_{\substack{k,\ell=1,\dots p_1\\ s,t=1,\dots p_2}}\big(\mathrm{E}|\boldsymbol{Q}_{\ell\ell}\left(\bar{k}\right)-\boldsymbol{T}_{\ell\ell}\left(\bar{k}\right)|\\ &\quad+\mathrm{E}|\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}|+\mathrm{E}|\boldsymbol{G}_{st}\left(\overline{k,\ell}\right)-\alpha^{-1}\boldsymbol{R}_{st}\left(\overline{k,\ell}\right)|\big)\\ &\quad+|\alpha|^{-1}\mathrm{E}|\boldsymbol{E}_{k\ell}|)\end{aligned} \tag{77}$$

where $0<c'<+\infty$ is a finite constant thanks to hypothesis H-2. A similar bound holds for $\mathrm{E}|\boldsymbol{Q}_{ii}-\boldsymbol{T}_{ii}|$. In fact, (70) and (75) yield

$$\begin{aligned}&\mathrm{E}|\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}|\\ &=\mathrm{E}|\boldsymbol{T}_{kk}\boldsymbol{T}_{kk}^{-1}\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}\boldsymbol{Q}_{kk}^{-1}\boldsymbol{Q}_{kk}|\\ &=\mathrm{E}|\boldsymbol{T}_{kk}\vec{\boldsymbol{A}}_k(\widetilde{\boldsymbol{R}}\left(\bar{k}\right)-\alpha\widetilde{\boldsymbol{G}}\left(\bar{k}\right))\vec{\boldsymbol{A}}_k\boldsymbol{Q}_{kk}-\boldsymbol{T}_{kk}\boldsymbol{E}_{kk}\boldsymbol{Q}_{kk}|\\ &\leq|\alpha|^{-1}\mathrm{E}\left(\sum_{i,j=1}^{p_2}|\boldsymbol{A}_{ki}(\alpha^{-1}\boldsymbol{R}_{ij}\left(\bar{k}\right)-\boldsymbol{G}_{ij}\left(\bar{k}\right))\boldsymbol{A}_{kj}^H|+|\alpha|^{-1}|\boldsymbol{E}_{kk}|\right)\\ &\leq|\alpha|^{-1}\left(\max_{ij=1,\dots,p_2}\mathrm{E}(|\alpha^{-1}\boldsymbol{R}_{ij}\left(\bar{k}\right)-\boldsymbol{G}_{ij}\left(\bar{k}\right)|)\right.\\ &\qquad\left.\times(\sum_j\|\boldsymbol{A}_{kj}\|^2)+|\alpha|^{-1}|\boldsymbol{E}_{kk}|\right)\end{aligned} \tag{78}$$

$$\leq c'|\alpha|^{-1}\left(\max_{ij=1,\dots,p_2}\mathrm{E}|\alpha^{-1}\boldsymbol{R}_{ij}\left(\bar{k}\right)-\boldsymbol{G}_{ij}\left(\bar{k}\right)|+|\alpha|^{-1}|\boldsymbol{E}_{kk}|\right). \tag{79}$$

Equation (78) is derived applying the triangular inequality (24). In (79), we use hypothesis H-2.

A similar inequality holds for $\mathrm{E}\left|\boldsymbol{G}_{ij}\left(\overline{k,\ell}\right)-\alpha^{-1}\boldsymbol{R}_{ij}\left(\overline{k,\ell}\right)\right|$.

For further study, we introduce the definitions, shown in the equations at the top of the next page. With the previous definitions and taking into account (77) and (79) we can write

$$\begin{aligned}a_{(0)}^{(0)}(\alpha)&\leq c|\alpha|^{-1}(\max(a_{(1)}^{(0)}(\alpha),b_{(1)}^{(0)}(\alpha),b_{(2)}^{(0)}(\alpha))\\ &\quad+|\alpha|^{-1}\max_{k,\ell}(\mathrm{E}|\boldsymbol{E}_{kk}|+\mathrm{E}|\boldsymbol{E}_{k\ell}|))\\ &\leq c|\alpha|^{-2}\varepsilon_0+c|\alpha|^{-1}\max(a_{(1)}^{(0)}(\alpha),b_{(1)}^{(0)}(\alpha),b_{(2)}^{(0)}(\alpha))\end{aligned} \tag{80}$$

with

$$\varepsilon_0=\varepsilon_0(n)=\varepsilon_0^0=\max_{k,\ell}\mathrm{E}(|\boldsymbol{E}_{kk}|+|\boldsymbol{E}_{k\ell}|)$$

and $c=3c'$. Considering relations similar to (77) and (79) for

$$\mathrm{E}\left|\boldsymbol{Q}_{p\ell}\left(\overline{\substack{j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s}}\right)-\boldsymbol{T}_{p\ell}\left(\overline{\substack{j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s}}\right)\right|$$

and

$$\mathrm{E}\left|\boldsymbol{G}_{p\ell}\left(\overline{\substack{j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s}}\right)-\alpha^{-1}\boldsymbol{R}_{p\ell}\left(\overline{\substack{j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s}}\right)\right|$$

$$a_{(0)}^{(0)}(\alpha) = \max_{p,\ell=1,\dots p_1} \mathrm{E}|\boldsymbol{Q}_{p\ell} - \boldsymbol{T}_{p\ell}|$$
$$a_{(s)}^{(r)}(\alpha) = \max_{p,\ell=1,\dots p_1} \max_{\substack{j_1,j_2,\dots,j_r\\k_1,k_2,\dots,k_s}} (\mathrm{E}|\boldsymbol{Q}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots j_r\\ k_1,k_2,\dots k_s\end{smallmatrix}\right) - \boldsymbol{T}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots j_r\\ k_1,k_2,\dots k_s\end{smallmatrix}\right)|)$$
$$b_{(0)}^{(0)}(\alpha) = \max_{p,\ell=1,\dots p_2} \mathrm{E}|\boldsymbol{G}_{p\ell} - \alpha^{-1}\boldsymbol{R}_{p\ell}|$$
$$b_{(s)}^{(r)}(\alpha) = \max_{p,\ell=1,\dots p_2} \max_{\substack{j_1,\dots j_r\\k_1,\dots k_s}} (\mathrm{E}|\boldsymbol{G}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right) - \alpha^{-1}\boldsymbol{R}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)|).$$

we can write

$$a_{(s)}^{(r)}(\alpha) \le c|\alpha|^{-1}\left(\max(a_{(s+1)}^{(r)}(\alpha), b_{(s+1)}^{(r)}, b_{(s+2)}^{(r)}(\alpha)) + |\alpha|^{-1}\varepsilon_{(s)}^{(r)}\right) \quad (81)$$

and

$$b_{(s)}^{(r)}(\alpha) \le c|\alpha|^{-1}\left(\max(a_{(s)}^{(r+1)}(\alpha), a_{(s)}^{(r+2)}(\alpha), b_{(s)}^{(r+1)}(\alpha)) + |\alpha|^{-1}\varepsilon_{(s)}^{(r)}\right) \quad (82)$$

with

$$\varepsilon_{(s)}^{(r)} = \max_{p,\ell} \max_{\substack{j_1,j_2,\dots,j_r\\k_1,k_2,\dots,k_s}} \left(\mathrm{E}\left|\boldsymbol{E}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right| + \mathrm{E}\left|\boldsymbol{E}_{\ell\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right| + \mathrm{E}\left|\boldsymbol{\mathcal{E}}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right| + \mathrm{E}\left|\boldsymbol{\mathcal{E}}_{\ell\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right|\right).$$

Here, $\mathrm{E}\left|\boldsymbol{\mathcal{E}}_{p\ell}(\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right|$ is defined analogously to $\mathrm{E}\left|\boldsymbol{E}_{p\ell}(\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right|$ but for the matrix

$$\mathrm{E}\left|\boldsymbol{G}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right) - \alpha^{-1}\boldsymbol{R}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right|.$$

Note that $\varepsilon_{(s)}^{(r)}$ depends on the parameter n. Let us substitute the upper bounds (81) and (82) for $a_{(1)}^{(0)}(\alpha), b_{(1)}^{(0)}(\alpha), b_{(2)}^{(0)}(\alpha)$ in[6] (80). We obtain

$$\begin{aligned} a_{(0)}^{(0)} \le\, & c|\alpha|^{-1}\Big[\max\Big(c|\alpha|^{-1}(\max(a_{(2)}^{(0)}, b_{(2)}^{(0)}, b_{(3)}^{(0)}) + |\alpha|^{-1}\varepsilon_{(1)}^{(0)}),\\ & c|\alpha|^{-1}(\max(a_{(0)}^{(2)}, a_{(0)}^{(3)}, b_{(0)}^{(2)}) + |\alpha|^{-1}\varepsilon_{(0)}^{(1)}),\\ & c|\alpha|^{-1}(\max(a_{(2)}^{(1)}, a_{(2)}^{(2)}, b_{(2)}^{(1)}) + |\alpha|^{-1}\varepsilon_{(2)}^{(0)})\Big) + |\alpha|^{-1}\varepsilon_{(0)}^{(0)}\Big]\\ \le\, & c^2|\alpha|^{-2}\Big(\max(a_{(2)}^{(0)}, b_{(2)}^{(0)}, b_{(3)}^{(0)}, a_{(0)}^{(2)}, a_{(0)}^{(3)}, b_{(0)}^{(2)}, a_{(2)}^{(1)}, a_{(2)}^{(2)}, b_{(2)}^{(1)})\\ & + |\alpha|^{-1}\max(\varepsilon_{(1)}^{(0)}, \varepsilon_{(0)}^{(1)}, \varepsilon_{(2)}^{(0)}) + c|\alpha|^{-2}\varepsilon_0\Big)\\ \le\, & c^2|\alpha|^{-2}\max(a_{(2)}^{(0)}, b_{(2)}^{(0)}, b_{(3)}^{(0)}, a_{(0)}^{(2)}, a_{(0)}^{(3)}, b_{(0)}^{(2)}, a_{(2)}^{(1)}, a_{(2)}^{(2)}, b_{(2)}^{(1)})\\ & + c^2|\alpha|^{-3}\max(\varepsilon_{(1)}^{(0)}, \varepsilon_{(1)}^{(0)}, \varepsilon_{(2)}^{(0)}) + c|\alpha|^{-2}\varepsilon_0. \end{aligned} \quad (83)$$

By denoting $\varepsilon_1 = \max(\varepsilon_{(1)}^{(0)}, \varepsilon_{(1)}^{(0)}, \varepsilon_{(2)}^{(0)}, \varepsilon_0)$ the quantity $a_{(0)}^{(0)}(\alpha)$ is bounded as follows:

$$\begin{aligned} a_{(0)}^{(0)} \le\, & c^2|\alpha|^{-2}\max(a_{(2)}^{(0)}, b_{(2)}^{(0)}, b_{(3)}^{(0)}, a_{(0)}^{(2)}, a_{(0)}^{(3)}, b_{(0)}^{(2)}, a_{(2)}^{(1)}, a_{(2)}^{(2)}, b_{(2)}^{(1)})\\ & + \varepsilon_1(c^2|\alpha|^{-3} + c|\alpha|^{-2}). \end{aligned} \quad (84)$$

By iterating the substitutions (81) and (82) in (80) and by considering that the upper bounds

$$\begin{aligned} & a_{(s)}^{(r)}(\alpha)\\ & \le \max_{p,\ell} \max_{i_1,i_2,\dots,i_k} \mathrm{E}\left|\boldsymbol{Q}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right| + \mathrm{E}\left|\boldsymbol{T}_{p\ell}\left(\begin{smallmatrix} j_1,j_2,\dots,j_r\\ k_1,k_2,\dots,k_s\end{smallmatrix}\right)\right|\\ & < 2|\alpha|^{-1} \end{aligned} \quad (85)$$

and similarly $b_{(s)}^{(r)}(\alpha) \le 2|\alpha|^{-1}$ at the mth iteration we obtain

$$a_{(0)}^{(0)}(\alpha) \le \frac{\varepsilon_m}{c}\sum_{i=2}^{m-1} c^i|\alpha|^{-i} + 2c^{k+1}|\alpha|^{-k-1}.$$

[6]In the following, $a_{(s)}^{(r)} = a_{(s)}^{(r)}(\alpha)$.

Note that $\varepsilon_{(s)}^{(r)} \to 0$ for $n \to \infty$, then also $\lim_{n\to\infty}\varepsilon_m = 0$. Since $a_{(0)}^{(0)}(\alpha) \ge 0$, for $c|\alpha|^{-1} < 1$

$$\lim_{m,n\to\infty} \frac{\varepsilon_m}{c}\sum_{i=2}^{m-1} c^i|\alpha|^{-i} + 2c^m|\alpha|^{-m} = 0.$$

Then

$$\lim_{m,n\to\infty} a_{(0)}^{(0)}(\alpha) = \lim_{n\to\infty}\max_{p,\ell} \mathrm{E}|\boldsymbol{Q}_{p\ell}(\alpha) - \boldsymbol{T}_{p\ell}(\alpha)| = 0 \quad (86)$$

for $c|\alpha|^{-1} < 1$. Since the matrices $\boldsymbol{Q}_{p\ell}(\alpha)$ and $\boldsymbol{T}_{p\ell}(\alpha)$, for $p, \ell = 1,\dots,p_1$ are analytical in $\alpha \in \mathbb{C}\backslash\mathbb{R}^-$ then the convergence holds for all $\alpha \in \mathbb{C}\backslash\mathbb{R}^-$.

The proof of the convergence in probability of the matrices $\boldsymbol{G}_{p\ell}$ to $\boldsymbol{R}_{p\ell}$ follows along the same line.

Lemma 11: Let us assume that the definitions of Lemma 10 hold and the conditions of Lemma 10 are satisfied.

Then, the $q_1 \times q_1$ matrices $\boldsymbol{C}_{kk}^{(1)}(\alpha), k = 1,\dots,p_1$, and the $q_2 \times q_2$ matrices $C_{\ell\ell}^{(2)}(\alpha), \ell = 1,\dots,p_2$, defined in (67) and (68), respectively, converge as $n \to \infty$ to the limit matrices

$$\lim_{n\to+\infty} \boldsymbol{C}_{kk}^{(1)} = \boldsymbol{\Psi}_{kk}^{(1)}, \qquad k = 1,\dots,p_1$$
$$\lim_{n\to+\infty} \boldsymbol{C}_{\ell\ell}^{(2)} = \boldsymbol{\Psi}_{\ell\ell}^{(2)}, \qquad \ell = 1,\dots,p_2$$

where $\boldsymbol{\Psi}_{kk}^{(1)}, k = 1,\dots,p_1$, and $\boldsymbol{\Psi}_{\ell\ell}^{(2)}, \ell = 1,\dots,p_2$ satisfy the canonical system of equations

$$\boldsymbol{\Psi}_{kk}^{(1)} = \alpha\boldsymbol{I} + \sum_{j=1}^{p_2}\mathrm{E}\left\{(\boldsymbol{\Xi}_{kj} - \boldsymbol{A}_{kj})\left\{\left[\widetilde{\boldsymbol{\Psi}}^{(2)} + \widetilde{\boldsymbol{A}}^H[\widetilde{\boldsymbol{\Psi}}^{(1)}]^{-1}\widetilde{\boldsymbol{A}}\right]^{-1}\right\}_{jj} \times (\boldsymbol{\Xi}_{kj} - \boldsymbol{A}_{kj})^H\right\}, \qquad k=1,\dots,p_1 \quad (87)$$

$$\boldsymbol{\Psi}_{\ell\ell}^{(2)} = \boldsymbol{I} + \sum_{j=1}^{p_1}\mathrm{E}\left\{(\Xi_{j\ell} - \boldsymbol{A}_{j\ell})^H\left\{\left[\widetilde{\boldsymbol{\Psi}}^{(1)} + \widetilde{\boldsymbol{A}}[\widetilde{\boldsymbol{\Psi}}^{(2)}]^{-1}\widetilde{\boldsymbol{A}}^H\right]^{-1}\right\}_{jj} \times (\boldsymbol{\Xi}_{j\ell} - \boldsymbol{A}_{j\ell})\right\}, \qquad \ell=1,\dots,p_2$$

$$\widetilde{\boldsymbol{A}} = (\boldsymbol{A}_{ij}),\ \widetilde{\boldsymbol{\Psi}}^{(1)} = \mathrm{diag}\{\boldsymbol{\Psi}_{kk}^{(1)}(\alpha)\},\ \widetilde{\boldsymbol{\Psi}}^{(2)} = \mathrm{diag}\{\boldsymbol{\Psi}_{\ell\ell}^{(2)}(\alpha)\}. \quad (88)$$

Proof: Let us consider the system of equations

$$\boldsymbol{C}_{kk}^{(1)} = \alpha\boldsymbol{I} + \sum_{j=1}^{p_2}\mathrm{E}\left(\boldsymbol{H}_{kj}(\boldsymbol{X})_{jj}\boldsymbol{H}_{kj}^H\right)_{\boldsymbol{X}=\alpha\widetilde{\boldsymbol{G}}}$$
$$\boldsymbol{C}_{kk}^{(2)} = \boldsymbol{I} + \sum_{j=1}^{p_1}\mathrm{E}\left(\boldsymbol{H}_{j\ell}^H(\boldsymbol{Y})_{jj}\boldsymbol{H}_{j\ell}\right)_{\boldsymbol{Y}=\widetilde{\boldsymbol{Q}}}.$$

Taking into account the convergence of $\boldsymbol{Q}_{p\ell}$ and $\alpha\boldsymbol{G}_{p\ell}$ to $\boldsymbol{T}_{p\ell}$ and $\boldsymbol{R}_{p\ell}$, respectively, shown in Lemma 10 (86) we can substitute $\boldsymbol{Q}_{p\ell}$ and $\alpha\boldsymbol{G}_{p\ell}$ with their limiting values. For finite n

$$\boldsymbol{C}_{kk}^{(1)} = \alpha\boldsymbol{I} + \sum_{j=1}^{p_2}\mathrm{E}\left(\boldsymbol{H}_{kj}((\widetilde{\boldsymbol{C}}^{(2)} + \widetilde{\boldsymbol{A}}^H(\widetilde{\boldsymbol{C}}^{(1)})^{-1}\widetilde{\boldsymbol{A}})^{-1})_{jj}\boldsymbol{H}_{kj}^H\right) + \boldsymbol{W}_{kk}^{(1)}$$
$$\boldsymbol{C}_{kk}^{(2)} = \boldsymbol{I} + \sum_{j=1}^{p_1}\mathrm{E}\left(\boldsymbol{H}_{j\ell}^H((\widetilde{\boldsymbol{C}}^{(1)} + \widetilde{\boldsymbol{A}}(\widetilde{\boldsymbol{C}}^{(2)})^{-1}\widetilde{\boldsymbol{A}}^H)^{-1})_{jj}\boldsymbol{H}_{j\ell}\right) + \boldsymbol{W}_{kk}^{(2)}$$

where $\widetilde{\boldsymbol{C}}^{(1)} = \mathrm{diag}(\boldsymbol{C}_{kk}^{(1)})_{k=1,\dots,p_1}$, $\widetilde{\boldsymbol{C}}^{(2)} = \mathrm{diag}(\boldsymbol{C}_{\ell\ell}^{(2)})_{\ell=1,\dots,p_2}$, and

$$\boldsymbol{W}_{kk}^{(1)} = \sum_{j=1}^{p_1} \mathrm{E}\left(\boldsymbol{H}_{kj}(\alpha(\alpha\boldsymbol{I}+\widetilde{\boldsymbol{\Xi}}^H\widetilde{\boldsymbol{\Xi}})^{-1} - (\widetilde{\boldsymbol{C}}^{(2)} + \widetilde{\boldsymbol{A}}^H(\widetilde{\boldsymbol{C}})^{-1}\widetilde{\boldsymbol{A}})^{-1})_{jj}\boldsymbol{H}_{kj}^H\right)$$

$$\boldsymbol{W}_{\ell\ell}^{(2)} = \sum_{j=1}^{p_2} \mathrm{E}\left(\boldsymbol{H}_{j\ell}^H((\alpha\boldsymbol{I}+\widetilde{\boldsymbol{\Xi}}\widetilde{\boldsymbol{\Xi}}^H)^{-1} - (\widetilde{\boldsymbol{C}}^{(1)} + \widetilde{\boldsymbol{A}}(\widetilde{\boldsymbol{C}}^{(2)})^{-1}\widetilde{\boldsymbol{A}}^H)^{-1})_{jj}\boldsymbol{H}_{j\ell}\right).$$

Let us consider $\boldsymbol{W}_{kk}^{(1)}$. By definition $\widetilde{\boldsymbol{G}} = (\alpha\boldsymbol{I}+\widetilde{\boldsymbol{\Xi}}^H\widetilde{\boldsymbol{\Xi}})^{-1}$ and $\widetilde{\boldsymbol{R}} = (\widetilde{\boldsymbol{C}}^{(2)} + \widetilde{\boldsymbol{A}}^H(\widetilde{\boldsymbol{C}}^{(1)})^{-1}\widetilde{\boldsymbol{A}})^{-1}$, thus

$$\begin{aligned}\max_k \mathrm{E}|\boldsymbol{W}_{kk}^{(1)}| &= \max_k \mathrm{E}\left|\sum_{j=1}^{p_1}(\boldsymbol{H}_{kj}(\alpha\widetilde{\boldsymbol{G}}-\widetilde{\boldsymbol{R}})_{jj}\boldsymbol{H}_{kj}^H)\right| \\ &\le \max_k \mathrm{E}\sum_{j=1}^{p_1}|\boldsymbol{H}_{kj}|^2|\alpha\boldsymbol{G}_{jj}-\boldsymbol{R}_{jj}| \\ &\le \max_k \mathrm{E}\left((\max_{j=1,\dots,p_2}|\alpha\boldsymbol{G}_{jj}-\boldsymbol{R}_{jj}|)\sum_{j=1}^{p_1}|\boldsymbol{H}_{kj}|^2\right).\end{aligned} \tag{89}$$

Here, (89) follows from the triangular inequality (24) and the submultiplicative inequality for the spectral norm (25). From hypothesis H-1 and the convergence shown in Lemma 10

$$\lim_{n\to\infty}\mathrm{E}\max_{j=1,\dots p_2}|\boldsymbol{G}_{jj}-\alpha^{-1}\boldsymbol{R}_{jj}| = 0$$

we obtain

$$\lim_{n\to\infty}\max_k \mathrm{E}|\boldsymbol{W}_{kk}^{(1)}| = 0.$$

Similarly, we can prove

$$\lim_{n\to\infty}\max_k \mathrm{E}|\boldsymbol{W}_{kk}^{(2)}| = 0.$$

Then, we get (90)–(91) at the bottom of the page. Let us consider the system of (87) and (88). The solution of this system coincides with the solution of the system

$$\boldsymbol{C}_{kk}^{(1)} = \alpha\boldsymbol{I}+\sum_{j=1}^{p_2}\mathrm{E}\left\{(\boldsymbol{\Xi}_{kj}-\boldsymbol{A}_{kj})\left\{\left[\widetilde{\boldsymbol{C}}^{(2)}+\widetilde{\boldsymbol{A}}^H[\widetilde{\boldsymbol{C}}^{(1)}]^{-1}\widetilde{\boldsymbol{A}}\right]^{-1}\right\}_{jj} \times(\boldsymbol{\Xi}_{kj}-\boldsymbol{A}_{kj})^H\right\}, \qquad k=1,\dots,p_1$$

$$\boldsymbol{C}_{\ell\ell}^{(2)} = \boldsymbol{I}+\sum_{j=1}^{p_1}\mathrm{E}\left\{(\boldsymbol{\Xi}_{j\ell}-\boldsymbol{A}_{j\ell})^H\left\{\left[\widetilde{\boldsymbol{C}}^{(1)}+\widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{C}}^{(2)}]^{-1}\widetilde{\boldsymbol{A}}^H\right]^{-1}\right\}_{jj} \times(\boldsymbol{\Xi}_{j\ell}-\boldsymbol{A}_{j\ell})\right\}, \qquad \ell=1,\dots,p_2.$$

The system of limits (90) and (91) guarantees that $\boldsymbol{C}_{kk}^{(1)}$ and $\boldsymbol{C}_{\ell\ell}^{(2)}$ defined in (67) and in (68) converge to the solutions of the system of equations defined by (87) and (88), $\boldsymbol{\Psi}_{kk}^{(1)}$ and $\boldsymbol{\Psi}_{\ell\ell}^{(2)}$, respectively.

Lemma 12: Let us assume that the definitions of Lemma 10 hold and the conditions of Lemma 10 are satisfied. Let us consider the system of canonical (87) and (88) with $\widetilde{\boldsymbol{A}}$ and $\boldsymbol{A}_{kj}$ defined in (xxiv). Then, the solution of the canonical system of (87) and (88) exists and it is unique in the class of nonnegative definite analytic matrices for $\mathrm{Re}(\alpha) > 0$.

This lemma can be proven along the same lines as the proofs of the existence and uniqueness of the solutions of canonical systems of equations in [21], [22]. For example, the reader can refer to the proof of Lemma 7.4 for the system of canonical equations K_7 in [21].

Appendix II
Proof of Theorem 1: Asymptotic Convergence of the Linear MMSE's SINR_k

The proof of Theorem 1 is based on the results of Lemmas 10–12. Then, let us verify that the matrix $\boldsymbol{\mathcal{H}}$ satisfies conditions H-1, H-2, H-3, and H-4 required by Lemmas 10 and 11. $\boldsymbol{\mathcal{H}}_{ij}$ is the $L\times 1$ block i,j of the matrix $\boldsymbol{\mathcal{H}}$. Consistently with the assumption on the linear MMSE detector that the channel gains are perfectly known at the receiver, we assume that the channel gains are given. Then, all blocks are independent since the spreading sequence elements are independent and condition H-4 is satisfied. Furthermore

$$\begin{aligned}\sup_N&\left[\max_{i=1,\dots N}\sum_{j=1}^{K}\mathrm{E}\{\|\boldsymbol{\mathcal{H}}_{ij}\|^2\} + \max_{i=1,\dots K}\sum_{j=1}^{N}\mathrm{E}\{\|\boldsymbol{\mathcal{H}}_{ji}\|^2\}\right] \\ &\le \sup_N\left[\frac{K}{N}\max_{j=1\dots K}\boldsymbol{l}_j^H\boldsymbol{l}_j + \max_{i=1\dots K}\boldsymbol{l}_i^H\boldsymbol{l}_i\right] \\ &\le \sup_N\left[(\beta+1)\max_{i=1,\dots,K}\boldsymbol{l}_i^H\boldsymbol{l}_i\right] < +\infty.\end{aligned} \tag{92}$$

The second inequality in (92) holds thanks to the assumption that $\|\boldsymbol{l}_i\|$ is uniformly bounded for all N. Then, $\boldsymbol{\mathcal{H}}$ satisfies condition H-1.

Condition H-2 is trivially verified since the entries of matrix $\boldsymbol{\mathcal{H}}$ are all zero mean.

In order to verify condition H-3, we focus on the limit

$$\lim_{K\to\infty}\sum_{j=1}^{K}\mathrm{E}\|\boldsymbol{\mathcal{H}}_{ij}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ij}\|>\tau) = 0 \tag{93}$$

for any $\tau > 0$. The limit

$$\lim_{N\to\infty}\sum_{j=1}^{N}\mathrm{E}\|\boldsymbol{\mathcal{H}}_{ji}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ji}\|>\tau) = 0 \tag{94}$$

can be computed in a similar way.

Let us observe that $\forall i,j$

$$\begin{aligned}\mathrm{E}\left(\|\boldsymbol{\mathcal{H}}_{ij}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ij}\|>\tau)\right) &= \boldsymbol{l}_j^H\boldsymbol{l}_j\int_{\left\{|s_{ij}|^2>\frac{\tau^2}{\boldsymbol{l}_j^H\boldsymbol{l}_j}\right\}}|s_{ij}|^2\mathrm{d}F(s_{ij}) \\ &\le \frac{(\boldsymbol{l}_j^H\boldsymbol{l}_j)^{1+\delta}}{\tau^{2\delta}}\int_{\{|s_{ij}|\ge 0\}}|s_{ij}|^{2+\delta}\mathrm{d}F(s_{ij})\end{aligned}$$

where $F(s_{ij})$ is the distribution function of s_{ij} and $\delta\in\mathbb{R}^+$. From the assumptions in Theorem 1, $\mathrm{E}(|s_{ij}|^4) \le \frac{1}{N^\gamma}$ with $\gamma > 1$. Then, for $\delta = 2$

$$\mathrm{E}\left(\|\boldsymbol{\mathcal{H}}_{ij}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ij}\|>\tau)\right) \le \frac{(\boldsymbol{l}_j^H\boldsymbol{l}_j)^3}{\tau^4N^\gamma}, \qquad \text{with } \gamma>1.$$

Since $\|\boldsymbol{l}_j\|$, for $j=1,\dots,K$ is uniformly bounded for all K, there exists a real number $m < +\infty$ such that $\max_{j=1,\dots,K}\boldsymbol{l}_j^H\boldsymbol{l}_j < m$ for all K and

$$\max_{i,j}\mathrm{E}\left(\|\boldsymbol{\mathcal{H}}_{ij}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ij}\|>\tau)\right) \le \frac{m^3}{\tau^4N^\gamma}, \qquad \text{with } \gamma>1.$$

$$\lim_{n\to\infty}\max_k\left|\boldsymbol{C}_{kk}^{(1)} - \alpha\boldsymbol{I} - \sum_{j=1}^{p_2}\mathrm{E}(\boldsymbol{H}_{kj}(\boldsymbol{C}^{(2)}+\widetilde{\boldsymbol{A}}^H(\boldsymbol{C}^{(1)})^{-1}\widetilde{\boldsymbol{A}})_{jj}^{-1}\boldsymbol{H}_{kj}^H)\right| = 0, \qquad k=1,\dots,p_1 \tag{90}$$

$$\lim_{n\to\infty}\max_\ell\left|\boldsymbol{C}_{\ell\ell}^{(2)} - \boldsymbol{I} - \sum_{j=1}^{p_1}\mathrm{E}(\boldsymbol{H}_{\ell j}^H(\boldsymbol{C}^{(1)}+\widetilde{\boldsymbol{A}}(\boldsymbol{C}^{(2)})^{-1}\widetilde{\boldsymbol{A}}^H)_{jj}^{-1}\boldsymbol{H}_{j\ell})\right| = 0, \qquad \ell=1,\dots,p_2. \tag{91}$$

Therefore

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \sum_{j=1}^{K} \mathrm{E}\left(\|\boldsymbol{\mathcal{H}}_{ij}\|^2\chi(\|\boldsymbol{\mathcal{H}}_{ij}\|>\tau)\right) \leq \lim_{N\to\infty} \frac{m^3\beta}{\tau^4 N^{\gamma-1}} = 0.$$

and H-3 is satisfied by the assumptions of Theorem 1.

The system of canonical equations (87) and (88) can be considerably simplified for the matrix $\boldsymbol{\mathcal{H}}^H\boldsymbol{\mathcal{H}}$. Equation (87) can be rewritten as

$$\boldsymbol{\Psi}_{kk}^{(1)} = \alpha\boldsymbol{I}_L + \sum_{j=1}^{K}\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{kj}([\widetilde{\boldsymbol{\Psi}}^{(2)}]^{-1})_{jj}\boldsymbol{\mathcal{H}}_{kj}^H\right\} \quad (95)$$

$$= \alpha\boldsymbol{I}_L + \sum_{j=1}^{K}([\widetilde{\boldsymbol{\Psi}}^{(2)}]^{-1})_{jj}\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{kj}\boldsymbol{\mathcal{H}}_{kj}^H\right\} \quad (96)$$

$$= \alpha\boldsymbol{I}_L + \boldsymbol{K}^{(1)} = \boldsymbol{\Psi}^{(1)}. \quad (97)$$

The step from (95) to (96) is justified by the fact that $([\widetilde{\boldsymbol{\Psi}}^{(2)}]^{-1})_{jj}$ is a scalar (1×1 matrix). Equation (97) emphasizes that the matrix

$$\boldsymbol{K}^{(1)} \triangleq \sum_{j=1}^{K}([\widetilde{\boldsymbol{\Psi}}^{(2)}]^{-1})_{jj}\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{kj}\boldsymbol{\mathcal{H}}_{kj}^H\right\} \quad (98)$$

is independent of k since $\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{kj}\boldsymbol{\mathcal{H}}_{kj}^H\right\} = \mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{k'j}\boldsymbol{\mathcal{H}}_{k'j}^H\right\}$ for all $k,k'=1,\ldots,N$.

Equation (88) can be specialized to system (1) as follows:

$$\boldsymbol{\Psi}_{ll}^{(2)} = 1 + \sum_{j=1}^{N}\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{jl}^H([\widetilde{\boldsymbol{\Psi}}^{(1)}]^{-1})_{jj}\boldsymbol{\mathcal{H}}_{jl}\right\} \quad (99)$$

$$= 1 + \sum_{j=1}^{N}\mathrm{E}\left\{\boldsymbol{l}_l^H([\boldsymbol{\Psi}^{(1)}]^{-1})\boldsymbol{l}_l|s_{jl}|^2\right\} \quad (100)$$

$$= 1 + \boldsymbol{l}_l^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_l. \quad (101)$$

Substituting (101) in (98), (97) can be rewritten as

$$\boldsymbol{\Psi}^{(1)} = \alpha\boldsymbol{I}_L + \sum_{j=1}^{K}\frac{\mathrm{E}\left\{\boldsymbol{\mathcal{H}}_{kj}\boldsymbol{\mathcal{H}}_{kj}^H\right\}}{1+\boldsymbol{l}_j^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_j} = \alpha\boldsymbol{I}_L + \frac{1}{N}\sum_{j=1}^{K}\frac{\boldsymbol{l}_j\boldsymbol{l}_j^H}{1+\boldsymbol{l}_j^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_j}.$$

Then, considering the limit for $K,N\to\infty$

$$\boldsymbol{\Psi}^{(1)} = \alpha\boldsymbol{I}_L + \beta\int\frac{\boldsymbol{l}\boldsymbol{l}^H}{1+\boldsymbol{l}^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}}\mathrm{d}\,F_{\boldsymbol{l}}(l_1,l_2,\ldots,l_L) \quad (102)$$

and substituting $(\boldsymbol{\Psi}^{(1)})^{-1} = \frac{\boldsymbol{A}}{\alpha}$ we obtain (9) for $\alpha=\sigma^2$.

Let $\boldsymbol{\mathcal{U}} = (\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H + \sigma^2\boldsymbol{I})^{-1}$ and let $\boldsymbol{U}_{ij}$, $i,j=1,\ldots,N$ be its $L\times L$ matrix-block elements. From Lemma 10

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \mathrm{E}|\boldsymbol{U}_{ij}-\boldsymbol{T}_{ij}| = 0$$

with $\boldsymbol{T}_{ij} = [\boldsymbol{C}_{ii}^{(1)}(\sigma^2)]^{-1}\delta_{ij}$ and

$$\boldsymbol{C}_{kk}^{(1)}(\sigma^2) = \sigma^2\boldsymbol{I} + \sum_{j=1}^{K}(\mathrm{E}|s_{jk}|^2\boldsymbol{l}_k^H\boldsymbol{X}_{jj}\boldsymbol{l}_k)|_{\boldsymbol{X}=\sigma^2(\boldsymbol{\mathcal{H}}^H\boldsymbol{\mathcal{H}}+\sigma^2\boldsymbol{I})^{-1}}.$$

Lemma 11 guarantees that

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \boldsymbol{T}_{ij} = [\boldsymbol{\Psi}^{(1)}]^{-1}\delta_{ij}.$$

Here, $\boldsymbol{\mathcal{L}}_k$ denotes the $LN\times K$ block-diagonal matrix whose blocks are identically equal to $\boldsymbol{l}_k$. Its maximum singular value is equal to $\sqrt{\boldsymbol{l}_k^H\boldsymbol{l}_k} < +\infty$ since $\|\boldsymbol{l}\|$ is uniformly bounded for all K. Then, $\boldsymbol{h}_k = \boldsymbol{\mathcal{L}}_k\boldsymbol{s}_k$, where $\boldsymbol{s}_k$ is the kth column of the spreading matrix $\boldsymbol{S}$.

The convergence in probability of $\mathrm{SINR}_k = \boldsymbol{h}_k^H\boldsymbol{\mathcal{U}}\boldsymbol{h}_k$ to the quantity $\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k$ is proven if $\eta_1 = \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{\mathcal{U}}\boldsymbol{h}_k - \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k|$ vanishes asymptotically, i.e.,

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \eta_1 = 0. \quad (103)$$

The rest of the proof is focused on showing (103). Let us observe

$$\eta_1 \leq \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{\mathcal{U}}\boldsymbol{h}_k - \boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k| + \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k - \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k|$$

where the triangular inequality of the spectral norm is applied and $\boldsymbol{T} = \mathrm{diag}([\boldsymbol{C}_{kk}^{(1)}(\sigma^2)]^{-1})_{k=1,\ldots,N}$.

By applying the submultiplicative inequality for spectral norms (25) and the triangular inequality (24) to the first term we obtain

$$\begin{aligned}\mathrm{E}|\boldsymbol{h}_k^H(\boldsymbol{\mathcal{U}}-\boldsymbol{T})\boldsymbol{h}_k| &= \mathrm{E}|\sum_{i,\ell}s_{ik}^*\boldsymbol{l}_k^H(\boldsymbol{\mathcal{U}}-\boldsymbol{T})_{i\ell}\boldsymbol{l}_k s_{\ell k}|\\ &\leq \sum_{i,\ell}\mathrm{E}|\boldsymbol{U}_{i\ell}-\boldsymbol{T}_{i\ell}|\boldsymbol{l}_k^H\boldsymbol{l}_k\mathrm{E}|s_{ik}^*s_{\ell k}|\\ &= \sum_i\mathrm{E}|\boldsymbol{U}_{ii}-\boldsymbol{T}_{ii}|\frac{\boldsymbol{l}_k^H\boldsymbol{l}_k}{N}\\ &\leq \boldsymbol{l}_k^H\boldsymbol{l}_k\max_i\mathrm{E}|\boldsymbol{U}_{ii}-\boldsymbol{T}_{ii}|.\end{aligned}$$

Thanks to Lemma 10 and the fact that $\boldsymbol{l}_k^H\boldsymbol{l}_k < +\infty$

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \mathrm{E}|\boldsymbol{h}_k^H(\boldsymbol{\mathcal{U}}-\boldsymbol{T})\boldsymbol{h}_k| = 0.$$

In order to prove the convergence to zero of $\eta_2 = \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k - \boldsymbol{l}_k^H\$[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k|$ we consider

$$\begin{aligned}\eta_2^2 &\leq \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k - \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k|^2\\ &= \mathrm{E}((\boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k)^2 - 2\boldsymbol{h}_k^H\boldsymbol{T}\boldsymbol{h}_k\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k + \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k)\\ &= \mathrm{E}\left(\sum_{ij}\boldsymbol{l}_k^H\boldsymbol{T}_{ii}\boldsymbol{l}_k\boldsymbol{l}_k^H\boldsymbol{T}_{jj}\boldsymbol{l}_k|s_{ik}|^2|s_{jk}|^2 - 2\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k\right.\\ &\qquad\left.\times\sum_i\boldsymbol{l}_k^H\boldsymbol{T}_{ii}\boldsymbol{l}_k|s_{ik}|^2 + (\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k)^2\right) \quad (104)\\ &= \sum_i(\boldsymbol{l}_k^H\boldsymbol{T}_{ii}\boldsymbol{l}_k)^2\frac{1}{N^\gamma} + \sum_{\substack{i,j\\ i\neq j}}(\boldsymbol{l}_k^H\boldsymbol{T}_{ii}\boldsymbol{l}_k)(\boldsymbol{l}_k^H\boldsymbol{T}_{jj}\boldsymbol{l}_k)\frac{1}{N^2}\\ &\quad - \frac{2}{N}\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k\sum_i\boldsymbol{l}_k^H\boldsymbol{T}_{ii}\boldsymbol{l}_k + (\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k)^2. \quad (105)\end{aligned}$$

From (104) to (105) we make use of the assumptions on the second and fourth moments of s_{ij}. Let us observe that the spectral norm of $[\boldsymbol{\Psi}^{(1)}]^{-1}$ and $\boldsymbol{T}_{ii}$, for any i, are bounded by $|[\boldsymbol{\Psi}^{(1)}]^{-1}| < \sigma^2$ and $|\boldsymbol{T}_{ii}| < \sigma^2$. Then, the first term in (105) vanishes as $N\to\infty$ since $\gamma > 1$. By appealing Lemma 11, for any i, $\boldsymbol{T}_{ii} \to [\boldsymbol{\Psi}^{(1)}]^{-1}$ as $K,N\to\infty$ with $\frac{K}{N}\to\beta$. Then, the second and third terms in (105) converge to $(\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k)^2$ and $-2(\boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k)^2$, respectively. We can conclude that

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \eta_2^2 = 0$$

and $\eta_2\to 0$ as $K,N\to\infty$ as $\frac{K}{N}\to\beta$. Therefore, (103) is proven. The Markov inequality implies that, $\forall\varepsilon > 0$

$$\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \Pr\{|\boldsymbol{h}_k^H\boldsymbol{\mathcal{U}}\boldsymbol{h}_k - \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k| > \varepsilon\} \leq \frac{1}{\varepsilon}\lim_{\substack{K,N\to\infty\\ \frac{K}{N}\to\beta}} \mathrm{E}|\boldsymbol{h}_k^H\boldsymbol{\mathcal{U}}\boldsymbol{h}_k - \boldsymbol{l}_k^H[\boldsymbol{\Psi}^{(1)}]^{-1}\boldsymbol{l}_k| = 0$$

and the convergence in probability stated in Theorem 1 is proven.

This concludes the proof of Theorem 1.

APPENDIX III
PROOF OF THEOREM 2: ASYMPTOTIC CONVERGENCE OF BAYESIAN FILTER RECEIVER'S SINR_k

Let us derive first the Bayesian filter. To this aim we calculate $\mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\}^{-1}$ and $\mathrm{E}\{b_k^*\boldsymbol{y}\}$ with the expectation taken over the noise, all transmitted signals, and over all transmitted powers, channel gains, and spreading sequences of all interferers. Then, $\mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\} = \mathrm{E}\{\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H\} + \boldsymbol{h}_k\boldsymbol{h}_k^H + \sigma^2\boldsymbol{I}_{NL}$. Because of the independence and zero mean of the elements of the spreading sequences, $\mathrm{E}\{\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H\}$ is a block-diagonal matrix with N blocks of size $L \times L$. Each block is given by $(\frac{K-1}{N})\boldsymbol{C_l}$ with $\boldsymbol{C_l} = \mathrm{E}\{\boldsymbol{l}\boldsymbol{l}^H\}$. It follows that

$$\mathcal{C}_{\boldsymbol{\mathcal{H}}} = \mathrm{E}\{\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H\} = \boldsymbol{I}_N \otimes \left(\frac{K-1}{N}\right)\boldsymbol{C_l}. \quad (106)$$

By applying the Sherman–Morrison equation we obtain

$$\begin{aligned}(\mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\})^{-1} &= (\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1} - (\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k^H \\ &\quad\times(1 + \boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k)\boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}.\end{aligned}$$

Let us observe that $\mathrm{E}\{b_k^*\boldsymbol{y}\} = \boldsymbol{h}_k$. The Bayesian receiver is given by

$$\boldsymbol{c}_k = \frac{(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k}{1 + \boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k}.$$

The energy of the useful signal k at the output of the Bayesian filter is given by

$$\mathrm{E}\{|\boldsymbol{c}_k^H\boldsymbol{h}_k b_k|^2\} = \left(\frac{\boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k}{1 + \boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k}\right)^2.$$

The energy of the noise at the output of the Bayesian filter is

$$\mathrm{E}\{|\boldsymbol{c}_k^H\boldsymbol{n}|^2\} = \sigma^2\frac{\boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-2}\boldsymbol{h}_k}{(1 + \boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k)^2}.$$

Finally, the energy of the interferers is

$$\begin{aligned}\mathrm{E}\left\{\sum_{\substack{j=1\\j\neq k}}^{K}|\boldsymbol{c}_k^H\boldsymbol{h}_j b_j|^2\right\} &= \mathrm{E}\left\{\sum_{\substack{j=1\\j\neq k}}^{K}\boldsymbol{c}_k^H\boldsymbol{h}_j\boldsymbol{h}_j^H\boldsymbol{c}_k\right\} \\ &= \boldsymbol{c}_k^H\mathrm{E}\left\{\sum_{\substack{j=1\\j\neq k}}^{K}\boldsymbol{h}_j\boldsymbol{h}_j^H\right\}\boldsymbol{c}_k \\ &= \boldsymbol{c}_k^H\mathrm{E}\{\boldsymbol{\mathcal{H}}_k\boldsymbol{\mathcal{H}}_k^H\}\boldsymbol{c}_k \\ &= \boldsymbol{c}_k^H\mathcal{C}_{\boldsymbol{\mathcal{H}}}\boldsymbol{c}_k.\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}\mathrm{SINR}_k &= \frac{\mathrm{E}\{|\boldsymbol{c}_k^H\boldsymbol{h}_k b_k|^2\}}{\mathrm{E}\{|\boldsymbol{c}_k^H\boldsymbol{n}|^2\} + \mathrm{E}\left\{\sum_{\substack{j=1\\j\neq k}}^{K}|\boldsymbol{c}_k^H\boldsymbol{h}_j b_j|^2\right\}} \\ &= \boldsymbol{h}_k^H(\mathcal{C}_{\boldsymbol{\mathcal{H}}} + \sigma^2\boldsymbol{I}_{NL})^{-1}\boldsymbol{h}_k \\ &= \boldsymbol{l}_k^H((\beta - \frac{1}{N})\boldsymbol{C_l} + \sigma^2\boldsymbol{I}_L)^{-1}\boldsymbol{l}_k\sum_{n=1}^{N}s_{nk}s_{nk}^*. \quad (107)\end{aligned}$$

Applying the strong law of large numbers we obtain the convergence of SNIR_k to $\boldsymbol{l}_k^H(\beta\boldsymbol{C_l} + \sigma^2\boldsymbol{I}_L)^{-1}\boldsymbol{l}_k$ with probability 1 as $N \to \infty$.

APPENDIX IV
PROOF OF THEOREM 3

The proof of Theorem 3 follows the same lines as the proof of Theorem 2, taking into account that $\boldsymbol{c}_k = \boldsymbol{h}_k$. Then

$$\begin{aligned}\mathrm{SINR}_k &= \frac{\mathrm{E}\{|\boldsymbol{h}_k^H\boldsymbol{h}_k b_k|^2\}}{\mathrm{E}\{|\boldsymbol{h}_k^H\boldsymbol{n}|^2\} + \mathrm{E}\{\sum_{\substack{j=1\\j\neq k}}^{K}|\boldsymbol{h}_k^H\boldsymbol{h}_j b_j|^2\}} \\ &= \frac{(\boldsymbol{h}_k^H\boldsymbol{h}_k)^2}{\sigma^2\boldsymbol{h}_k^H\boldsymbol{h}_k + \boldsymbol{h}_k^H\mathcal{C}_k\boldsymbol{h}_k} \\ &= \frac{(\boldsymbol{l}_k^H\boldsymbol{l}_k\sum_{n=1}^{N}s_{nk}s_{nk}^*)^2}{\boldsymbol{l}_k^H\left[\left(\beta - \frac{1}{N}\right)\boldsymbol{C_l} + \sigma^2\boldsymbol{I}_L\right]\boldsymbol{l}_k\sum_{n=1}^{N}s_{nk}s_{nk}^*}. \quad (108)\end{aligned}$$

Applying the strong law of large numbers, we obtain the almost sure convergence of SINR_k as $K, N \to \infty$ with $\frac{K}{N} \to \beta$. More specifically, we obtain

$$\lim_{K=\beta N\to\infty}\mathrm{SNIR}_k \overset{a.s.}{=} \frac{(\boldsymbol{l}_k^H\boldsymbol{l}_k)^2}{\boldsymbol{l}_k^H(\beta\boldsymbol{C_l} + \sigma^2\boldsymbol{I}_L)\boldsymbol{l}_k}.$$

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Blind OFDM Channel Estimation Using FIR Constraints: Reduced Complexity and Identifiability

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***Abstract*—In this correspondence, blind channel estimators exploiting finite alphabet constraints are discussed for orthogonal frequency-division multiplexing (OFDM) systems. Considering the channel and data jointly, a joint maximum-likelihood (JML) algorithm is described, along with identifiability conditions in the noise-free case. This approach enables development of general identifiability conditions for the minimum-distance (MD) finite alphabet blind algorithm of Zhou and Giannakis. Both the JML and MD algorithms suffer from high numerical complexity, as they rely on exhaustive search methods to resolve a large number of ambiguities. We present a substantially more efficient blind algorithm, the reduced complexity minimum distance (RMD) algorithm, by exploiting properties of the assumed finite-length impulse response (FIR) channel. The RMD algorithm exploits constraints on the unwrapped phase of FIR systems and results in significant reductions in numerical complexity over existing methods. In many cases, the RMD approach is able to completely eliminate the exhaustive search of the JML and MD approaches, while providing channel estimates of the same quality.**

***Index Terms*—Blind channel estimation, finite-length impulse response (FIR), orthogonal frequency-division multiplexing (OFDM).**

I. Introduction

Orthogonal frequency-division multiplexing (OFDM) is a popular modulation technique used in a variety of high-rate digital communication standards, including digital audio and video broadcasting (DAB, DVB) [1], [2] and high-speed broadband wireless local area networks (IEEE 802.11a and HIPERLAN/2) [3], [4]. Similar to other transmission schemes, such as single-carrier time-division multiple access (TDMA) and code-division multiple access (CDMA), OFDM systems require knowledge of the channel impulse response, or channel state information (CSI), in order to perform equalization. For the purpose of channel estimation, training data may proceed or be inserted into a data block at a modest expense of system throughput and various pilot-aided channel estimation algorithms have been developed [5]–[8].

Either to reduce the overhead of training data, or for applications such multicasting or eavesdropping, for which training is either inconvenient or unavailable, blind algorithms aim to perform equalization in the absence of training sequences or pilots, and a number of blind channel estimation algorithms have been proposed and investigated in the literature [9], [10]. For OFDM systems, the cyclic prefix can also be effectively exploited for blind identification. Correlation-matching methods have been developed in [11] and [12], by taking advantage of the cyclostationarity of the transmitted OFDM signal. In [13], the structure induced in the autocorrelation matrix of the received signal due to the cyclic prefix is used to develop a subspace-based blind identification algorithm. However, this method fails to guarantee identifiability for channels with nulls located on subcarriers (discrete Fourier transform (DFT) bins of the associated channel impulse response that are identically zero). A subspace algorithm that guarantees channel identifiability was proposed for OFDM systems with zero padding in [14] at the expense of structural changes to the transmitter and receiver. As is common to many subspace approaches, numerical complexity can become an issue due to the necessary matrix multiplications and singular value decompositions (SVD) of matrices of degree proportional to the number of subcarriers used [11], [12], restricting their use.

Joint channel and data estimation methods have also been employed in a number of scenarios, including that shown in [15], in which statistical sampling methods are used for single-carrier systems with turbo-equalization-based receivers, and [16] in which sequential Monte Carlo (SMC) methods are studied for OFDM systems. Even though such SMC methods can be shown to be asymptotically optimal for certain criteria, they suffer a number of potentially major drawbacks. First, they typically require high numerical complexity, due to their iterative nature. Second, their performance can be sensitive to the amount and quality of prior knowledge available about the channel, which is often rather limited. Decision-directed approaches use quantized outputs from the equalizer or the decoder, as does the blind algorithm in [17], [18], which makes use of the output of an error correction decoder to refine a channel estimate starting from a coarse initial channel estimate. This comes at the expense of numerical complexity, storage, and latency involved in transferring the data between the additional encoding and decoding.

finite alphabet constraints are often employed in decision-directed strategies, where hard decisions (quantization of equalizer or decoder outputs to valid symbols) facilitate channel estimation when such symbols are largely correct. The existence of a finite alphabet constraint on the transmitted data has been applied to enable factorization of the data matrix in multiple-input multiple-output blind channel estimation for single-carrier communication systems [19], [20]. Recently, in [21] and [22], a minimum distance (MD) blind algorithm exploiting finite alphabet constraints was proposed for OFDM systems and shown to have a number of benefits, including guaranteeing channel identifiability, regardless of the existence of null subcarriers, and requiring fewer samples for convergence under phase-shift keying (PSK) signaling. However, the MD algorithm relies on exhaustive search techniques for resolving ambiguities for each subcarrier. As such, the numerical complexity becomes exponential in the number of subcarriers, making it rapidly become intractable for systems with large numbers of subcarriers. To combat the complexity of these approaches that rely on exhaustive search, in [21], a number of reduced complexity modifications

Manuscript received May 20, 2005; revised November 28, 2006.

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Communicated by X. Wang, Associate Editor for Detection and Estimation.

Color versions of Figures 3–6 in this correspondence are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2006.890774