Diversity and Coding Gain of Linear and Decision-Feedback Equalizers for Frequency-Selective SIMO Channels

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***Abstract*— Since the introduction of the diversity-rate tradeoff by Zheng and Tse for ML reception in frequency-flat MIMO channels, some results have been obtained also for the diversity behavior of suboptimal receivers such as linear and decision-feedback equalizers for frequency-selective SIMO channels. However, these results are limited to infinite length equalizers. Furthermore, so far attention has focused mostly on just diversity order aspects of diversity. In this paper we analyze the diversity of more practical FIR equalizers. We show in particular that in the case of multiple subchannels, the diversity of infinite length filters can also be attained by FIR equalizers of sufficient length. Increasing the filter lengths improves the coding gain though. Whereas the diversity order determines the slope of the asymptote at high SNR, the coding gain determines its position.**

I. Introduction

Consider a linear modulation scheme and single-carrier transmission over a Single Input Single Output (SISO) or Multiple Output (SIMO) linear channel with additive white noise. The multiple (subchannel) outputs will be mainly thought of as corresponding to multiple receive antennas. After a Rx filter (possibly noise whitening), we sample the received signal to obtain a discrete-time system at symbol rate[1]. After stacking the samples corresponding to multiple subchannels in column vectors, the discrete-time communication system is described by

$$\underbrace{\boldsymbol{y}_k}_{p\times 1} = \underbrace{\mathbf{h}[q]}_{p\times 1}\,\underbrace{a_k}_{1\times 1} + \underbrace{\boldsymbol{v}_k}_{p\times 1} \tag{1}$$

where p is the number of subchannels, the noise power spectral density matrix is $S_{\boldsymbol{vv}}(z) = \sigma_v^2\, I$, q^{-1} is the unit sample delay operator: $q^{-1}\, a_k = a_{k-1}$, and $\mathbf{h}[z] = \sum_{i=0}^{L} \mathbf{h}_i\, z^{-i}$ is the channel transfer function in the z domain. The channel delay spread is L symbol periods. In the Fourier domain we get the vector transfer function $\mathbf{h}(f) = \mathbf{h}[e^{j2\pi f}]$.

Let $\mathbf{h} = [\mathbf{h}_0^T \cdots \mathbf{h}_L^T]^T$ contain all channel elements. Assume the energy normalization $\mathrm{tr}\{R_{\mathbf{hh}}\} = p$ with $R_{\mathbf{hh}} = \mathrm{E}\,\{\mathbf{h}\mathbf{h}^H\}$ ($.^H$ denotes Hermitian transpose). By default we shall assume the i.i.d. complex Gaussian channel model: $\mathbf{h} \sim \mathcal{CN}(0, \frac{1}{L+1} I_{p(L+1)})$ so that spatio-temporal diversity of order $p(L+1)$ is available. The average per subchannel SNR is $\rho = \frac{\sigma_a^2}{\sigma_v^2}$. In this paper we consider full channel state information at the receiver (Rx) (CSIR) and none at the transmitter (Tx) (CSIT).

Whereas in non-fading channels, probability of error P_e decreases exponentially with SNR, for a given symbol constellation, in fading channels the probability of error averaged over the channel distribution $P_e \sim \rho^{-d}$ for large SNR ρ, where d is the diversity order. On the other hand, at high SNR the channel capacity increases with SNR as $\log \rho$, which can be achieved with adaptive modulation. In [1] it was shown however that both benefits at high SNR cannot be attained simultaneously and a compromise has to be accepted. In [1] the frequency-flat MIMO channel was considered. These results were extended to the frequency-selective SISO channel in [2] and the frequency-selective MIMO channel in [3]. In [4], it was shown for the frequency-selective SIMO channel that a Decision-Feedback Equalizer (DFE) with unconstrained feedforward filter allows to attain the optimum diversity and similar results for the MIMO frequency-flat channel case, with a linear MIMO prefilter and a MMSE MIMO DFE appears in [5]. These last results confirm the interpretation of the DFE as canonical Rx [6].

In practice also the Linear Equalizer (LE) is used often since its settings are easier to compute and there is no error propagation. Also in practice, for both LE and DFE, only a limited degree of non-causality (delay) can be used and the filters are usually of finite length (FIR). Some initial investigation via simulations into the diversity aspects of FIR LE Rx's appears in [7]. Analytical investigations into the diversity for SISO with LEs are much more recent, see [8] for linearly precoded OFDM and [9] for Single-Carrier with Cyclic Prefix (SC-CP). Earlier on the mean LE SINR for broadband SIMO was investigated in [10]. The use of the DFE appears in [11] (FIR) and [12],[13] (SC-CP) where in the last two references diversity behavior is investigated through simulations.

[1] In the case of additional oversampling with integer factor, we would vectorize the samples to get a per antenna vector received signal sequence at symbol rate.

To describe transmission over frequency-selective channels with time-invariant filters and frequency-domain formulas implies the use of infinite block lengths. This represents of course a strong simplification in the context of time-varying wireless systems. In practice time-invariant filters can be used over finite block lengths if guard intervals or prefixes are introduced, or the exact treatment of transmission over finite blocks requires time-varying operators.

II. SINR of Optimal and Infinite-Length Non-Causal Suboptimal Receivers

We get for the Matched Filter Bound (MFB)

$$\text{MFB} = \rho \, \|\mathbf{h}\|^2 \, , \, \rho = \frac{\sigma_a^2}{\sigma_v^2} \, , \, \|\mathbf{h}\|^2 = \sum_{i=0}^{L} \|\mathbf{h}_i\|_2^2 = \int_{-0.5}^{0.5} \|\mathbf{h}(f)\|_2^2 \, df$$

where e.g. $\|\mathbf{h}_i\|_2^2 = \mathbf{h}_i^H \mathbf{h}_i$ and $\mathbf{h}^\dagger[z] = \mathbf{h}^H[1/z^*]$ denotes the paraconjugate (matched filter) ($.^*$ denotes complex conjugate). The MFB corresponds to Maximum Ratio Combining (MRC) of all energy in the spatio-temporal channel. The MFB is a close approximation (up to a propagation of errors type of deviation) for the performance of Maximum Likelihood Sequence Detection (MLSD). In fact [14], MFB $= \frac{\sigma_a^2}{\text{CRB}_a}$ in which appears the Cramer-Rao Bound (CRB) for estimating the symbol of interest without any constraint on this symbol, and with finite alphabet constraints on all other symbols. In practice one is often forced to resort to suboptimal Rx's when the delay spread and or the constellation size get large. Two popular classes of suboptimal Rx's are linear and decision-feedback equalizers (LE and DFE). Both types of equalizers are in fact linear estimators of the transmitted symbol sequence, one is based on the received signal only whereas the other is also based on the past decisions.

The goal of these suboptimal Rx's is to transform the frequency-selective channel into a frequency-flat channel the performance of which depends on the Signal-to-Interference-plus-Noise Ratio (SINR) at its output. For Mutual Information (C) purposes, the channel-equalizer cascade is treated as an AWGN channel, hence C $= \log(1 + \text{SINR})$. The MFB can alternatively be interpreted as the SINR at the output of an unbiased MMSE (UMMSE) non-causal DFE, in which the feedback involves all other symbols but the symbol of interest [15], a Rx structure that is now standard in turbo equalization. In the SIMO multichannel context considered here, a zero-forcing (ZF) LE is not unique, since $\mathbf{h}(f)$ has a non-empty orthogonal complement. Among all the ZF equalizers, there is one that will minimize the noise enhancement (MSE), which hence can be called the MMSE-ZF design. For a DFE, which has a feedforward and a feedback filter, this non-uniqueness already arises for a SISO channel. To simplify notation, we shall henceforth refer to the MMSE-ZF design as the ZF design. Introduce

$$\delta = \begin{cases} 0 & \text{, MMSE-ZF design,} \\ 1 & \text{, MMSE design.} \end{cases} \tag{2}$$

For infinite-length non-causal (feedforward) filters, we get the following SINR results

- $\text{MFB} = \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} \|\mathbf{h}(f)\|^2 \, df = \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} (\|\mathbf{h}(f)\|^2 + \frac{\delta}{\rho}) \, df - \delta$ *arithmetic* average
- $\text{SINR}_{DFE}^{\delta} = \rho \, \exp\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(\|\mathbf{h}(f)\|^2 + \frac{\delta}{\rho}) \, df\right] - \delta$ *geometric* average
- $\text{SINR}_{LE}^{\delta} = \rho \, \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} (\|\mathbf{h}(f)\|^2 + \frac{\delta}{\rho})^{-1} \, df\right]^{-1} - \delta$ *harmonic* average

with inequalities

$$\text{SINR}_{LE}^{\delta} \, \leq \, \text{SINR}_{DFE}^{\delta} \, \leq \, \text{MFB} \, , \; \text{SINR}^0 \, \leq \, \text{SINR}^1 \tag{3}$$

where the last inequality holds for either LE or DFE. For the case of MMSE design, the SINR here corresponds to the SINR computed correctly (SINR $= \frac{\sigma_a^2}{\text{MSE}} - 1$) which might be more easily interpreted in terms of Unbiased MMSE (UMMSE) design [6].

III. Outage-Rate Tradeoff

The SINR is random due to its dependence on the random channel $\mathbf{h}$. In [16], it was demonstrated that at high SNR outage only depends on the SINR distribution behavior near zero (this was also observed in [1]). This result is quite immediate. Indeed, let us introduced the normalized SINR γ through SINR $= \rho \, \gamma$ and consider the dominating term in the cumulative distribution function (cdf) of γ:

$$\text{Prob}\{\gamma \leq \epsilon\} = c \, \epsilon^k \tag{4}$$

for small $\epsilon > 0$. Then the outage probability for a certain outage threshold α is

$$\text{Prob}\{\text{SINR} \leq \alpha\} = c \left(\frac{\alpha}{\rho}\right)^k = \left(\frac{\alpha}{g \, \rho}\right)^k \tag{5}$$

from which we see that k is the diversity order and $g = c^{-1/k}$ is the coding gain (reduction in SNR required for identical outage probability).

Now consider outage in terms of outage capacity. Since at high SNR the SINR will tend to be proportional to ρ, the mutual information $C = \log(1 + \text{SINR})$ will tend to be $\log \rho$. So consider the rate $R = r \log \rho$ (in nats, assuming natural logarithm) where $r \in [0, 1]$ is the normalized rate. Then the outage probability at high SNR is

$$\begin{aligned} P_o \; &= \; \text{Prob}\{C < R\} = \text{Prob}\{\log(1 + SINR) < \log(\rho^r)\} \\ &= \; \text{Prob}\{\rho\gamma < \rho^r - 1\} = \text{Prob}\{\gamma < \tfrac{1}{\rho^{(1-r)}} - \tfrac{1}{\rho}\} \\ &= \; \text{Prob}\{\gamma < \tfrac{1}{\rho^{(1-r)}}\} \, , \text{ for } r > 0 \\ &= \; c \, \frac{1}{\rho^{(1-r)k}} \; = \; \frac{1}{(g \, \rho)^{(1-r)k}} \end{aligned} \tag{6}$$

Hence for the system with the SINR considered, we get for $r \in (0, 1]$:

$$d(r) = (1 - r) \, k \;\; , \;\; g(r)(\text{dB}) = -\frac{10}{(1 - r)k} \log_{10} c \tag{7}$$

where $d(r)$ is the diversity(order)-rate tradeoff and $g(r)$ is the tradeoff dependent coding gain. As $c > 1$ usually, the coding gain is actually a coding loss that decreases with increasing

diversity order k and decreasing rate r. The case $r = 0$ (fixed rate) requires separate investigation. The above picture may be somewhat oversimplified since possibly $\gamma = \gamma(\rho)$ and in the tradeoff analysis, $\epsilon = \epsilon(\rho)$. As a result it may happen that c and k vary with r also. This occurs in the MIMO case [1] but does not appear to occur in the single stream case considered here.

The mutual information for the frequency-selective channel with white Gaussian input is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(\|\mathbf{h}(f)\|^2 + \frac{\delta}{\rho})\, df = \log(1 + \text{SINR}_{DFE}^{MMSE}) \qquad (8)$$

which reconfirms that canonical character of the MMSE DFE Rx. In [4] it was shown that $\text{SINR}_{DFE}^{MMSE} \geq \beta\,\text{MFB}$ for some constant β that only depends on the delay spread L. As a result we have (4) with $k = p(L{+}1)$ and we get the optimal tradeoff

$$d^*(r) = (1 - r)\, p\, (L + 1) \quad , \quad r \in [0, 1] \qquad (9)$$

which will be valid for any distribution of $\mathbf{h}$ that has finite positive density everywhere (e.g. Gaussian with non-singular covariance matrix $R_{\mathbf{hh}}$). This tradeoff can be achieved by transmitting i.i.d. QAM symbols from a constellation of size $e^R = \rho^r$ and using a MMSE DFE Rx. As shown in [5], at high SNR the probability of (symbol or frame) error is dominated by the outage probability (see also [1]). In [5] it is also shown that residual inter-symbol interference (ISI) component in the MSE of a MMSE DFE design at high SNR becomes negligible compared to the noise component. Finally, (8) assumes a DFE with ideal error-free feedback. The error propagation in an actual DFE can be either ignored by considering frame error (as in [5]), can be mitigated by multi-block coding as in [17], or can be accepted since according to [18] it leads to an increase in probability of error by at most a factor $L{+}1$ and hence it only leads to coding loss. The performance of an ideal DFE can only be approximated though since an infinitely long non-causal feedforward filter would be required. The optimal tradeoff can also be attained by transmitting the same QAM symbol sequence and performing ML reception. Indeed, the optimal tradeoff is clearly attained when substituting SINR by MFB in (6). The deviation of the ML performance from the MFB is limited to a coding gain loss, as can be inferred from [19]. Since no Tx/Rx system can do better than the channel's outage capacity, this ML coding loss must exceed the difference in coding gain between the MFB and SINR_{DFE}^{MMSE}. Of course, using QAM constellations instead of Gaussian constellation leads to an additional coding (shaping) loss.

The coding gain also depends on the channel distribution. For instance for a non i.i.d. Gaussian distribution with covariance matrix $R_{\mathbf{hh}}$, the coefficient c gets reduced by a factor $\frac{\det(R_{\mathbf{hh}})}{(\frac{1}{p(L+1)}\,\text{tr}\{R_{\mathbf{hh}}\})^{p(L+1)}}$, see e.g. [20] (at least, this is correct for the MFB).

The analysis of the coding gain allows to compare and classify schemes that have an identical diversity-rate tradeoff. Diversity order and coding gain may e.g. depend on the filter lengths and delay used in LE and DFE. At high SNR, the outage behavior is dominated by the diversity order. At moderate SNRs however, it is possible that a scheme with a worse tradeoff but a much better coding gain can perform better.

IV. Outage Analysis of Suboptimal Receiver SINR

A perfect outage occurs when $\text{SINR} = 0$. For the MFB this can only occur if $\mathbf{h} = 0$. For a suboptimal Rx however (or also the MI), the SINR can vanish for any $\mathbf{h}$ on the *Outage Manifold* $\mathcal{M} = \{\mathbf{h} : \text{SINR}(\mathbf{h}) = 0\}$. At fixed rate R, the diversity order is the codimension of (the tangent subspace of) the outage manifold, assuming this codimension is constant almost everywhere and assuming a channel distribution with finite positive density everywhere (e.g. Gaussian with non-singular convariance matrix). For example, for the MFB the outage manifold is the origin, the codimension of which is the total size of $\mathbf{h}$. The codimension is the (minimum) number of complex constraints imposed on the complex elements of $\mathbf{h}$ by putting $\text{SINR}(\mathbf{h}) = 0$. Some care has to be excercised with complex numbers. Valid complex constraints (which imply two real constraints) are such that their number becomes an equal number of real constraints if the channel coefficients were to be real. A constraint on a coefficient magnitude however, which is in principle only one real constraint, counts as a valid complex constraint (at least if the channel coefficient distributions are insensitive to phase changes). Examples will follow. An actual outage occurs whenever $\mathbf{h}$ lies in the *Outage Shell*, a (thin) shell containing the outage manifold. The thickness of this shell shrinks as the rate increases.

V. Linear Equalization (LE) in Single Carrier Cyclic Prefix (SC-CP) Systems

The diversity of LE for SC-CP systems has been studied in [9] for the SISO case with i.i.d. Gaussian channel elements, fixed rate R and block size $N = L{+}1$. Consider a block of N symbol periods preceded by a cyclic prefix (CP) of length L (as a result of the CP insertion, actual rates are reduced by a factor $\frac{N}{N+L}$). The channel input-output relation over one block can be written as

$$\mathbf{Y} = \mathbf{H}\, A + \mathbf{V} \qquad (10)$$

where $\mathbf{Y} = \mathbf{Y}_k = [\mathbf{y}_k^T\ \mathbf{y}_{k+1}^T \cdots \mathbf{y}_{k+N-1}^T]^T$ etc. and $\mathbf{H}$ is the banded block-circulant matrix appearing in (13). Now apply an N-point DFT (with matrix F_N) to each subchannel received signal, then we get

$$\underbrace{F_{N,p}\mathbf{Y}}_{\mathbf{U}} = \underbrace{F_{N,p}\mathbf{H}\,F_N^{-1}}_{\mathcal{H}}\ \ \underbrace{F_N\, A}_{X} + \underbrace{F_{N,p}\mathbf{V}}_{\mathbf{W}} \qquad (11)$$

where $F_{N,p} = F_N \otimes I_p$ (Kronecker product: $A \otimes B = [a_{ij}B]$), $\mathcal{H} = \text{blockdiag}\,\{\boldsymbol{h}_0, \ldots, \boldsymbol{h}_{N-1}\}$ with $\boldsymbol{h}_n = \mathbf{h}(f_n)$, the $p \times 1$ channel transfer function at tone n: $f_n = \frac{n}{N}$, at which we have

$$\mathbf{u}_n \ = \ \boldsymbol{h}_n\, x_n + \mathbf{w}_n\ . \qquad (12)$$

The x_n are i.i.d. and independent of the i.i.d. $\mathbf{w}_n$ with $\sigma_x^2 = N\,\sigma_a^2,\ \sigma_w^2 = N\,\sigma_v^2$.

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_0 & & \mathbf{h}_L & \cdots & \mathbf{h}_1 \\ \vdots & \mathbf{h}_0 & & \ddots & \vdots \\ \vdots & & & & \mathbf{h}_L \\ \mathbf{h}_L & \vdots & & & \\ & \mathbf{h}_L & \ddots & & \\ & & \ddots & & \\ & & \mathbf{h}_L & \cdots & \mathbf{h}_0 \end{bmatrix} \tag{13}$$

A ZF ($\delta = 0$) or MMSE ($\delta = 1$) LE produces per tone $\widehat{x} = (\boldsymbol{h}^H\boldsymbol{h} + \frac{\delta}{\rho})^{-1}\boldsymbol{h}^H\mathbf{u}$ from which $\widehat{a}$ is obtained after IDFT with

$$\text{SINR}_{CP-LE}^{\delta} = \rho\left(\frac{1}{N}\sum_{n=0}^{N-1}(\|\boldsymbol{h}_n\|^2 + \frac{\delta}{\rho})^{-1}\right)^{-1} - \delta\ . \tag{14}$$

Consider now the case $N = L{+}1$. If the elements of $\mathbf{h}$ are i.i.d. Gaussian, then so are the elements in $\boldsymbol{h}_{0:L}$. For i.i.d. $\alpha_n = \|\boldsymbol{h}_n\|^2$, we have for sufficiently small $\epsilon > 0$

$$\text{Prob}\{\frac{1}{\sum_{n=0}^{N-1}\frac{1}{\alpha_n}} \le \epsilon\} = N\,\text{Prob}\{\alpha_0 \le 0\} = \text{Prob}\{\alpha_{min} \le 0\}$$

from which one can derive

$$d_{CP-LE}(r) = (1-r)\,p\ \ ,\quad r \in (0,1] \tag{15}$$

and the coding gain. (15) holds for both ZF and MMSE. It also holds for ZF for $r = 0$. So the LE, be it ZF or MMSE, has only spatial diversity for any normalized rate $r > 0$, any frequency diversity is lost! For the MMSE however, one can show (for any N, and also for the other LE's to be considered below) that $d_{CP-LE}^{MMSE}(0) = p\,(L+1)$. So the MMSE LE has full diversity at constant rate R. Since this holds for any $r \ll \frac{1}{\log\rho}$, one can expect that at finite SNR the MMSE LE may show significant diversity over a limited range of rates R as the simulations in [9] show.

These diversity order results (15) hold immediately also for arbitrary $N \ge L{+}1$. Indeed, the outage manifold is the collection of manifolds for which $\boldsymbol{h}_n = 0$ for some n. As a result the codimension is p.

VI. Non-Causal Infinite Length Linear Equalizer

For the infinite length (ZF) LE case, the outage manifold is clearly $\{\mathbf{h} :\ \mathbf{h}(f) = 0 \text{ for any } f\}$. For any given f the codimension is again p. In spite of the ambiguity on f, the diversity order is p. Consider e.g. the case $L = 1$: $\{\mathbf{h} :\ \mathbf{h}_0 = \mathbf{h}_1\,e^{-j(2\pi f+\pi)}\}$ which for the SISO case becomes $|h_o| = |h_1|$, for which it is easy to verify that the diversity order is 1.

VII. FIR Linear Equalization

Consider now the use of an FIR LE of length N. For SIMO channels, there exist indeed FIR equalizers for FIR channels, due to the Bezout identity [21], as long as $N \ge \frac{L}{p-1}$. The LE design is based on a banded block Toeplitz input-output matrix $\overline{\mathbf{H}}$ which can be obtained by starting from a block circulant $\mathbf{H}$ as in (13) of size $N{+}L$ and removing the top L block rows. We obtain for a certain equalizer delay

$$\text{SINR}_{FIR-LE}^{\delta} + \delta = \frac{\rho}{e^H(\overline{\mathbf{H}}^H\overline{\mathbf{H}} + \frac{\delta}{\rho})^{-1}e} = \frac{\rho}{\sum_i \frac{1}{\lambda_i + \frac{\delta}{\rho}}|V_{i,d}|^2} \tag{16}$$

where e is a standard unit vector containing a 1 in the position corresponding to the delay and zeros elsewhere, and we introduced the SVD $\overline{\mathbf{H}}^H\overline{\mathbf{H}} = V\Lambda V^H = \sum_i \lambda_i V_i V_i^H$. The outage manifold is determined (again) by $\lambda_{min} = 0$. For $N \ge \frac{L}{p-1}$, singularity of $\overline{\mathbf{H}}^H\overline{\mathbf{H}}$ occurs whenever $\overline{\mathbf{H}}$ loses full column rank. This occurs whenever $\mathbf{h}[z_o] = 0$ for some z_o, in other words, the subchannel transfer functions have a zero in common. This imposes on the $p{-}1$ other subchannels to have a zero equal to a zero of the first subchannel. Hence the codimension of the outage manifold is $p{-}1$. So

$$d_{FIR-LE}(r) = (1-r)\,(p-1)\ \ ,\quad r \in (0,1] \tag{17}$$

regardless of equalizer delay. For $r = 0$, the MMSE version has diversity $d = p\ \min\{N, L{+}1\}$ (for appropriately chosen delay). The eigenvector V_{min} associated to $\lambda_{min} = 0$ is a Vandermonde vector built from powers of z_o. As can be seen from (16), there is a weighting with $|V_{min,d}|^2$. The reason why (for $r > 0$) the diversity jumps from $p{-}1$ to p in the limit as $N \to \infty$ is that with some intermediate delay, and due to the normalization of the eigenvector energy, as $N \to \infty$, $|V_{min,d}|^2 \to 0$ unless $|z_o| = 1$.

VIII. DFE with Ideal Feedforward and Reduced Feedback Filters

So the DFE reaches full diversity as long as the feedback filter order $M = L$. Consider the MSE in a DFE design, after optimization of the unconstrained feedforward filter, with the feedback filter $b(f)$ still to be designed

$$\text{MSE} = \sigma_v^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{|b(f)|^2}{\|\mathbf{h}(f)\|^2 + \frac{\delta}{\rho}}\,df \tag{18}$$

With $M = 0$ ($b(f) = 1$), the MSE explodes whenever $\mathbf{h}(f) = 0$ (outage manifold LE). When $M = L$, as $\mathbf{h}(f) = 0$ can be zero only in at most L frequencies, the optimized feedback filter $b(f)$ will put zeros in those frequencies and hence can always prevent the MSE from exploding. When $M < L$, we have an outage whenever $\mathbf{h}(f)$ has $M{+}1$ zeros. So the diversity for any M is $d_{DFE}^{ZF}(r) = p(M{+}1)(1{-}r)$. See also [22] for IIR DFE design.

IX. DFE in Single Carrier Cyclic Prefix Systems

The problem of infinite length non-causal feedforward filters (FFFs) in the DFE can be overcome by introducing a CP and performing the FFF'ing in the frequency domain (SC-CP) and the feedback filtering in the time domain, see [12],[13] (where oversampling leads to increased DFT size which is not necessary). The same expressions as for the infinite length FFF case are obtained by replacing integration in the frequency domain by averaging over tones. The same diversity results hold.

X. FIR Decision-Feedback Equalization

Consider now a FFF of length N, $M = L$ and equalization delay equal to $N - 1$. For $N = 1$, $d^{ZF}(0) = p$ whereas $d^{ZF}(0) = p(L+1)$ for $N \to \infty$. For intermediate values of N intermediate diversity orders are obtained. For instance with $L = 1$, for $N = 2$, $d^{ZF}(0) = 2p-1$ and fractional diversities between $2p-1$ and $2p$ are obtained for $N > 2$.

XI. Concluding Remarks

The diversity results reported herein have been confirmed by simulations. An important question is whether anything can be done about the bad diversity available in LE's. In [8] full diversity even for a MMSE ZF LE is obtained by, on top of the CP, introducing L guard symbols and linear precoding which appears to lead to high complexity. It may be important to look also at finite SNR behavior as in [23]

Looking further at outage analysis, one may distinguish 3 SNR regions. At high SNR, in the diversity-rate tradeoff, a family of transmit signal constellations is considered, that varies with SNR and depends on the normalized rate. At any rate though, the diversity is determined by the distribution of the MI near zero.

In a medium SNR range, one can apply a Gaussian approximation of SINR, see e.g. [24], or the "amount of fading" in [16]. This leads to a case of both additive and multiplicative noise. See e.g. [10] where for large delay spread L, only the mean of the SINR is taken into account. Clark obtains $\text{E SINR}_p^{ZF-LE} = \text{E SINR}_{p-1}^{MFB}$.

At low SNR: by duality, if in the high SNR region P_e depends on the distribution of the SINR near zero, in the low SNR region it depends on the tail behavior of the SINR pdf and hence also on the tail behavior of the distribution of the channel coefficients. Actually, what counts is the cascade of input and channel (see e.g. [25]), which leads to the choice of peaky or low-rank input (decrease of the number of streams in the MIMO case).

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