

DIVERSITY-MULTIPLEXING TRADEOFF FOR FREQUENCY FADING CHANNELS WITH CSIT

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ABSTRACT

In this contribution¹, we analyse the performance limits of a multiband frequency fading channels when perfect channel state information is available at the transmitter (CSIT). For various power allocation (P.A) schemes such as water-filling, channel inversion and Haye's policy, we derive optimal constant-rate coding schemes that minimizes the information outage probability and show the potential gain of such techniques with respect to classical uniform power allocation.

1. INTRODUCTION

Cognitive radio is an emerging approach to optimize the use of the spectrum. Indeed, even if many radio resource management (RRM) techniques are available in the literature, the utilization of the spectrum remains sub-optimal. Many works are now oriented to design smart terminals able to detect the available bands and to allocate in a smart way, based on channel state information (CSI), the available power. Another target point for cognitive radio is the optimization of the quality of service (QoS) in terms of rate and bit error rate (BER) which is directly related to multiplexing and diversity gains [3]. In this work we focus our analysis on a slow (block) frequency selective fading wideband channel (or multiband channel) where the transmission bandwidth is greater than the coherence bandwidth W_c (frequency selective channel). Usually, when the coherence time T_c is smaller than the codeword length (fast fading), the relevant performance metric is the ergodic capacity, namely [1]:

$$C_{erg} = \mathbf{E} \left\{ \log_2 \left(1 + \text{SNR} |h|^2 \right) \right\} \quad (b/s/Hz) \quad (1)$$

When this is not the case (slow fading scenario), the decoding error probability can not be made arbitrarily small and the relevant metric in this case is the *information outage*

probability defined as the probability that the instantaneous mutual information of the channel is below the transmitted code rate [2]. Accordingly, the outage probability is:

$$P_{out}(R) = P \{ I(\mathbf{x}; \mathbf{y}) \leq R \} \quad (2)$$

Where $I(\mathbf{x}; \mathbf{y})$ is the mutual information of the channel between the transmitted vector \mathbf{x} and the received vector \mathbf{y} and R is the data rate in (*bits/s/Hz*) given by:

$$R = r \log_2 \text{SNR}; \quad r \in [0, 1] \quad (3)$$

Reliable communication can therefore be achieved when the mutual information of the channel is strong enough to support the target rate R . Traditionally, this problem has been studied by considering the diversity-multiplexing tradeoff. Thus, in [3], Zheng and Tse define a multiplexing gain r and a diversity gain d if the data rate, R , and the probability of outage, P_{out} , as functions of SNR, satisfy:

$$d = \lim_{\text{SNR} \rightarrow \infty} - \frac{\log_2(P_{out}(\text{SNR}))}{\log_2(\text{SNR})} \quad (4)$$

$$r = \lim_{\text{SNR} \rightarrow \infty} \frac{\log_2(R(\text{SNR}))}{\log_2(\text{SNR})} \quad (5)$$

According to these definitions, one can interpret the diversity-multiplexing tradeoff as a tradeoff between reliability and data rate of a given system.

However, one major difference between our work and [3] is that we consider that perfect CSI is available at the transmitter and at the receiver while they do not consider CSI at the transmitter. Therefore, an additional important definition related to outage probability is that of *delay-limited capacity*, sometimes refereed to as *zero-outage capacity*. This is the maximum rate for which the minimum outage probability is zero for a given power constraint [5]. We adopt this framework to characterize the performance of frequency selective channels and at the same time, how to best exploit the inherent frequency diversity/multiplexing capability provided by the multiband frequency selective channel. A common way to study the underlying tradeoff is to compute the reliability function from the theory of error exponents [7]. In [8], partial CSI has been considered and this

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tradeoff has been studied via error exponent approach and exponential diversity decay (diversity gain infinite) was obtained for a fast fading channel at very low SNR regime. In [6], authors show that the ISI channel achieves the matched filter bound in terms of the optimal diversity-multiplexing tradeoff in a slow fading context without CSIT. This result appears as a counter intuitive result since one would expect that the ISI will degrade the received signal. This suggests that ISI channel would have good performances especially when CSIT is available.

In this work, we consider a slow fading channel in each sub-band and derive optimal constant-rate coding schemes that minimizes the information outage probability of some commonly used power allocation policies.

The rest of the paper is organized as follows: Section 2 describes the system model. In section 3, we analyse the performance limits of such a system when considering perfect CSIT. Simulation results are provided in Section 4 and Section 5 concludes the paper.

2. FREQUENCY FADING MODEL

Consider a wireless point to point multipath system:

$$y(f) = h(f).x(f) + n(f) \quad (6)$$

We statistically model h to be i.i.d Rayleigh distributed over the L frequency bands. The additive gaussian noise n at the receiver is i.i.d circularly symmetric with power spectral density $\frac{N_0}{2}$. We assume that the channel h stays constant over each block fading length ($h(f_i) = h_i$), i.e. the slow fading scenario and is known by the receiver (CSIR). Such a channel model is especially suitable for wireless communication systems with slow moving terminals. Under these assumptions, the channel, within one block fading, can be written as:

$$y_i = h_i x_i + n_i; \quad i \in [1, L] \quad (7)$$

Where $h_i \sim \mathcal{CN}(0, 1)$ and $n_i \sim \mathcal{CN}(0, N_0)$. Such a system can be viewed as a multi-carrier/multi-band systems where diversity can be obtained by coding across the symbols in different sub-carriers available, like in Orthogonal Frequency Division Multiplexing (OFDM) or Sub-band Division Multiplexing (SDM) [4]. On the other hand, considering that both the transmitter and the receiver have CSI, the decoding rate will depend on the channel realization. Thus, depending on the CSIT, the transmitter decides appropriate power control policy such that the outage probability goes to zero. Note that the power control policy is such that the long term average transmit power is equal to \bar{P} .

3. RATE VERSUS OUTAGE PROBABILITY PERFORMANCE

In this section, we will analyse the performance limits of the underlying system. Two cases should be dissociated:

- When CSI is available at the receiver only: the relevant metric is the diversity-multiplexing tradeoff. Our goal here is to minimize the outage probability with respect to a fixed target rate R .
- When CSI is available at the receiver and at the transmitter: the transmitter can adapt its transmission strategy relative to this CSI. The relevant metric in this case is zero-outage capacity. Our goal here is to find the transmission scheme that minimizes the outage probability under a given average power constraint.

3.1. CSI at the receiver only

It was shown in [3] that the outage probability, in absence of CSI at the transmitter of a basic single antenna slow fading channel, decays like SNR^{-1} at high SNR-regime when CSI. In a general system with L random fading coefficients, one can design a scheme that achieve the maximal diversity gain (L) by averaging over the total number of fading gain available at a fixed data rate R . In this case, the outage probability decays like SNR^{-L} . Yet, in such coding scheme, increasing the diversity gain d is done at the expense of the multiplexing gain r . As a consequence, the question of how to optimally exploit diversity gain and degrees of freedom of a given system needs to be answered. In particular, it is of major interest to quantify the variations of the diversity gain with respect to the multiplexing gain.

3.1.1. Single Carrier System

Let us, firstly, consider a single fading system and derive the corresponding optimal diversity-multiplexing tradeoff. The probability of outage at a fixed target rate R is:

$$\begin{aligned} P_{out} &= P \left\{ \log_2 \left(1 + \text{SNR} |h_i|^2 \right) \leq R \right\} \\ &= P \left\{ |h_i|^2 \leq \frac{\text{SNR}^r - 1}{\text{SNR}} \right\} \end{aligned}$$

Notice that $|h_i|^2$ is exponentially distributed with probability density function $p_{|h_i|^2} = e^{-t}$, yielding at high SNR-regime:

$$\begin{aligned} P_{out}(r, \text{SNR}) &\approx [1 - \exp(-\text{SNR}^r)] \\ &\approx \text{SNR}^{r-1} \end{aligned}$$

The optimal diversity-multiplexing tradeoff in this case is:

$$d(r) = (1 - r); \quad r \in [0, 1]$$

Hence, this scheme provides a diversity gain of order 1. This suggests that a multi-path fading system would have better performances particularly in terms of optimal diversity-multiplexing tradeoff.

3.1.2. Multiple Carrier System

Let us now determine the outage probability of the system presented in (7):

$$P_{out}(r, \text{SNR}) = P \left\{ \sum_{i=1}^L \log_2 (1 + \text{SNR} |h_i|^2) \leq LR \right\} \quad (8)$$

Considering that outage occurs when each of the fading channel available can not support the target rate R , equation (8) can be tightly upper bounded by:

$$\begin{aligned} P_{out}(r, \text{SNR}) &\approx P \left\{ \log_2 (1 + \text{SNR} |h_i|^2) \leq R \right\}^L \quad (9) \\ &\approx P \left\{ |h_i|^2 \leq \frac{2^{R-1}}{\text{SNR}} \right\}^L \\ &\approx [1 - \exp(-\text{SNR}^{r-1})]^L \\ &\approx \text{SNR}^{L(r-1)} \end{aligned}$$

The quality of this approximation as well as the parameters which make the approximation valid will be discussed in section 4. The optimal diversity-multiplexing tradeoff is:

$$d(r) = L(1-r); \quad r \in [0, 1]$$

Thus, frequency fading channels achieve an L -fold diversity gain over the single carrier performance at every multiplexing gain r . Note here that the practical data rate at which the whole information is sent is $Lr \log_2(\text{SNR})$. For large bandwidths (when L is supposed to be infinite), the law of large number shows that:

$$\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^L \log_2 (1 + \text{SNR} |h_i|^2) = C_{erg} \quad (10)$$

Average here is done over the stationary ergodic distribution of the fading channel. This result suggests that by coding over a large number of independent frequency bins ($L \geq 20$), one can achieve a reliable communication at a rate equal to the ergodic capacity. The tradeoff in this case is not necessary since we know exactly at which rate, data should be transmitted with arbitrarily small outage probability.

3.2. CSI at the receiver and the trasmitter

In this section, the system model considered is the same as in section 3.1.2 with the assumption that CSI is available at the receiver and at the transmitter. In this case, probability of outage is defined as below:

$$P_{out} = P \left\{ \frac{1}{L} \sum_{i=1}^L \log_2 (1 + \text{SNR} P_i |h_i|^2) \leq R \right\} \quad (11)$$

where P_i is the power allocated to the i th block fading subject to the long term average power constraint:

$$\mathbb{E} \left\{ \sum_{i=1}^L P_i \right\} = \bar{P} \quad (12)$$

Without loss of generality, we take $\bar{P} = 1$ and analyze performances of some commonly used power-allocation policies in terms of zero-outage capacity.

3.2.1. Truncated Channel Inversion policy (TCI)

Let us firstly consider a sub-optimal power adaptation strategy where the transmitter uses the CSIT to maintain the received SNR constant irrespective of the channel gain. Thus, with exact channel inversion, there is zero outage probability. However, this strategy would not be efficient especially when the channel is very bad. Consequently, we will allow to inverse the channel below a certain cutoff value γ_0 . The truncated channel inversion (TCI) power allocation in this context is:

$$P_i^{TCI} = \frac{1}{\gamma_i}, \quad \gamma_i \geq \gamma_0 \quad (13)$$

Where γ_i is defined as:

$$\gamma_i = \frac{|h_i|^2}{N_0}; \quad i \in [1, L] \quad (14)$$

By solving the power constraint on γ_0 in (12), and from the asymptotic expansion of $E_i(x)$ in [9], we obtain²:

$$\begin{aligned} \text{SNR} &= L E_i \left(\frac{\gamma_0}{\text{SNR}} \right) \\ &\approx -L \log \left(\frac{\gamma_0}{\text{SNR}} \right); \quad \text{at high SNR regime} \end{aligned}$$

Then, $\gamma_0 \approx \text{SNR} \cdot \exp \left(-\frac{\text{SNR}}{L} \right)$.

3.2.2. Water-filling policy (WF)

The optimal power allocation which maximizes the transmission rate here is solution of the optimization problem:

$$\max_{P_1, \dots, P_L} \left\{ \frac{1}{L} \sum_{i=1}^L \log_2 (1 + \text{SNR} P_i |h_i|^2) \right\}$$

subject to the average power constraint (12). The optimal solution is computed applying Lagrange's method which leads in this case to the well known water-filling (WF) power allocation [4]:

$$P_i^{WF} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_i}, & \gamma_i \geq \gamma_0 \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

² $E_i(x)$ is the exponential integral function defined as: $E_i(x) = \int_x^\infty \frac{e^{-t}}{t} dt$

Where γ_i is defined as in (14) and γ_0 is the the Lagrange's multiplier satisfying³:

$$\mathbb{E} \left\{ \sum_{i=1}^L \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right)^+ \right\} = 1 \quad (16)$$

We obtain a closed-form expression for the optimal cutoff value γ_0 . Numerical root finding is needed to determine different values of γ_0 [10]. Our numerical results, in section 4, show that γ_0 increases as SNR increases, and γ_0 always lies in the interval $[0,1]$. On the other hand, an asymptotic expansion of (16) shows that at very high SNR-regime $\gamma_0 \rightarrow 1$.

3.2.3. Hayes' Policy (H.P)

Instead of analyzing the policy that maximizes the rate (water-filling), let us now focus on the strategy that minimizes the BER. The optimum power allocation that minimizes the BER of an uncoded system on fading channel was studied in [11] and defined as following:

$$P_i^{HP} = \begin{cases} \frac{1}{\gamma_i} \ln \left(\frac{\gamma_i}{\gamma_0} \right), & \gamma_i \geq \gamma_0 \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

Where γ_i is defined as in (14) and γ_0 is chosen to satisfy the average power constraint in (12). Note that even in this case, the cut-off value, γ_0 , can not be analytically determined. Though, we will show in section 4 by numerical results that γ_0 lies near zero as SNR increases.

4. SIMULATIONS AND RESULTS

In order to check how and under which conditions does approximation done in (8) is true, we run Monte-Carlo simulations. We consider a point to point multipath system over the L rayleigh fading coefficients. In figure 1, we compute the difference between the approximated expression and the outage probability in (9) for $r = 0.5$ and SNR = 10 dB. We find that the approximation is very tight since $L \leq 16$. If it is not the case, the difference increases exponentially with respect to L and the approximation is not valid any more. In our case, we considered low values of L since for high values of L , we reach the ergodic capacity (see eq.(2)) and the tradeoff is not necessary since we know exactly at which data rate we should transmit with arbitrarily small error.

In figure 2, we compare the difference between the approximated expression and the outage probability for $L = 5$. It is clear here that at high SNR region, the approximation is true even when multiplexing gain r is close to 1.

Figure 3 depicts Outage Probabilities of respective P.A policies used in presented in this paper. The cut-off values

³ $(x)^+ = \max(0, x)$.

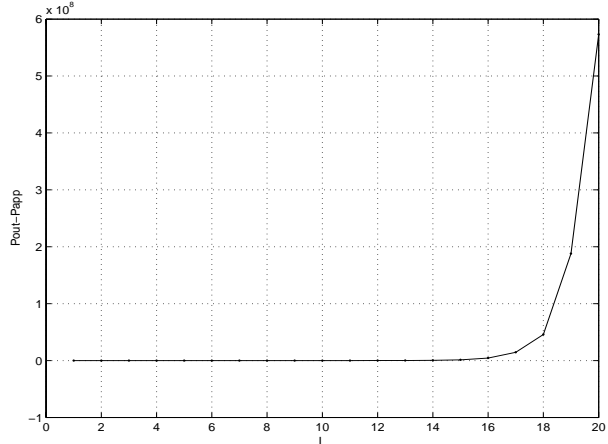


Fig. 1. Difference between the approximated expression in (8) and the outage probability in (9) over L with $r = 0.5$ and SNR = 10 dB.

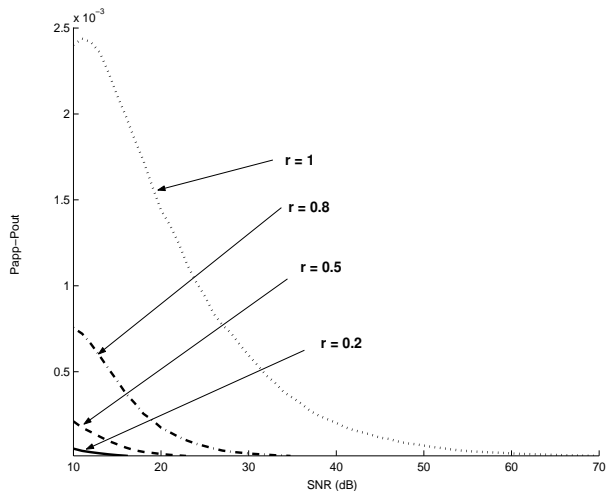


Fig. 2. Difference between the approximated expression in (8) and the outage probability in (9) over SNR with $L = 5$.

γ_0 were numerically obtained through a dichotomical algorithm. One observes that TCI policy presents the worst behavior as SNR increases. This is due to the power control policy chosen here. In fact, by maintaining the received SNR constant irrespective of the fading gain, channel inversion policy does not exploit the available diversity. While Hayes' policy bad behavior can be explained by the optimization problem which focuses on minimizing the BER. On the other hand, the water-filling strategy affords a significant performance gain over the constant-power strategy at low SNR. The intuition is that when there is little transmit power, it is much more effective to expend it on the strongest fading gain of the system rather than spread the

power evenly across all modes. Next, we consider the high SNR-regime. We see that as a result already noted in , as $\text{SNR} \rightarrow \infty$, it is well known [12] that the water-filling and the constant power strategies yield almost the same performance.

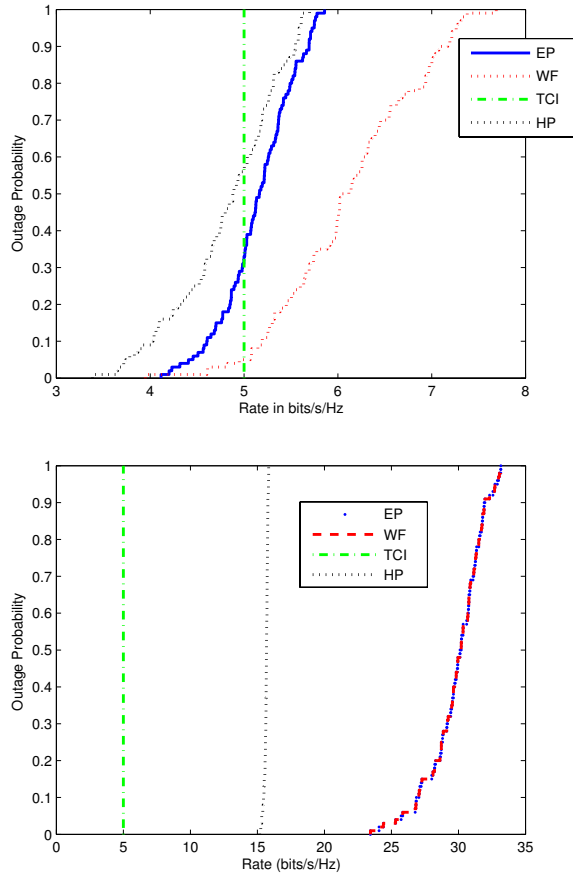


Fig. 3. Outage Probabilities of different Power allocation policies for $L = 5$ at $\text{SNR} = 5$ dB and $\text{SNR} = 20$ dB respectively.

5. CONCLUSION

An important issue in cognitive radio systems is the design of techniques that exploit the inherent variability of the channel across time, frequency, and space. Diversity and multiplexing schemes appear as a useful solution to exploit the wireless variations of the channel. In this paper, we focus our attention on the trade-off diversity versus multiplexing over a wideband or multi-band channel when CSIT is available. Interestingly, with appropriate power allocation, one can increase the performance of classical trade-off. Thus, we showed that W.F achieves the best diversity orders but, at high SNR-regime, reach the same diversity diversity

gain provided by the multipath channel without CSIT. We also showed that TCI achieves infinite diversity gain within a specified data rate region.

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