Maximizing the Capacity of Large Wireless Networks: Optimal and Distributed Solutions

Saad G. Kiani and David Gesbert
Mobile Communications Department, Eurecom Institute,
06560 Sophia Antipolis, France
Email: {kiani, gesbert}@eurecom.fr

***Abstract*— We analyze the sum capacity of multicell wireless networks with full resource reuse and channel-driven opportunistic scheduling in each cell. We address the problem of finding the co-channel (throughout the network) user assignment that results in the optimal joint multicell capacity, under a resource-fair constraint and a standard power control strategy. This problem in principle requires processing the complete co-channel gain information, and thus, has so far been justly considered unpractical due to complexity and channel gain signaling overhead. However, we expose here the following key result: The multicell optimal user scheduling problem admits a remarkably simple and fully distributed solution for large networks. This result is proved analytically for an idealized network. From this constructive proof, we propose a practical algorithm that is shown to achieve near maximum capacity for realistic cases of simulated networks of even small sizes.**

I. Introduction

In wireless networks featuring multiple simultaneous transmission links (cellular or ad-hoc), there exists a well known trade-off between the reuse of spectral resource across these links and the interference created to one another by co-channel transmissions. Power control strategies were proposed to limit interference effects by targeting a given signal-to-interference ratio (SIR) [1], [2] or a received signal power-level [3], leading to a truncated channel gain inversion. Combining power control with cell diversity was subsequently shown to increase the number of supported users in the uplink [4]. Recently proposed resource allocation techniques [5], [6], [7], [8], [9] attempt to mitigate this problem by exploiting directional antennae, sectorization or clever location-dependent power transmission profiles to reduce the interference. However, to fully benefit from all degrees of freedom provided by the multi-user fading channel, a more promising solution lies in the concept of *co-channel user scheduling*. In this setting, the assignment of the users to the spectral resource is done not just to maximize the capacity of each individual cell [10], but rather to maximize the capacity of all links and cells in a joint fashion, thus giving rise to an extension of the concept of multi-user diversity to a full network.

In principle the complexity of the above problem is high: the number of degrees of freedom is governed by the number of cells × the number of users per cell × the number of possible scheduling slots to which a user can be assigned. Additionally, it can be shown easily that the co-channel scheduling problem makes sense only if some form of power control is used. In this paper, we first formulate the co-channel user scheduling problem for an arbitrary network given the complete network-wide co-channel gain information and a standard power control rule (gain inversion-based power control). Next, we propose an idealization for a large network coined *interference-ideal network*, that can be exploited to simplify the problem formulation. We then obtain the following striking results:

- For interference-ideal networks, maximum network capacity can be reached by using a low-complexity *fully distributed* scheduling protocol, based on local channel gains. This result admits a theoretical constructive proof which we further exploit to propose a multicell scheduling algorithm for realistic (non-ideal) networks.
- For fast-fading, the algorithm is a generalization of the single cell maximum capacity scheduler [10] to the multicell case. As a result, per-cell throughput maximization and multicell interference avoidance are shown to go hand in hand and multi-user diversity scheduling can also be throughput optimal in a multicellular scenario.

From the analysis above we derive a practical co-channel scheduling algorithm that can trade-off resource fairness for system capacity. These results have applications in cellular/ad-hoc networks with interference-limited transmission. In this paper we test the algorithms over finite-size non-ideal cellular-type networks and show the throughput gains over a non-coordinated co-channel scheduler in the presence of interference.

II. Network Capacity Model

Consider a multicell system with N access points (AP) communicating with U user terminals (UT) in each cell. We consider the downlink in which the AP sends data to the UT, but the results presented in this paper can be generalized to the uplink. We assume a multiple access scheme in which spectral resource is orthogonally divided into units called (e.g. code-, time-, frequency- etc.) slots. Slot assignment occurs simultaneously in all cells and is used to separate the transmissions to the users of any given cell. We enforce K-th order resource fairness, where $1 \leq K \leq U$. This means that a scheduling frame consists of K slots assigned to K distinct users per cell. Note that this does not necessarily yield throughput fairness, even with $K = U$, as users may not enjoy an equal throughput due to local channel conditions. Moreover, because of concurrent transmissions in all cells in any one slot, an assigned user "sees" interference from all co-channel cells.

A. Signal Model

To preserve light notation we focus on the single antenna case. For a user u_n in cell n, the downlink is a typical interference channel [11], the received signal for which is given by

$$Y_{u_n} = \sqrt{G_{u_n \leftarrow n}} X_{u_n} + \sum_{i \neq n}^{N} \sqrt{G_{u_n \leftarrow i}} X_{u_i} + Z_{u_n},$$

where X_{u_n} is the signal from the serving AP n and Z_{u_n} is additive white Gaussian noise. The signal to interference-plus-noise ratio (SINR), Γ is given by,

$$\Gamma_{u_n} = \frac{G_{u_n \leftarrow n} \mathbb{E}|X_{u_n}|^2}{\mathbb{E}|Z_{u_n}|^2 + \sum_{i \neq n}^{N} G_{u_n \leftarrow i} \mathbb{E}|X_{u_i}|^2}.$$

If transmit power used by an AP to serve u_n is P_{u_n}, we have $\mathbb{E}|X_{u_n}|^2 = P_{u_n}$. Note that $G_{u_n \leftarrow m} \in \mathbb{R}^+$ reflects the composite channel gain possibly including fast-fading.

B. Power Control

As is seen later, power control plays a key role in enabling the gains of network coordination. We assume each AP has a peak transmission power constraint, P_{MAX} and a multiplicative power control factor $0 < \rho \leq 1$ is used to adjust the transmitted power such that $P_{u_n} = \rho_{u_n} P_{MAX}$. The AP transmit power is adjusted in order to achieve a target received power R^* at the receiver. If it is not achievable, the AP transmits at full power. Assuming each user can measure and communicate back the power received from the serving AP, $\rho_{u_n} = \frac{R^*}{G_{u_n \leftarrow n} P_{MAX}}$. But, since there is a peak power constraint P_{MAX}, ρ is upper bounded by one:

$$\rho_{u_n} = \min \left\{ \frac{R^*}{G_{u_n \leftarrow n} P_{MAX}}, 1 \right\}. \tag{1}$$

We point out here that the capacity optimal scheduling policy should be jointly optimized with the power control policy. Such issues are, however, beyond the scope of this paper and will be addressed in a later paper.

Power control setting: Depending on the value of R^* and the intra-cell channel gain, a user will be receiving in full ($\rho = 1$) or reduced ($\rho < 1$) power mode. We consider three network scenarios. (1) *fully power controlled* (FPC) network: All users achieve R^* after power control. (2) *mixed power controlled* (MPC) network: Only a fraction of users achieve R^*. (3) *no power controlled* (NPC) network: $\rho = 1$ for all users.

III. The Co-Channel User Matching Problem

We assume that channel gains do not vary over the scheduling frame duration which is sized in accordance with the coherence period of the channel. Under the K-th order resource fairness constraint, the co-channel user matching problem consists in selecting K users in each cell and assigning these users to K slots so as to optimize the system utility function (joint capacity). To facilitate the formulation of the problem, we state the following definitions:

Definition 1: A ***scheduling policy*** φ is a bijective mapping of the subset $\mathcal{U}_n$, consisting of K users chosen from the set of all users in cell n, onto $\mathcal{K}$ the set of slots, $\varphi_n : \mathcal{U}_n \mapsto \mathcal{K}$.

Definition 2: A ***scheduling vector*** $\mathscr{I}^{(k)}$ contains the set of users scheduled in slot k across all cells (based on φ):

$$\mathscr{I}^{(k)} = \left[u_1^{(k)} \, u_2^{(k)} \, \cdots u_n^{(k)} \, \cdots \, u_N^{(k)} \right]^{\mathrm{T}} \in [1, K]^N,$$

where $[\mathscr{I}^{(k)}]_n = u_n^{(k)}$ is the user scheduled during slot k in cell n. Note that because φ is a bijection, scheduling vectors are element-wise disjoint, $\mathscr{I}^{(a)} \bigcap \mathscr{I}^{(b)} = \varnothing \ \forall \ a \neq b$. The scheduling vector is the ensemble of users which interfere with each other and thus it determines the sum capacity for slot k.

Definition 3: A ***scheduling matrix*** $\mathbf{S}$ is a K-column matrix composed of scheduling vectors given by the scheduling policy φ.

$$\mathbf{S} = \left[\mathscr{I}^{(1)} \, \mathscr{I}^{(2)} \, \cdots \, \mathscr{I}^{(K)} \right].$$

This matrix describes the complete ordering of all users during one frame.

A. System Performance

The SINR for users scheduled in slot k will depend on the scheduling vector $\mathscr{I}^{(k)}$. We can express the SINR in cell n as

$$\Gamma(\mathscr{I}^{(k)}, n) = \frac{G_{u_n^{(k)} \leftarrow n} \rho_{u_n^{(k)}} P_{MAX}}{\sigma^2 + \sum_{i \neq n}^{N} G_{u_n^{(k)} \leftarrow i} \rho_{u_i^{(k)}} P_{MAX}}. \tag{2}$$

Assuming an ideal link adaptation protocol, the per cell capacity in slot k can be expressed in bits/sec/Hz/cell using the Shannon capacity,

$$C(\mathscr{I}^{(k)}) = \frac{1}{N} \sum_{n=1}^{N} \log \left(1 + \Gamma(\mathscr{I}^{(k)}, n) \right). \tag{3}$$

The network capacity of the system is a function of the overall scheduling matrix $\mathbf{S}$ given by

$$\begin{aligned} \mathcal{C}(\mathbf{S}) &\triangleq \frac{1}{K} \sum_{k=1}^{K} C(\mathscr{I}^{(k)}) \qquad (4) \\ &\triangleq \frac{1}{NK} \sum_{k=1}^{K} \sum_{n=1}^{N} \log \left(1 + \frac{G_{u_n^{(k)} \leftarrow n} \rho_{u_n^{(k)}} P_{MAX}}{\sigma^2 + \sum_{i \neq n}^{N} G_{u_n^{(k)} \leftarrow i} \rho_{u_i^{(k)}} P_{MAX}} \right). \end{aligned}$$

B. Round Robin Scheduling

A standard approach for resource fair scheduling is round robin (RR) in which users are given slots turn by turn in each frame. Letting $\mathbb{S}$ be the set of all scheduling matrices, the network capacity for RR is

$$\mathcal{C}_{RR} \triangleq \mathbb{E}_{(\mathbf{S} \in \mathbb{S})} \left\{ \mathcal{C}(\mathbf{S}) \right\}. \tag{5}$$

C. Optimal Co-channel Scheduling

On the other hand, the scheduling policy for optimum network capacity (4) can be stated as

$$\mathbf{S}^* = \underbrace{\text{argmax}}_{\mathbf{S}\in\mathbb{S}}\{\mathcal{C}(\mathbf{S})\}. \quad (6)$$

As $\mathbf{S}^*$ gives the optimal network capacity, we have in general: $\mathcal{C}(\mathbf{S}^*) \geq C_{RR}$. Inequality will be strict in most cases, thus showing the gain of coordinated networks over uncoordinated ones.

Multicell scheduling gain in NPC system: It is easy to see that some scenarios will result in no gain at all as shown below:

Lemma 1: For a no power control (NPC) network, the network capacity gain associated with multicell scheduling is zero.

Proof: With no power control $\rho_{u_n} = 1 \ \forall \ u_n$, and thus all BS transmit at same (maximum) power. Substituting this in (2) we obtain

$$\Gamma(\mathscr{I}^{(k)}, n) = \frac{G_{u_n^{(k)} \leftarrow n} P_{MAX}}{\sigma^2 + \sum_{i\neq n}^{N} G_{u_n^{(k)} \leftarrow i} P_{MAX}}, \quad (7)$$

which is independent of the choice of co-channel users in other cells. It follows that the capacity will be the same no matter which users are scheduled with each other. □

This result indicates that the gain can be intuitively expected to depend much on the degree of *variability* of channel *and* power control coefficients across the network users, as well as on the number of cells and users. We now turn to the issue of *finding* the optimal $\mathbf{S}$.

IV. Optimum Sum Capacity Scheduling

As $\mathbb{S}$ is a discrete finite set, (6) is a non-linear *combinatorial optimization problem* for which, finding optimal solutions is NP-hard. $|\mathbb{S}| = (U!)^{N-1}$, considering that a set of K scheduling vectors can be ordered in $K!$ ways without changing the network capacity. Even for a small network this method remains prohibitive: for $N = 7$ and $U = 5$, $|\mathbb{S}| \approx 2.9 \times 10^{12}$. Collecting and processing all path gain information within the coherence time will pose significant signaling and delay problems. In order to find a *distributed* multicell scheduling algorithm instead, we introduce a simplified model for network capacity used to later approximate the actual capacity.

A. Interference-Ideal Networks

Due mostly to the large number of interference sources adding up at the receiver, we can offer a simpler model for interference in large full reuse networks. We define the concept of an *interference-ideal network* as one in which, *for any cell user, the total received interference is independent of its location in the cell*. Mathematically, a network is interference-ideal if, for any user u_n and cell n:

$$\sum_{i\neq n}^{N} G_{u_n \leftarrow i} \rho_{u_i} P_{MAX} = G \sum_{i\neq n}^{N} \rho_{u_i} P_{MAX}, \quad (8)$$

where G does not depend on the location of u_n. Fortunately, the interference-ideal network is a good model for a full reuse network with a large number of cells: We have, due to the interference channel gain and the power control coefficients being uncorrelated:

$$\sum_{i\neq n}^{N} G_{u_n \leftarrow i} \rho_{u_i} P_{MAX} \approx 1/N \sum_{i\neq n}^{N} G_{u_n \leftarrow i} \sum_{i\neq n}^{N} \rho_{u_i} P_{MAX},$$

and for a large N, due to the law of large numbers,

$$\Rightarrow \frac{1}{N} \sum_{i\neq n}^{N} G_{u_n \leftarrow i} \approx G \ \forall \ u_n \text{ (for large } N\text{)},$$

where G can be thought of as the mean channel gain from an interferer to any user position. The value of G will naturally depend on the distribution of interferers and the channel statistics but, as will be seen, this value need not be known. Moreover, variation of the interference from the cell center to the cell boundary in a dense network can be shown to be quite small [12] and from an algorithmic design point of view, we can consider that all users get the average channel gain G from each AP.

B. Optimum Scheduling in Interference-Ideal Networks

We characterize the solution to the optimal network scheduling problem in an interference-ideal network in an FPC scenario. Using (8) and (1) we can rewrite (2) as

$$\Gamma(\mathscr{I}^{(k)}, n) = \frac{R^*}{\sigma^2 + GR^* \sum_{i\neq n}^{N} \frac{1}{G_{u_i^{(k)} \leftarrow i}}}. \quad (9)$$

The network capacity will be given by

$$\mathcal{C} = \frac{1}{NK} \sum_{k=1}^{K} \sum_{n=1}^{N} \log\left(1 + \frac{R^*}{\sigma^2 + GR^* \sum_{i\neq n}^{N} \frac{1}{G_{u_i^{(k)} \leftarrow i}}}\right). \quad (10)$$

We define a vector $\mathscr{U}_n\!\downarrow$, containing the K users of $\mathcal{U}_n$ ordered in descending order of intra-cell channel gains,

$$\mathscr{U}_n\!\downarrow = [u_{1,n} \ldots u_{j,n} \ldots u_{K,n}]^{\mathrm{T}},$$

where $G_{u_{1,n} \leftarrow n} \geq \ldots \geq G_{u_{j,n} \leftarrow n} \geq \ldots \geq G_{u_{K,n} \leftarrow n}$. We now present the following result:

Theorem 1: Let $\mathbf{S}\downarrow = [\mathscr{U}_1\downarrow \ \ldots \mathscr{U}_n\downarrow \ \ldots \mathscr{U}_N\downarrow]^{\mathrm{T}}$ *and* $\pi(\mathbf{S}\downarrow)$ *be the scheduling matrix obtained by applying any column-wise permutation on* $\mathbf{S}\downarrow$. *Then, for an interference-ideal network,* $\pi(\mathbf{S}\downarrow)$ *is an optimal scheduling matrix,* $\mathbf{S}^*$ *for the problem* (6).

Proof: See Appendix. □

Based on Theorem 1 an optimal scheduling policy is for each cell to rank its users by, say, decreasing order of channel gain and assign the best K users to the K available slots, regardless of the channel gains in other cells. As co-channel users are matched based on the rank of their channel gain, we call this scheduling policy *Power Matched Scheduling* (PMS).

PMS is completely distributed as local channel gain is the only scheduling criteria. Note that a side-effect of the policy is to group users with similar channel quality levels, possibly creating an unfair service.

V. Multi-user Diversity And Fairness

Interestingly, when we choose $K = 1$ (no resource fairness), Theorem 1 leads to scheduling the user with the best channel gains in each cell. This can be interpreted as a generalization of multi-user diversity scheduling to the multicell case. Clearly, for $K > 1$ there is an unfair repartition of interference among the different slots with best users getting also the least interference. Thus, network capacity is optimized at the expense of throughput fairness which is reasonable from an information theoretic point of view and extends the well-known capacity/fairness trade-off known in single cell scenarios [10], [13]. Notice that K can be selected (between 1 and U) by the service provider to vary the resource fairness/throughput trade-off. For $K = 1$ only multi-user diversity gain is obtained without regard for resource fairness, while $K = U$ provides full resource fairness at the cost of capacity.

VI. Numerical Results

The performance of PMS is compared with RR through Monte Carlo simulations under a full resource fairness constraint ($K = U$). An 1800 MHz hexagonal cellular system with 1 km. radius cells is considered, with 30 users/cell randomly spread according to a uniform distribution. *Both inter-cell and intra-cell* AP-UT links are based on the COST-231 [14] path loss model including zero mean lognormal shadowing with a standard deviation of 10 dB and fast-fading $\sim \mathcal{CN}(0, 1)$. R^* corresponds to an SNR target of 30 dB and $P_{MAX} = 1$W. *These parameters result in an MPC network which serves to compare the schemes in a realistic setting.*

For PMS, the scheduling matrix is given by Theorem 1 and, in accordance with (5), RR is modeled by selecting a random permutation of the scheduling matrix for each frame. The comparison of the two scheduling policies is represented by the Network Capacity Gain τ, given by $\tau = \frac{\mathcal{C}(\mathbf{S}^*)}{\mathcal{C}_{RR}}$. We show traces of network capacity obtained with $N = 3$ and $N = 19$ in figs. 1 & 2 respectively. As the number of cells increases, interference averaging reduces variation in network capacity yielding an increase in gain. As expected, PMS outperforms RR in all cases and moreover, the gain increases with system size (fig. 3). Moreover, the gain is greater in the presence of both shadowing and fast-fading. This leads to the conclusion that increase in system size, as well as greater channel variation, improves performance.

VII. Conclusion

We address the problem of multicell scheduling for wireless networks. We show that large gains are obtained from inter-cell coordination due to the inter-cell interference variability that stems from power control and fading. We show that the optimal scheduler can be efficiently approximated by a fully distributed multicell scheduler. In the optimal scheduler each cell ranks its users according to decreasing channel gains. The multi-cell scheduler is also consistent with maximizing the capacity of each cell independently through multi-user diversity.

References

[1] J. Zander, "Distributed cochannel interference control in cellular radio systems," *IEEE Trans. Veh. Technol.*, vol. 41, pp. 305–311, Aug. 1992.

[2] G. J. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence," *IEEE Trans. Veh. Technol.*, vol. 42, 1993.

[3] J. F. Whitehead, "Signal-level-based dynamic power control for cochannel interference management," in *Proc. IEEE Vehicular Technology Conference*, May 1993, pp. 499–502.

[4] S. V. Hanly, "An algorithm for combined cell-site selection and power control to maximize cellular spread spectrum capacity," *IEEE J. Select. Areas Commun.*, vol. 13, pp. 1332–1340, 1995.

[5] K. K. Leung and A. Srivastava, "Dynamic allocation of downlink and uplink resource for broadband services in fixed wireless networks," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 990–1006, May 1999.

[6] K. Chawla and X. Qiu, "Quasi-static resource allocation with interference avoidance for fixed wireless systems," *IEEE J. Select. Areas Commun.*, vol. 17, pp. 493–504, Mar. 1999.

[7] I. Koutsopoulos and L. Tassiulas, "Channel state-adaptive techniques for throughput enhancement in wireless broadband networks," in *Proc. IEEE INFOCOM*, Alaska, Apr. 2001.

[8] V. Tralli, R. Veronesi, and M. Zorzi, "Power-shaped advanced resource assignment for fixed broadband wireless access systems," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 2207–2220, Nov. 2004.

[9] T. Bonald, S. Borst, and A. Proutière, "Inter-cell scheduling in wireless data networks," in *Proc. European Wireless*, Cyprus, Apr. 2005.

[10] R. Knopp and P. Humblet, "Information capacity and power control in single-cell multiuser communications," in *Proc. IEEE ICC*, Seattle, June 1995, pp. 331–335.

[11] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: John Wiley & Sons, 1991.

[12] S. G. Kiani and D. Gesbert, "Optimal and distributed scheduling for multicell capacity maximization," *Journal version submitted*.

[13] P. Viswanath, D. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1277–1294, June 2002.

[14] *Urban Transmission Loss Models for Mobile Radio in the 900 and 1800 MHz Bands*, EURO-COST Std. 231, 1991.

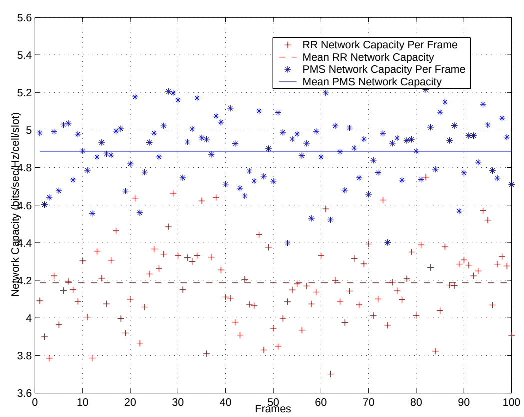

Fig. 1. Trace of network capacity values for 3 cells and 30 users per cell. Independent channel realizations based on shadowing and fast-fading are generated on a frame by frame basis.

Appendix
Proof of Theorem 1

We prove the optimality of $\mathbf{S}\downarrow$ by first showing that it is valid for N cells and two slots. This is then extended to K slots.

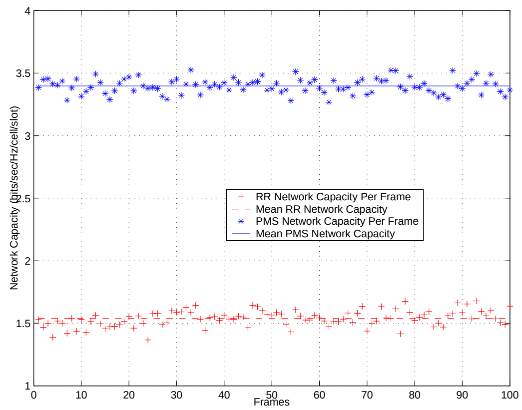

Fig. 2. Trace of network capacity values for 19 cells and 30 users per cell. Independent channel realizations based on shadowing and fast-fading are generated on a frame by frame basis.

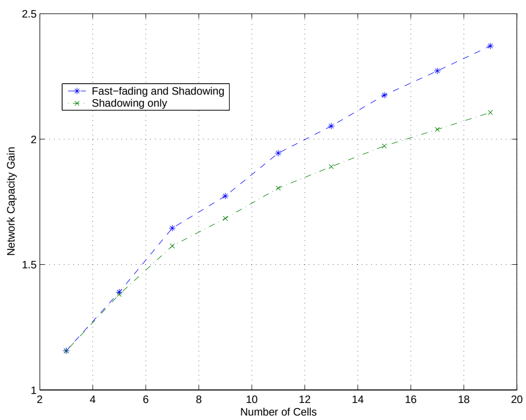

Fig. 3. Network capacity gain versus number of cells for different propagation scenarios. Gain increases with system size as optimization space increases. Greater channel variation increases performance gap between the two scheduling policies thereby increasing gain.

Lemma 2: For an arbitrary number of cells N and two slots, let

$$\mathbf{S}{\downarrow}^{N\times 2} = \begin{pmatrix} u_{1,1} & u_{2,1} \\ u_{1,2} & u_{2,2} \\ \vdots & \vdots \\ u_{1,N} & u_{2,N} \end{pmatrix}.$$

The optimal scheduling matrix for (6), $\mathbf{S}^* = \mathbf{S}{\downarrow}^{N\times 2}$.

Proof: We show that interchanging users in $M < N$ cells will result in either no change or a decrease in network capacity ($M = N$ will result in same capacity). Without loss of generality let these be the first M cells. We employ lighter notation by letting $G_{k\leftarrow n}$ represent the channel gain between user scheduled in slot $k = 1, 2$ and it's serving AP n. Capacity before the swapping is given by

$$\mathcal{C}^* = \sum_{k=1}^{2}\sum_{n=1}^{N} \log\left(1 + \frac{R^*}{\sigma^2 + GR^*\left[\sum_{\substack{i=1\\ i\neq n}}^{M} \frac{1}{G_{k\leftarrow i}} + \sum_{\substack{j=M+1\\ j\neq n}}^{N} \frac{1}{G_{k\leftarrow j}}\right]}\right),$$

and after the swap

$$\begin{aligned}\mathcal{C}' &= \sum_{n=1}^{N} \log\left(1 + \frac{R^*}{\sigma^2 + GR^*\left[\sum_{\substack{i=1\\ i\neq n}}^{M} \frac{1}{G_{2\leftarrow i}} + \sum_{\substack{j=M+1\\ j\neq n}}^{N} \frac{1}{G_{1\leftarrow j}}\right]}\right) \\ &+ \sum_{n=1}^{N} \log\left(1 + \frac{R^*}{\sigma^2 + GR^*\left[\sum_{\substack{i=1\\ i\neq n}}^{M} \frac{1}{G_{1\leftarrow i}} + \sum_{\substack{j=M+1\\ j\neq n}}^{N} \frac{1}{G_{2\leftarrow j}}\right]}\right).\end{aligned}$$

As $G_{1\leftarrow n} \geq G_{2\leftarrow n} \forall\ n$, we declare

$$\left(\beta_{1,n} = \sum_{\substack{i=1\\ i\neq n}}^{M} \frac{1}{G_{1\leftarrow i}}\right) \leq \left(\beta_{2,n} = \sum_{\substack{i=1\\ i\neq n}}^{M} \frac{1}{G_{2\leftarrow i}}\right),$$

$$\left(\alpha_{1,n} = \sum_{\substack{j=M+1\\ j\neq n}}^{N} \frac{1}{G_{1\leftarrow j}}\right) \leq \left(\alpha_{2,n} = \sum_{\substack{j=M+1\\ j\neq n}}^{N} \frac{1}{G_{2\leftarrow j}}\right).$$

Letting

$$\begin{aligned} g_n(x) &= \log\left(1 + \frac{R^*}{\sigma^2 + GR^*(x + \beta_{1,n})}\right) \\ &\quad - \log\left(1 + \frac{R^*}{\sigma^2 + GR^*(x + \beta_{2,n})}\right),\end{aligned}$$

we need to show

$$\mathcal{C}^* - \mathcal{C}' = \sum_{n=1}^{N}\Big(g_n(\alpha_{1,n}) - g_n(\alpha_{2,n})\Big) \geq 0\ \forall\ \alpha_{1,n} \leq \alpha_{2,n}.$$

Fortunately, it can be shown that the differential of $g_n(x)$ is negative [12], making it a decreasing function. Thus $\mathcal{C}^* - \mathcal{C}' \geq 0$. This proves that $\mathbf{S}^* = \mathbf{S}{\downarrow}^{N\times 2}$. □

Next, we define an operator $\mathcal{Q}_{l,k}(\mathbf{S})$ which orders the users in columns (slots) l and k of the scheduling matrix in decreasing order of channel gain.

$$\begin{aligned}\mathcal{Q}_{l,k}(\mathbf{S}) = \Big[&\mathscr{I}^{(1)}\ \mathscr{I}^{(2)} \cdots \mathscr{I}^{(l-1)}\ \zeta(\mathscr{I}^{(l)}, \mathscr{I}^{(k)})_{:,1}\ \mathscr{I}^{(l+1)} \cdots \\ &\mathscr{I}^{(k-1)}\ \zeta(\mathscr{I}^{(l)}, \mathscr{I}^{(k)})_{:,2}\ \mathscr{I}^{(k+1)} \cdots \mathscr{I}^{(K)}\Big],\end{aligned}$$

where $\zeta(u, v) \in \mathbb{N}^{N\times 2}$ obtained through

$$\begin{aligned}\zeta(u, v)_{i,1} &= \max\left(G_{u_i\leftarrow i}, G_{v_i\leftarrow i}\right), \\ \zeta(u, v)_{i,2} &= \min\left(G_{u_i\leftarrow i}, G_{v_i\leftarrow i}\right).\end{aligned}$$

Lemma 3: For an arbitrary scheduling matrix $\mathbf{S}$, $\mathcal{C}(\mathcal{Q}_{l,k}(\mathbf{S})) \geq \mathcal{C}(\mathbf{S})$

Proof: As only columns l and k are manipulated, the capacity due to other columns remains unchanged. From Lemma 2, the capacity of two slots arranged in decreasing order of channel gains will be more than when they are arranged in any other fashion. Thus, $\mathcal{C}(\mathcal{Q}_{l,k}(\mathbf{S})) \geq \mathcal{C}(\mathbf{S})$. □

Lemma 4: For an arbitrary scheduling matrix $\mathbf{S}$

$$\mathcal{Q}_{K-1,K} \cdots \mathcal{Q}_{2,K} \cdots \mathcal{Q}_{2,3}\mathcal{Q}_{1,K} \cdots \mathcal{Q}_{1,3}(\mathcal{Q}_{1,2}(\mathbf{S})) = \mathbf{S}{\downarrow}.$$

Proof: From Lemma 3, the capacity of the scheduling matrix after each $\mathcal{Q}$ operation will be greater than the previous. The successive $\frac{K(K-1)}{2}$ $\mathcal{Q}$ operations will result in the perfectly ordered matrix $\mathbf{S}{\downarrow}$. □

Since there is an increase in capacity at every step, $\mathcal{C}(\mathbf{S}{\downarrow}) \geq \mathcal{C}(\mathbf{S})$. This concludes the proof. ■