Linear Multiuser Detection for Asynchronous CDMA Systems: Chip Pulse Design and Time Delay Distribution.

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***Abstract*— The large system performance analysis of linear multiuser detectors (e.g. MMSE, MSWF, multistage detectors) for asynchronous CDMA systems is provided. While the performance of synchronous systems with square-root waveforms is independent of the chip bandwidth, the performance of asynchronous systems depends on the pulse shape and the bandwidth. It increases as the bandwidth increases beyond half on the chip rate and, in such a case, asynchronous systems outperform the synchronous ones.**

I. Introduction

The large system analysis of linear multiuser detectors with random spreading sequences is mainly focused on synchronous code division multiple access (CDMA) systems. Only few works analyze linear multiuser detectors in asynchronous scenarios [1]–[7]

In [3], [4], the analysis of asynchronous CDMA systems with linear detectors is decomposed in (i) the analysis of asynchronous CDMA systems with symbol asynchronous but chip synchronous signal, i.e. the time delay of the users is a multiple of the chip interval, and (ii) in the analysis of the effects of chip asynchronism. While the effects of symbol asynchronism (case (i)) are well-understood and analytic results are available [4]–[6] the effects of chip asynchronism are still unknown in their whole generality.

In [3], [7] the effects of chip asynchronism are analyzed assuming band limited chip pulses. In [7] the chip waveform is assumed to be an ideal Nyquist sinc function. The baseband received signal is filtered by a low pass filter (or, equivalently a filter matched to the chip waveform) and subsequently sampled at the arrival time of the signal of the user of interest with a frequency equal to the chip rate. [7] proves that the signal to interference and noise ratio (SINR) at the output of the linear minimum mean square error (MMSE) detector converges in mean square sense to the SINR in an equivalent chip-synchronous system. In [3] the wider class of square root Nyquist waveform is considered. For a sufficiently long scalar linear interference equalizer, adjusted according to the MMSE criterion, chip asynchronism does not lead to a significant degradation compared to the chip synchronous transmission.

This work is focused on the analysis of symbol quasi synchronous but chip asynchronous systems, i.e. systems with time delays non greater than the chip interval. The results hold also for general asynchronous systems making use of the analysis of symbol asynchronous and chip synchronous systems in [5], [6].

A general result for the performance analysis of linear detectors for chip asynchronous systems is provided. The chip pulse waveforms are assumed to be identical for all users.

Asynchronous CDMA systems using chip pulse waveforms with bandwidth B not greater than half of the chip rate $\frac{1}{T_c}$, i.e. $B \leq \frac{1}{2T_c}$, have the same asymptotic performance, in terms of SINR, as the correspondent synchronous systems. This generalizes the equivalence result for the ideal Nyquist sinc waveform shown in [7]. Additionally, the performance is independent of the initial sampling instant and of the delay distribution. It depends on the chip pulse waveform with Fourier transform $\Xi(j2\pi f)$ through the coefficients $\mathcal{E}_s(y) = \frac{1}{T_c^{2s}} \int_{-y}^{y} |\Xi(j2\pi \frac{x}{T_c})|^{2s} \mathrm{d}x$, with $y = 1/2$ and s positive integer.

Increasing the bandwidth of the chip waveform above $\frac{1}{2T_c}$ the behaviour of CDMA systems changes substantially. It depends on the time delay distribution and the equivalence between synchronous and asynchronous systems does not hold. Focusing on chip pulse waveforms with bandwidth $\frac{1}{2T_c} \leq B \leq \frac{1}{T_c}$, under general constraints on the chip pulse waveform and on the time delay distribution, the performance of a linear multiuser detector is independent of the time delay and depends on the chip pulse waveform through the coefficients $\mathcal{E}_s(1)$. The asymptotic performance analysis applied to square root raised cosine chip pulse waveforms points out interesting effects of the time delay distribution. As it is well known, the performance of synchronous CDMA systems with square root Nyquist waveforms is independent of the bandwidth. In contrast, the output SINR of linear detectors optimum in a MMSE sense increases if the system is asynchronous and the time delay is uniformly distributed. The gap in performance between synchronous and asynchronous systems is relevant and increases as the SNR at the detector input increases.

II. System Model

Let us consider an asynchronous CDMA system with K users in the uplink channel and spreading factor N. The channel is flat fading and impaired by additive white Gaussian noise. Then, the signal received at the base station, in complex base-band notation, is given by

$$y(t) = \sum_{k=1}^{K} a_{kk} s_k(t-\tau_k) + n(t) \qquad t \in [-\infty, +\infty]$$

where a_{kk} is the received signal amplitude of user k and takes into account the transmitted symbol amplitude, the effects of flat fading channel, and the carrier phase offset. τ_k is the time delay of user k. $n(t)$ is a zero mean complex Gaussian process with two-sided power spectral density, N_0. $s_k(t)$ is the spread signal of user k,

$$s_k(t) = \sum_{m=-\infty}^{+\infty} b_k[m] c_k^{(m)}(t).$$

$b_k[m]$ is the m-th transmitted symbol of user k and

$$c_k^{(m)}(t) = \sum_{u=0}^{N-1} s_{km}[u]\psi(t - mT_s - uT_c)$$

is its spreading waveform. $s_{km}[u]$, $u \in [0, \ldots, N-1]$, are elements of the signature sequence of user k in the m^{th} symbol interval. T_s and T_c are the symbol and chip periods, respectively.

The users' symbols $b_k[m]$ are uncorrelated and identically distributed random variables with $\mathrm{E}\{|b_k[m]|^2\} = 1$ and $\mathrm{E}\{b_k[m]\} = 0$. The spreading sequences $s_{km}[u]$ are assumed to be i.i.d. random variables with $\mathrm{E}\{|s_{km}[u]|^2\} = \frac{1}{N}$ and $\mathrm{E}\{s_{km}[u]\} = 0$.

$\psi(t)$ is the band limited chip waveform with bandwidth B and energy $E_\psi = \int_{-\infty}^{+\infty} |\psi(t)|^2 \mathrm{d}\, t$. Thanks to the normalization of the chip signature the energy of the signature waveform satisfies $\int_{-\infty}^{+\infty} |c_k^{(m)}(t)|^2 \mathrm{d}\, t = E_\psi$.

The front-end of the multiuser detector performs:

- A low pass filtering $G(f)$ with low pass band $|f| \le \frac{r}{2Tc}$ where $r \in \mathbb{Z}^+$ satisfies the constraint $B \le \frac{r}{2T_c}$ so that the sampling theorem is satisfied. The impulse response of the filter is normalized to obtain an amplification factor for the information bearing signal equal to one, i.e.

$$G(f) = \begin{cases} \frac{1}{\sqrt{E_\psi}} & |f| \le \frac{r}{2T_c} \\ 0 & |f| > \frac{r}{2T_c}. \end{cases}$$

- A subsequent continuous-discrete time conversion by conventional sampling at rate $\frac{r}{T_c}$.

With this choice of the front end we obtain sufficient statistics. Additionally, the discrete noise is still white with zero mean and variance $\sigma^2 = \frac{N_0 r}{E_s}$.

In this work we consider symbol quasi-synchronous but chip asynchronous systems, i.e. the time delays τ_k, $k = 1, \ldots, K$, satisfy the constraints $\tau_k \le T_c$. Additionally, we assume that the chip pulse $\psi(t)$ is much shorter than the symbol waveform, i.e. $\psi(t)$ becomes negligible for $|t| > t_0$ and $t_0 \ll T_s$. This is usually verified in systems with large spreading factor. Thus, we can neglect the useful signal outside the symbol interval $[0, T_s]$ and we can focus on the transmission of a single symbol per user as in [7]. The discrete signal at the front-end output is given by

$$y[p] = \sum_{k=1}^{K} a_{kk} b_k \sum_{u=0}^{N-1} s_k[u] \widetilde{\psi}\left(\left(\frac{p}{r} - u\right) T_c - \tau_k\right) + n[p] \quad (1)$$

where $p = \ldots, -1, 0, 1, \ldots$ and $\widetilde{\psi}(t)$ is the pulse shape $\psi(t)$ normalized to have unitary energy, i.e. $\widetilde{\psi}(t) = \frac{\psi(t)}{\sqrt{E_\psi}}$. The system model (1) with $p = 0, 1, \ldots, Nr-1$ reduces to

$$\widetilde{\boldsymbol{y}} = \sum_{k=1}^{K} a_k b_k \boldsymbol{v}_k + \widetilde{\boldsymbol{n}}.$$

$\widetilde{\boldsymbol{y}}$ and $\widetilde{\boldsymbol{n}}$ are the Nr dimensional vectors of received signal and zero mean, complex-valued, circular symmetric, white Gaussian noise with variance $\sigma^2 = \frac{rN_0}{E_s}$, respectively. $\boldsymbol{v}_k$ is the Nr dimensional virtual spreading sequence of user k given by

$$\boldsymbol{v}_k = \widetilde{\boldsymbol{\Psi}}_k \boldsymbol{s}_k.$$

$\boldsymbol{s}_k = (s_k[0] \ldots s_k[N-1])^T$ and $\widetilde{\boldsymbol{\Psi}}_k$ is an $Nr \times N$ matrix taking into account the effects of the pulse shape and the time delay of user k. Its (i,j)-element is given by $(\widetilde{\boldsymbol{\Psi}}_k)_{ij} = \widetilde{\psi}\left(\frac{(i-1)T_c}{r} - (j-1)T_c - \tau_k\right)$.

Let $\widetilde{\boldsymbol{S}}$ be the $rN \times N$ matrix of virtual spreading, i.e. $\widetilde{\boldsymbol{S}} = (\widetilde{\boldsymbol{\Psi}}_1 \boldsymbol{s}_1, \widetilde{\boldsymbol{\Psi}}_2 \boldsymbol{s}_2, \ldots, \widetilde{\boldsymbol{\Psi}}_K \boldsymbol{s}_K)$, $\boldsymbol{A}$ the $K \times K$ diagonal matrix of received amplitudes, and $\boldsymbol{b}$ the vector of transmitted symbols. Then, the system model in matrix notation is given by

$$\widetilde{\boldsymbol{y}} = \widetilde{\boldsymbol{S}} \boldsymbol{A} \boldsymbol{b} + \widetilde{\boldsymbol{n}} = \widetilde{\boldsymbol{H}} \boldsymbol{b} + \widetilde{\boldsymbol{n}} \quad (2)$$

with $\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{S}} \boldsymbol{A}$. Additionally, $\widetilde{\boldsymbol{h}}_k$ denotes the k^{th} column of the matrix $\widetilde{\boldsymbol{H}}$. $\widetilde{\boldsymbol{T}}$ and $\widetilde{\boldsymbol{R}}$ are the correlation matrices defined as $\widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{H}} \widetilde{\boldsymbol{H}}^H$ and $\widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{H}}^H \widetilde{\boldsymbol{H}}$, respectively. $\beta = \frac{K}{N}$ is the system load.

III. Linear Multiuser Detection

We consider the large class of linear multiuser detectors that can be expressed as multistage detectors of rank $M \in \mathbb{Z}^+$ in the Krylov subspace $\chi_{M,k}(\widetilde{\boldsymbol{H}}) = \mathrm{span}(\widetilde{\boldsymbol{T}}^m \widetilde{\boldsymbol{h}}_k)|_{m=0}^{M-1}$, i.e.

$$\widehat{b}_k = \sum_{m=0}^{M-1} (\boldsymbol{w}_k)_m \widetilde{\boldsymbol{h}}^H \widetilde{\boldsymbol{T}}^m \widetilde{\boldsymbol{y}} \quad (3)$$

This class includes the linear MMSE detectors, the linear Parallel Interference Cancelling (PIC) detectors, multistage Wiener filters (MSWF), polynomial expansion detectors [8]. The weighting vectors $\boldsymbol{w}_k$ and M in (3) define completely the detector and, eventually, can be determined by enforcing an optimality criterion. The framework for the performance analysis provided in this work can be applied to any multistage detector in $\chi_{M,k}(\widetilde{\boldsymbol{H}})$. However, we devote special attention to multistage detectors with weights $\widetilde{\boldsymbol{w}}_k$ for the

k-th user such that the mean square error (MSE) $\mathrm{E}\{\|b_k - \sum_{m=0}^{M-1}(\widetilde{\boldsymbol{w}}_k)_m \widetilde{\boldsymbol{h}}^H \widetilde{\boldsymbol{T}}^m \widetilde{\boldsymbol{y}}\|^2\}$ is minimized. We refer to them as Type J-I detectors. They coincide with the linear MMSE detector for $M = \min(rN, K)$ (e.g. [8]). Additionally, for $M < K$, their output SINR converge exponentially in the rank M to the output SINR of the linear MMSE detectors, as assessed by simulations in [9] and analytically proven in [10]. An 8-stages detector achieves near linear MMSE performance independently of the system size [9]. Therefore, the analysis of Type J-I detectors yields also the analysis of linear MMSE detectors. The weights $\widetilde{\boldsymbol{w}}_k$ are given by

$$\widetilde{\boldsymbol{w}}_k = \widetilde{\boldsymbol{\Phi}}_k^{-1} \widetilde{\boldsymbol{\varphi}}_k$$

where $\widetilde{\boldsymbol{\Phi}}_k = \left((\widetilde{\boldsymbol{R}}^{2i+j})_{kk} + \sigma^2(\widetilde{\boldsymbol{R}}^{i+j-1})_{kk}\right)_{i,j=1\dots M}$ and $\widetilde{\boldsymbol{\varphi}}_k = \left((\widetilde{\boldsymbol{R}}^j)_{kk}\right)_{j=1\dots M}$.

The SINR of multistage detectors with arbitrary weights $\boldsymbol{w}_k = ((\boldsymbol{w}_k)_0 \dots (\boldsymbol{w}_k)_{M-1})$ is given by

$$\mathrm{SINR}_k = \frac{\boldsymbol{w}_k^H \widetilde{\boldsymbol{\varphi}}_k \widetilde{\boldsymbol{\varphi}}_k^H \boldsymbol{w}_k}{\boldsymbol{w}_k^H (\widetilde{\boldsymbol{\Phi}}_k - \widetilde{\boldsymbol{\varphi}}_k \widetilde{\boldsymbol{\varphi}}_k^H) \boldsymbol{w}_k}$$

For a Type J-I detector

$$\mathrm{SINR}_{JI,k} = \frac{\widetilde{\boldsymbol{\varphi}}_k^T \widetilde{\boldsymbol{\Phi}}_k^{-1} \widetilde{\boldsymbol{\varphi}}_k}{1 - \widetilde{\boldsymbol{\varphi}}_k^T \widetilde{\boldsymbol{\Phi}}_k^{-1} \widetilde{\boldsymbol{\varphi}}_k}.$$

The large system analysis, i.e. the performance analysis as $K, N \to \infty$ with $\frac{K}{N} \to \beta$, reduces to the computation of the asymptotic values $\widetilde{R}^s_{kk,\infty} = \lim_{K=\beta N \to \infty} (\widetilde{\boldsymbol{R}^s})_{kk}$.

The convergence of the diagonal elements of $\widetilde{\boldsymbol{R}}^\ell$ to deterministic values is established in the following theorem whose assumptions summarize the properties of system model (2).

Theorem 1: Assume

(a) $\boldsymbol{s}_k^{(N)}$, for $k = 1, \dots K$, are K independent N-dimensional column vectors with i.i.d. random elements $s_{nk} \in \mathbb{C}$ such that $\mathrm{E}\{s_{nk}^{(N)}\} = 0$, $\mathrm{E}\{|s_{nk}^{(N)}|^2\} = \frac{1}{N}$, and $\lim_{N\to\infty} \mathrm{E}\{N^3 |s_{nk}^{(N)}|^6\} < +\infty$.[1]

(b) $(\tau_1, \tau_2, \dots, \tau_K)$ is a sequence of K elements with $\tau_k \in [0, T_c)$ and T_c positive real[2].

(c) $\boldsymbol{A}^{(K)} \in \mathbb{C}^{K \times K}$ is a diagonal matrix with k^{th} element a_{kk}.

(d) The sequence of the empirical joint distributions $F^{(K)}_{|\boldsymbol{A}|^2,T}(\lambda,\tau) = \frac{1}{K}\sum_{k=1}^K 1(\lambda - |a_{kk}|^2) 1(\tau - \tau_k)$ converges almost surely, as $K \to \infty$, to a non-random distribution function $F_{|\boldsymbol{A}|^2,T}(\lambda,\tau)$ with bounded support[3].

(e) Given the function $\widetilde{\psi}(t) : \mathbb{R} \to \mathbb{R}$ with unitary Fourier transform $\Xi(j2\pi f)$, the sequence $\{\widetilde{\psi}(T_c n - \tau)\}$ is square root summable, for any $\tau \in [0, T_c]$.

(f) Let $\widetilde{\boldsymbol{\Psi}}_k^{(N)} = \left(\widetilde{\psi}\left(\frac{(i-1)T_c}{r} - (j-1)T_c - \tau_k\right)\right)_{i=1,\dots rN}^{j=1\dots N}$.

(g) $\widetilde{\boldsymbol{S}}^{(N)} = (\widetilde{\boldsymbol{\Psi}}_1^{(N)} \boldsymbol{s}_1, \widetilde{\boldsymbol{\Psi}}_2^{(N)} \boldsymbol{s}_2, \dots \widetilde{\boldsymbol{\Psi}}_K^{(N)} \boldsymbol{s}_K)$.

(h) $\widetilde{\boldsymbol{H}}^{(N)} = \widetilde{\boldsymbol{S}}^{(N)} \boldsymbol{A}^K$.

(i) The spectral radius of the matrix $\widetilde{\boldsymbol{R}}^{(N)} = (\widetilde{\boldsymbol{H}}^{(N)})^H \widetilde{\boldsymbol{H}}^{(N)}$ is almost surely upper bounded as $K, N \to +\infty$ with $\frac{K}{N} \to \beta$[4].

(j) $K = K(N)$ with $\lim_{N\to\infty} \frac{K(N)}{N} = \beta$.

Then, given $(|a_{kk}|^2, \tau_k)$, the k^{th} diagonal element of the matrix $(\widetilde{\boldsymbol{R}}^{(N)})^\ell = ((\widetilde{\boldsymbol{H}}^{(N)})^H \widetilde{\boldsymbol{H}}^{(N)})^\ell$ converges with probability one to a deterministic value, conditionally on $(|a_{kk}|^2, \tau_k)$,

$$\lim_{K=\beta N\to\infty} (\widetilde{\boldsymbol{R}}^{(N)})^\ell_{kk} \stackrel{a.s.}{=} \widetilde{R}^\ell(|a_{kk}|^2, \tau_k)$$

with $\widetilde{R}(|a_{kk}|^2, \tau_k)$ determined by the following recursion

$$\widetilde{R}^\ell(\lambda,\tau) = \sum_{s=0}^{\ell-1} g(\widetilde{\boldsymbol{T}}^{\ell-s-1}, \lambda, \tau) \widetilde{R}^s(\lambda,\tau)$$

and, for $0 \le x \le 1$,

$$\widetilde{\boldsymbol{T}}^\ell(x) = \sum_{s=0}^{\ell-1} \mathbf{f}(\widetilde{\boldsymbol{R}}^{\ell-s-1}, x) \widetilde{\boldsymbol{T}}^s(x)$$

$$\mathbf{f}(\widetilde{\boldsymbol{R}}^s, x) = \beta \int \lambda \boldsymbol{\Delta}_\tau(x) \boldsymbol{\Delta}_\tau^H(x) \widetilde{R}^s(\lambda,\tau) \mathrm{d}\, F_{|\boldsymbol{A}|^2,T}(\lambda,\tau)$$

$$g(\widetilde{\boldsymbol{T}}^s, \lambda, \tau) = \lambda \int_0^1 \boldsymbol{\Delta}_\tau^H(x) \widetilde{\boldsymbol{T}}^s(x) \boldsymbol{\Delta}_\tau(x) \mathrm{d}\, x$$

where

$$\boldsymbol{\Delta}_\tau(x) = \left(\xi_\tau(x), \xi_{\tau - \frac{T_c}{r}}(x) \dots \xi_{\tau - \frac{T_c(r-1)}{r}}(x)\right)^T.$$

with $\xi_\tau(x) \stackrel{\triangle}{=} \frac{1}{T_c} \sum_{s=-\infty}^{+\infty} \mathrm{e}^{-j2\pi \frac{\tau}{T_c}(x+s)} \Xi^*\left(\frac{j2\pi}{T_c}(x+s)\right)$.

The recursion is initialized by setting $\overline{\boldsymbol{T}}^0(x) = \boldsymbol{I}_r$ and $\overline{R}^0(\lambda,\tau) = 1$.

Theorem 1 is proven in [11].

IV. Effects of Asynchronism and of Chip Pulse Waveforms

In the following we focus on two cases:

- Chip pulse waveforms with bandwidth $B \le \frac{1}{2T_c}$.
- Chip pulse waveforms with bandwidth $\frac{1}{2T_c} \le B \le \frac{1}{T_c}$.

A. Chip pulses waveforms with $B \le \frac{1}{2T_c}$

Let us consider chip pulse waveforms $\widetilde{\psi}(t)$ with bandwidth B non greater than $\frac{1}{2T_c}$. We sample the received signal at a rate equal or multiple than the chip rate.

Theorem 1, applied to this case, yields the following algorithm to derive $\widetilde{R}^\ell(\lambda)$ and $m_{\widetilde{\boldsymbol{R}}}$, the asymptotic eigenvalue moments of the matrix $\widetilde{\boldsymbol{R}}$.

Algorithm 1:

1st step Let $\rho_0(z) = 1$ and $\mu_0(y) = \frac{1}{r}$.

ℓ^{th} step
- Define $u_{\ell-1}(y) = y\mu_{\ell-1}(y)$ and write it as a polynomial in y.
- Define $v_{\ell-1}(z) = z\rho_{\ell-1}(z)$ and write it as a polynomial in z.

[1] These random column vectors model the spreading sequences.

[2] τ_k corresponds to the time delay of user k.

[3] $1(\cdot)$ denotes the indicator function.

[4] This condition can be replaced by the less restrictive condition that the integer positive eigenvalue moments are upper bounded.

- Define $\mathcal{E}_s = \frac{1}{T_c^{2s}} \int_{-1/2}^{1/2} \left| \Xi(j2\pi \frac{x}{T_c}) \right|^{2s} \mathrm{d}\, x$ and replace all monomials $y, y^2, \dots, y^\ell$ in the polynomial $u_{\ell-1}(y)$ by $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_\ell$, respectively. Assign the result to $U_{\ell-1}$.
- Define $m^s_{|\boldsymbol{A}|^2} = \mathrm{E}\{|a_{kk}|^{2s}\}$ and replace all monomials $z, z^2, \dots, z^\ell$ in the polynomial $v_{\ell-1}(z)$ by the moments $m^1_{|\boldsymbol{A}|^2}$, $m^2_{|\boldsymbol{A}|^2}, \dots$, $m^\ell_{|\boldsymbol{A}|^2}$, respectively. Assign the result to $V_{\ell-1}$.
- Set
$$\rho_\ell(z) = \sum_{s=0}^{\ell-1} r^2 z U_{\ell-s-1} \rho_s(z)$$
$$\mu_\ell(y) = r \sum_{s=0}^{\ell-1} \beta y V_{\ell-s-1} \mu_s(y).$$
- Assign $\rho_\ell(\lambda)$ to $\widetilde{R}^\ell(\lambda)$
Replace all monomials $z, z^2, \dots, z^\ell$ in the polynomial $\rho_\ell(z)$ by the moments $m^1_{|\boldsymbol{A}|^2}$, $m^2_{|\boldsymbol{A}|^2}, \dots$, $m^\ell_{|\boldsymbol{A}|^2}$, respectively, and assign the result to $m^\ell_{\widetilde{\boldsymbol{R}}}$.

Algorithm 1 is derived in [11].

For ideal Nyquist sinc functions with bandwidth $B = \frac{1}{2T_c}$, $\mathcal{E}_s = 1$, $s = 1, 2, \dots$. Then, $\widetilde{R}^\ell(|a_{kk}|^2) = r^\ell R^\ell(|a_{kk}|^2)$, where $R^\ell(|a_{kk}|^2)$ are the asymptotic diagonal elements obtained by Algorithm 1 in [8] for synchronous systems. It can be verified that the asymptotic performance is independent of r and coincides with the performance of synchronous systems.

In the general cases, the eigenvalue moments of $\widetilde{\boldsymbol{R}}$ depend only on the system load β, the sampling rate $\frac{r}{T_c}$, the eigenvalue distribution of the matrix $\boldsymbol{A}^H \boldsymbol{A}$, and $\mathcal{E}_s$, $s \in \mathbb{Z}^+$. These last coefficients take into account the effects of the shape of the chip pulse $\widetilde{\psi}(t)$. The diagonal elements $\widetilde{R}^\ell(|a_{kk}|^2)$ and the eigenvalue moments $m^\ell_{\widetilde{\boldsymbol{R}}}$ are also independent of the delay distribution. In particular, Algorithm 1 can be applied also to synchronous systems with or without oversampling and any kind of chip-pulse waveform. Therefore, the performance of asynchronous and synchronous systems coincides. Asynchronism does not cause any performance degradation on the system. In this way we have generalized the results obtained in [7] for systems using an ideal Nyquist sinc waveform to any kind of chip pulse waveforms with bandwidth $B \leq \frac{1}{2T_c}$.

The output SINR is also independent of the initial sampling time. Therefore, the system does not incur any degradation in SINR if, for all signals of interest, we consider a discrete statistic obtained by sampling the received signal starting at a random instant and with a proper sampling rate, instead of considering K different statistics obtained by sampling the received signal at the exact arrival time of each signal of interest. *Therefore, without performance degradation we can replace a bank of K different samplers and K different multiuser detectors by a single sampler followed by a single multiuser detector processing jointly all users.*

B. Chip pulse waveform with $\frac{1}{2T_c} \leq B \leq \frac{1}{T_c}$

Let $\widetilde{\psi}(t)$ be an *even* chip waveform with *real* unitary Fourier transform $\Xi(j2\pi f)$ and bandwidth $\frac{1}{2T_c} \leq B \leq \frac{1}{T_c}$. Sufficient statistics are obtained sampling at rate $\frac{2}{T_c}$. Additionally, let us assume that the received powers $|a_{kk}|^2$ and the time delays are statistically independent random variables and $f_T(\tau)$, the marginal eigenvalue probability density function of the time delay, is symmetric around $\tau = \frac{T_c}{2}$, i.e. $f_T(\tau - \frac{T_c}{2})$ is an even function. Then, applying Theorem 1, we obtain the following algorithm to derive the asymptotic values $R^s_{kk,\infty}$.

Algorithm 2:

1st step Let $\rho_0(z) = 1$ and $\mu_0(y) = 1$.

ℓth step
- Define $u_{\ell-1}(y) = y\mu_{\ell-1}(y)$ and write it as a polynomial in y.
- Define $v_{\ell-1}(z) = z\rho_{\ell-1}(z)$ and write it as a polynomial in z.
- Define
$$\mathcal{E}_s = \left(\frac{2}{T_c^2}\right)^s \int_{-1}^{1} \Xi^{2s}\left(j\frac{2\pi}{T_c}x\right) \mathrm{d}\, x \qquad (4)$$
and replace all monomials $y, y^2, \dots, y^\ell$ in the polynomial $u_{\ell-1}(y)$ by $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_\ell$, respectively. Assign the result to $U_{\ell-1}$.
- Define $m^s_{|\boldsymbol{A}|^2} = \mathrm{E}\{|a_{kk}|^{2s}\}$ and replace all monomials $z, z^2, \dots, z^\ell$ in the polynomial $v_{\ell-1}(z)$ by the moments $m^1_{|\boldsymbol{A}|^2}$, $m^2_{|\boldsymbol{A}|^2}, \dots$, $m^\ell_{|\boldsymbol{A}|^2}$, respectively, Assign the result to $V_{\ell-1}$.
- Set
$$\rho_\ell(z) = \sum_{s=0}^{\ell-1} z U_{\ell-s-1} \rho_s(z)$$
$$\mu_\ell(y) = \sum_{s=0}^{\ell-1} \beta y V_{\ell-s-1} \mu_s(y).$$
- Assign $\rho_\ell(\lambda)$ to $\widetilde{R}^\ell(\lambda)$.
Replace all monomials $z, z^2, \dots, z^\ell$ in the polynomial $\rho_\ell(z)$ by the moments $m^1_{|\boldsymbol{A}|^2}$, $m^2_{|\boldsymbol{A}|^2}, \dots$, $m^\ell_{|\boldsymbol{A}|^2}$, respectively, and assign the result to $m^\ell_{\widetilde{\boldsymbol{R}}}$.

Algorithm 2 is derived in [11].

Interestingly the recursive equations do not depend on the arrival time τ_k of the signal of the user k: the performance of such a CDMA system is independent of the time delay.

$\widetilde{R}^\ell(\lambda)$ depends on $\Xi(j2\pi f)$ through the quantities $\mathcal{E}_s$, $s = 1, 2, \dots$, defined in (4).

The performance of synchronous CDMA systems with square root Nyquist chip-pulse waveforms is well-known to be independent of the roll-off and given in [12].

Figure 1 shows the large system performance, in terms of asymptotic output SINR, of detectors Type J-I with $M = 8$ and increasing roll-off versus $\frac{E_s}{N_0}$, in case of both synchronous (lines with markers) and asynchronous CDMA systems (lines without markers). The SINR is obtained assuming equal received powers at the receiver, i.e. $\boldsymbol{A} = \boldsymbol{I}$, and system load $\beta = \frac{3}{4}$.

While the SINR of synchronous systems with square root Nyquist waveforms is independent of the roll-off γ, the output SINR of asynchronous systems increases as the roll-off increases. For $\gamma = 0$, i.e. for an ideal Nyquist sinc

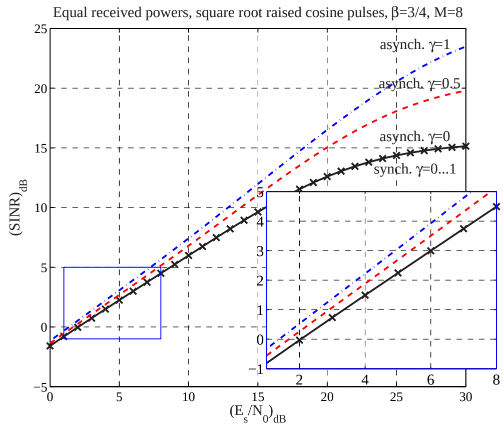

Fig. 1: Output SINR of a Type J-I detector with $M = 8$ versus $\frac{E_s}{N_0}$ for synchronous systems (lines with markers) and chip asynchronous but quasi synchronous symbol systems (lines without markers).

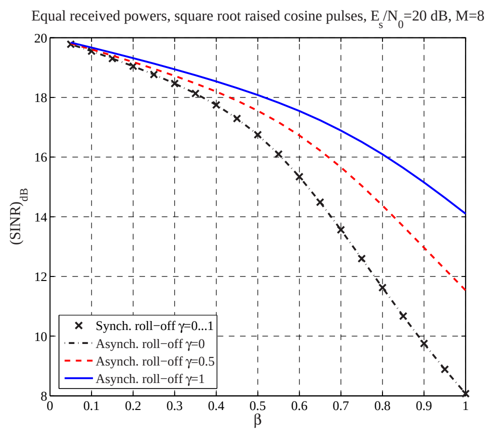

Fig. 2: Output SINR of a Type J-I detector with $M = 8$ versus the system load β for synchronous systems (lines with markers) and chip asynchronous but quasi synchronous symbol systems (lines without markers).

chip pulse waveform, the performance of synchronous and asynchronous systems coincides as already observed in the previous section. For $\gamma > 0$ asynchronous systems outperform the correspondent synchronous systems with equal roll-off.

In Figure 2 the SINR is plotted as a function of the system load, parametric in the roll-off, for $E_S/N_0 = 20$ dB. The improvement achievable by asynchronous systems over synchronous systems increases as the system load increases.

The results obtained for symbol quasi synchronous and chip asynchronous systems can be generalized to asynchronous systems. A chip synchronous and symbol asynchronous system with linear detection is equivalent in performance to the corresponding synchronous system if the observation window is sufficiently large or infinite [4]–[6]. We conjecture that this property extends to chip asynchronous systems. Then, an asynchronous system and a chip asynchronous but symbol quasi synchronous CDMA system have the same performance. This is verified in [11] by simulations.

V. Conclusions

In this work we provided a general framework for the large system performance analysis of asynchronous CDMA systems using a wide class of linear multiuser detectors including linear MMSE, linear PIC detectors, MSWF, polynomial expansion detectors.

Under general conditions verified for systems of practical interest, the effects of chip shape is captured by the coefficients $\mathcal{E}_s$, $s = 1, 2, \ldots$ simply related to the chip waveform. For $B \leq \frac{1}{2T_c}$ the performance of synchronous and asynchronous systems coincides independently of the chip pulse waveform and the delay distribution. For $B > \frac{1}{2T_c}$ detectors optimum in a MMSE sense (MMSE, MSWF, Type J-I detectors), and for square root raised cosine waveforms asynchronous CDMA systems outperform the corresponding synchronous systems. The gap in SINR increases as the bandwidth of the waveforms, or the system load β, or $\frac{E_s}{N_0}$ increases. For $\frac{E_s}{N_0} = 20$ dB a system with roll-off $\gamma = 0$ and load β outperforms a system with roll-off $\gamma = 1$ and load 2β (see Figure 2). This gap decreases as $\frac{E_s}{N_0}$ increases.

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