Ziv-Zakai Lower Bound On The Time Delay Estimation of UWB Signals

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***Abstract*— A unique feature of ultra wideband (UWB) technology lies in its dual capabilities of communication and ranging. As UWB pulses are very narrow, very strict synchronization requirements are incurred as timing errors induce severe performances degradation. In this work, using second order statistics of the received signals, we present the Ziv-Zakai lower bound (ZZLB) for the time delay estimation of UWB signals. This bound is known to be more accurate in the low SNR regime than the Cramer-Rao lower bound (CRLB).**

I. Introduction

The use of Ultra-Wide Band (UWB) signaling techniques are being considered for short-range indoor communications, primarily for next generation high bit-rate *Wireless Personal Area Networks (WPAN)*. Initial works in this direction were carried out by Scholtz [1,2], using the most common form of signaling based duty cycle transmission, where information is carried in pulse position. Such techniques, as well as others are being considered in the standardization process of the IEEE 802.15.3a WPAN proposal (see http://grouper.ieee.org). At the same time, regulatory aspects have quickly been defined by the FCC.

Due to low power density, duty cycle transmission and dense UWB multipath channel [3,4], synchronization is very crucial for reliable transmission. Many works treated this critical issue and derived different Cramer Rao lower bounds to set a limit on the attainable mean-square estimation errors [5]–[8]. Its well known that the CRLB yields an accurate answer only for large signal-to-noise ratios [9]. As UWB systems operate at very low SNR using duty cycle transmission, CRLB represents a very optimistic performance limit. This motivates our effort to derive a Ziv Zakai Lower Bound (ZZLB) [10,11] as it is shown to be more realistic at low SNR and include explicitly the dependence on the a priori interval. Fundamental work on ZZLB for time delay estimation was applied in broadband acoustic channel [12,13]. To derive this bound in our context we use second order statistics approach. Indeed, in UWB systems the channel coherence time is very large (typically $100\mu s$), this characteristic can be very helpful to estimate the statistics of the received signal in order to assist the synchronization task.

II. System Model

Let $s(t) = \sqrt{\frac{E_p}{T_p}}p(t)$ be the transmitted IR-UWB single-pulse one-shot signal, with E_p been the pulse energy and $p(t)$ is the transmitted pulse of duration T_p with $\int_0^{T_p} p(t)^2 dt = 1$. Propagation studies for IR-UWB signals have shown that they undergo dense muptipath environment producing large number of resolvable paths [3]. A typical model for the impulse response of a multipath channel is given by

$$h(t) = \sum_{i=1}^{L} h_i \delta(t - \tau_i) \tag{1}$$

Where τ_i is the $i-th$ path delay and h_i is random variable modeling signal attenuation at τ_i.
The received signal during an observation period of duration T_f can then be written as

$$r(t) = \begin{cases} y(t) + n(t) & t \in [\theta_0,\ \theta_0 + T_d], \\ n(t) & t \in [0,\ \theta_0] \cup [\theta_0 + T_d,\ T_f] \end{cases}$$

Where θ_0 is the time delay parameter to be estimated, T_d the channel delay spread, $n(t)$ is additive white gaussian noise process with zero mean and one sided spectral density $N_o/2$, and

$$\begin{aligned} y(t) &= s(t) * h(t) \\ &= \sqrt{\frac{E_p}{T_p}} \sum_{i=1}^{L} h_i p(t - \tau_i) \end{aligned} \tag{2}$$

Since $y(t)$ is a combination of many significant random variables we model it as a non-sationary guassian process. We assume then the knowledge of the second order satistics $K_y(t,u)$ of $y(t)$. Our interest is then to assess the uncertainty in estimating time delay using IR-UWB in one-shot procedure.

III. Ziv-Zakai Lower Bound

The Ziv-Zakai formulation of the lower bound is based on the probability of deciding correctly between two possible values (θ_1) and (θ_2) of the signal delay. The derivation of this bound relies on result from detection theory [10]. An optimal detection scheme which minimize the probability of error performs a likelihood ratio test between the two hypothesized delays. On the other hand, a suboptimal procedure will be to apply, first, some estimation procedure to estimate the delay $\hat{\theta_0}$ of the received signal then decide between the to hypothesis by comparing $\hat{\theta}$ with $(\theta_1+\theta_2)/2$.

By comparing the performance the two schemes, one obtains the improved Ziv-Zakai lower bound [11] on mean square error of the delay estimate

$$E[(\hat{\theta_0}-\theta_0)^2] \geq \int_0^{T_f-T_d} x dx \int_0^{T_f-T_d-x} \frac{P_e(\theta_1,\theta_1+x)}{T_f-T_d} d\theta_1 \quad (3)$$

Where $P_e(\theta_1,\theta_1+x)$ denotes the probability of error of the likelihood ratio test when deciding between θ_1 and θ_1+x.

Assuming that $P_e(\theta_1,\theta_1+x)$ is independent of θ_1, the expression in (3) becomes

$$E[(\hat{\theta_0}-\theta_0)^2] \geq \int_0^{T_f-T_d} x \frac{T_f-T_d-x}{Tf-Td} P_e(\theta_1,\theta_1+x) dx \quad (4)$$

We start first by deriving the detcetion error probability in our context, then we use (4) to express the ZZLB.

The received signal during the observation interval $[0,\ T_f]$ is first projected on a Fourier basis as

$$\begin{aligned} r(t) &= \sum_{i=1}^{N} R_i \psi_i(t) && (5)\\ R_i &= \int_0^{T_f} r(t)\psi_i(t) dt && (6) \end{aligned}$$

$\psi_i(t) = \frac{1}{\sqrt{T_f}} e^{-\frac{j2\pi it}{T_f}}$ are elements of the Fourier basis defined in the interval $[0,\ T_f]$ and $j=\sqrt{-1}$.

R_i are complex gaussian variables with zero mean and covariance matrix given as

$$\begin{aligned} & K_0(i,k) = E[R_i R_k^\dagger] \\ = & E\left[\frac{1}{T_f}\int_0^{T_f}\int_0^{T_f} r(t)r(u)\psi_i(t)\psi_k^\dagger(u) dtdu\right] \\ = & \frac{N_0}{2}\delta(i-k) + \\ & \frac{1}{T_f}\int_{\theta_0}^{\theta_0+T_d}\int_{\theta_0}^{\theta_0+T_d} K_y(t,u) e^{-\frac{j2\pi(it-ku)}{T_f}} dtdu \end{aligned} \quad (7)$$

Let θ_1 and θ_2 denote the hypothesized delays. We have then

$$r(t) = \begin{cases} y(t-(\theta_1-\theta_0)) + n(t) & t \in [\theta_1,\ \theta_1+T_d] \mid H_1 \\ y(t-(\theta_2-\theta_0)) + n(t) & t \in [\theta_2,\ \theta_2+T_d] \mid H_2 \end{cases} \quad (8)$$

Let $\Delta_1 = \theta_1-\theta_0$ and $x=\theta_2-\theta_1$, we get then $\Delta_2 = \theta_1-\theta_0 = \Delta_1+x$.

Under Hypothesis H_1, R has a covariance matrix given as

$$\begin{aligned} & {}_1K_{\theta_1,x}(i,k) = E[R_i R_k^\dagger | H_1] \\ = & E\left[\frac{1}{T_f}\int_0^{T_f}\int_0^{T_f} r(t)r(u)\psi_i(t)\psi_k^\dagger(u) dtdu\right] \\ = & \frac{N_0}{2}\delta_{i,k} + \frac{1}{T_f}\int_{\theta_1}^{\theta_1+T_d} \\ & \int_{\theta_1}^{\theta_1+T_d} K_y(t-\Delta_1,u-\Delta_1) e^{-\frac{j2\pi it}{T_f}} e^{\frac{j2\pi ku}{T_f}} dtdu \\ = & \frac{N_0}{2}\delta_{i,k} + \frac{1}{T_f}\int_{\theta_0}^{\theta_0+T_d} \\ & \int_{\theta_0}^{\theta_0+T_d} K_y(t',u') e^{-\frac{j2\pi i(t'+\Delta_1)}{T_f}} e^{\frac{j2\pi k(u'+\Delta_1)}{T_f}} dt'du' \\ = & e^{-\frac{j2\pi\Delta_1(i-k)}{T_f}} K_0(i,k) + \frac{N_0}{2}\left[1-e^{-\frac{j2\pi\Delta_1(i-k)}{T_f}}\right]\delta_{i,k} \end{aligned} \quad (9)$$

Similarly, under hypothesis H_2, R has a covariance matrix given as

$$\begin{aligned} & {}_2K_{\theta_1,x}(i,k) = E[R_i R_k^\dagger | H_2] \\ = & e^{-\frac{j2\pi\Delta_2(i-k)}{T_f}} K_0(i,k) + \frac{N_0}{2}\left[1-e^{-\frac{j2\pi\Delta_2(i-k)}{T_f}}\right]\delta_{i,k} \end{aligned} \quad (10)$$

The resultant log-likelihood function is then

$$\begin{aligned} L(\theta_1,x) &= \ln\left\{\frac{P(R|H1)}{P(R|H2)}\right\} \\ &= R^\dagger Q_{\theta_1,x} R - D_{\theta_1,x} && (11) \end{aligned}$$

Where

$$\begin{aligned} Q_{\theta_1,x} &= {}_1K_{\theta_1,x}^{-1} - {}_2K_{\theta_1,x}^{-1} && (12)\\ D_{\theta_1,x} &= \ln det({}_2K_{\theta_1,x}) - \ln det({}_1K_{\theta_1,x}) && (13) \end{aligned}$$

The two hypothesis are then compared according to the decision rule

$$L(\theta_1, x) \underset{H_1}{\overset{H_2}{\gtrless}} D_{\theta_1,x} \tag{14}$$

The resulting probability of detection error is then

$$P_e(\theta_1, \theta_1+x) = \frac{1}{2}[P(Z > D_{\theta_1,x}|H_1)+P(Z < D_{\theta_1,x}|H_2)] \tag{15}$$

Where $Z = R^{\dagger}Q_{\theta_1,x}R$.
In the following we derive the distribution of Z under $H1$. The same procedure is used under $H2$ so it is omitted here.
Under H_1, $\Delta_1 = 0$ so $K_1 = K_0$. As ${}_1K_{\theta_1,x}$ and ${}_2K_{\theta_1,x}$ are Hermitian, Q is also Hermitian.
We begin by making a Karhunen-Loeve decomposition of R in the basis of its covariance matrix.

$$K_0 = U\Lambda U^{\dagger} \tag{16}$$

where Λ is a diagonal matrix with eigenvalues of K_0 as diagonal elements, and U is a unitary matrix formed by the corresponding eigenvectors.
R can be written then as

$$R = U\Lambda^{\frac{1}{2}}\dot{R} \tag{17}$$

Where

$$\dot{R} = \Lambda^{-\frac{1}{2}}U^{\dagger}R \quad , \quad K_{\dot{R}} = I \tag{18}$$

So we get

$$Z = \dot{R}^{\dagger}\dot{Q}_{\theta_1,x}\dot{R} \tag{19}$$

Whith

$$\dot{Q}_{\theta_1,x} = \Lambda^{\frac{1}{2}}U^{\dagger}Q_{\theta_1,x}U\Lambda^{\frac{1}{2}} \tag{20}$$

$\dot{Q}_{\theta_1,x}$ is hermitian and can be decomposed also as $VMV^{\dagger}$ where V is an orthonormal matrix of eigenvectors of $\dot{Q}_{\theta_1,x}$ and M is a diagonal matrix of corresponding eigenvalues $\mu^i_{\theta_1,x}$.
We can thus write

$$Z = (V^{\dagger}\dot{R})^{\dagger}M(V^{\dagger}\dot{R}) = \sum \mu^i_{\theta_1,x}|\ddot{R}_i|^2 \tag{21}$$

Where

$$\ddot{R} = V^{\dagger}\dot{R} = V^{\dagger}\Lambda^{-\frac{1}{2}}U^{\dagger}R \quad , \quad K_{\ddot{R}} = VK_{\dot{R}}V^{\dagger} = I \tag{22}$$

We have thus expressed Z as weighted sum of squares of uncorrelated random variables with unit variance.
The moment generating function of Z can then be written as

$$\begin{aligned} G_Z(s) &= \frac{1}{det\,(I - sK_0Q_{\theta_1,x})} \\ &= \prod_{i=0}^{N} \frac{1}{1-s\mu^i_{\theta_1,x}} \quad \text{for}|\Re(s)| < 1/max(|\mu^i_{\theta_1,x}|) \\ &= \sum_{i=0}^{N} \frac{\pi^i_{\theta_1,x}}{1-s\mu^i_{\theta_1,x}} \quad \text{for} \quad \mu^i_{\theta_1,x} \text{ distinct} \end{aligned} \tag{23}$$

Where the coefficients $\pi^i_{\theta_1,x}$ are the residues given by

$$\pi^i_{\theta_1,x} = \prod_{\substack{k=0 \\ k\neq i}}^{N} \frac{\mu^i_{\theta_1,x}}{\mu^i_{\theta_1,x} - \mu^i_{\theta_1,x}} \tag{24}$$

This can be inverted to yield the distribution of Z:

$$F_Z(z,\theta_1,x) = \begin{cases} \sum_{\{i|\mu^i_{\theta_1,x}>0\}} & \pi^i_{\theta_1,x}e^{-z/\mu^i_{\theta_1,x}} \quad z > 0 \\ \sum_{\{i|\mu^i_{\theta_1,x}<0\}} & \pi^i_{\theta_1,x}(1-e^{-z/\mu^i_{\theta_1,x}}) \\ \quad + \sum_{\{i|\mu^i_{\theta_1,x}>0\}} & \pi^i_{\theta_1,x} \quad z \leq 0 \end{cases} \tag{25}$$

Under $H1$, $\Delta_1 = 0$ so $K_1 = K_0$. Similarly, Under $H2$, $\Delta_2 = 0$ so $K_2 = K_0$. As the probability error is independent from θ_1 we remove the dependence on θ_1. The decision threshold can then be written as

$$D_{\theta_1,x} = \begin{cases} {}_1z_x = \ln det({}_2K_{\theta_1,x}) - \ln det(K_0) & | H1 \\ {}_2z_x = \ln det(K_0) - \ln det({}_1K_{\theta_1,x}) & | H2 \end{cases} \tag{26}$$

We can now express the ZZLB as

$$\begin{aligned} E[(\hat{\theta_0} - \theta_0)^2] &\geq \int_0^{T_f-T_d} \frac{x(T_f-T_d-x)}{Tf-Td}P_e(\theta_1,\theta_1+x)dx \\ &\geq \frac{1}{2(Tf-Td)}\int_0^{T_f-T_d} x(T_f-T_d-x) \\ &\quad [1-F_Z({}_1z_x,x)+F_Z({}_2z_x,x)]\,dx \end{aligned} \tag{27}$$

IV. Numerical Results

In this section, we evaluate the ZZLB for IR-UWB one shot signal where the received signal $y(t)$ is modeled as a non-stationary gaussian process. For numerical purpose, we assume the knowledge of a degenerate kernel of the second order statistics $K_y(t,u)$ characterized by a finite number of eigenmodes. In Fig. 1, we plot the obtained standard deviation of ZZLB for different delay spread durations T_d Vs. Average transmitted SNR.The pulse is of duration $T_p = 1ns$ and the observation period is of length $T_f = 100ns$. The average SNR is defined as $SNR_{avg} = \frac{E_pT_p}{T_fN_0}$.
As predicted by the bound, we observe three different operating regions:

1) The full ambiguity region corresponding to a very small SNR, in this region the receiver see

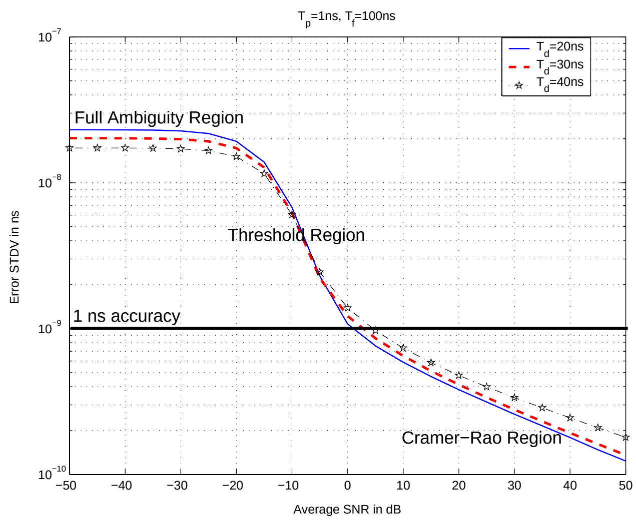

Fig. 1. Estimation error standard deviation

the signal as noise and the error in this case is uniformly distributed over the a priori interval $[0,\ T_f - T_d]$.

2) The Cramer-rao region correponds to a high SNR, in this case the receiver success to match well the signal with uncertainty in the order of T_p. We observe however that for increasing delay spread T_d, and for the same pulse energy, the error variance increases as the energy is more spread which is intuitivly comprehensible .
3) The threshold region is located just between the two regions cited above. The estimation error in this case exceeds the CRLB by a large factor and describes more precisely the limit of the estimation error. It is then more realistic bound, especially for UWB systems that are supposed to operate on this range of SNR.

V. Conclusion

In this work, we presented the Ziv-Zakai lower bound on the mean square error of the time delay estimation for IR-UWB signals. The obtained bound is more tight than the Cramer-rao lower bound in the operating SNR region of UWB systems, showing that 1ns accuracy range is achievable even at low average SNR in one shot attempt. This result is encouraging for the design of real applications based on UWB signaling technology.

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