On the Optimality of LAttice Space-Time (LAST) Coding

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Abstract — **In this paper, we introduce the class of LAttice Space-Time (LAST) codes. We show that these codes achieve the optimal diversity-vs-multiplexing tradeoff defined by Zheng and Tse under generalized minimum Euclidean distance lattice decoding. Our scheme is based on a generalization of Erez and Zamir mod-Λ scheme to the MIMO case. This result settles the open problem posed by Zheng and Tse on the construction of *explicit* coding and decoding schemes that achieve the optimal diversity-vs-multiplexing tradeoff. Moreover, our results shed more light on the structure of optimal coding/decoding techniques in delay limited MIMO channels. In particular; 1) we show that MMSE-GDFE plays a fundamental role in approaching the limits of delay limited MIMO channels in the high SNR regime, unlike the AWGN channel case and 2) our random coding arguments represent a major departure from traditional space-time code designs based on the rank and/or mutual information design criteria.**

I. The Diversity-Multiplexing Tradeoff

We consider the quasi-static frequency-flat fading M-transmit N-receive multiple-input multiple-output (MIMO) channel with no channel knowledge at the transmitter and perfect channel knowledge at the receiver. The problem of characterizing the optimal diversity-vs-multiplexing tradeoff was well-posed and solved by Zheng and Tse in [1]. For given M, N and block length T, the authors considered a family of space-time codes $\{\mathcal{C}_\rho\}$ indexed by their operating SNR ρ, such that the code $\mathcal{C}_\rho$ has rate $R(\rho)$ bits per channel use and error probability $P_e(\rho)$. The multiplexing gain r and the diversity gain d are defined as

$$r \triangleq \lim_{\rho\to\infty} \frac{R(\rho)}{\log\rho} \quad \text{and} \quad d \triangleq -\lim_{\rho\to\infty} \frac{\log(P_e(\rho))}{\log\rho}. \qquad (1)$$

In [1], the optimal tradeoff curve $d^\star(r)$, yielding for each r the maximum possible d, was found for unrestricted coding and ML decoding. The conclusion of [1] provides the motivation for this paper: *"It should be noted that other than for the* 2×1 *channel (for which the Alamouti scheme is optimal), there is no explicitly constructed coding scheme that achieves the optimal tradeoff curve for any* $r > 0$. *This remains an open problem."* [1]. Here, we exhibit explicit coding schemes that achieve $d^\star(r)$ for any M and N, and $T \geq M + N - 1$.

II. LAST Coding

An $m = 2MT$-dimensional lattice code $\mathcal{C}(\Lambda, \mathbf{u}, \mathcal{R})$ is the finite subset of the lattice translate $\Lambda + \mathbf{u}$ inside the *shaping region* $\mathcal{R}$, i.e., $\mathcal{C} = \{\Lambda + \mathbf{u}\} \cap \mathcal{R}$, where $\mathcal{R}$ is a bounded measurable region of $\mathbb{R}^m$. We say that a space-time coding scheme is a *full-dimensional* LAST code if its underlying codebook is a lattice code, i.e., if $\mathcal{C} = \mathcal{C}(\Lambda, \mathbf{u}, \mathcal{R})$, for some $2MT$-dimensional lattice Λ, translation vector $\mathbf{u}_0$ and shaping region $\mathcal{R}$. We further say that a LAST code is nested if the underlying lattice code is nested (please refer to [2, 3] for the definition of nested lattice codes). The proposed mod-Λ scheme works as follows. Consider the nested LAST code $\mathcal{C}$ defined by Λ_c (the *coding lattice*) and by its sublattice Λ_s (the *shaping lattice*) in $\mathbb{R}^{2MT}$. Assume further that Λ_s has a second-order moment $\sigma^2(\Lambda_s) = 1/2$. The transmitter selects a codeword $\mathbf{c} \in \mathcal{C}$, generates a random translate $\mathbf{u}$ (dither) with uniform distribution over $\mathcal{V}_s$ and computes

$$\mathbf{x} = [\mathbf{c} - \mathbf{u}] \mod \Lambda_s \qquad (2)$$

The signal $\mathbf{x}$ is then transmitted on the MIMO channel (i.e., periodic multiplexing in space and time). Let $\mathbf{y}$ denote the corresponding channel output. We replace the *scalar* scaling of [2] by a matrix multiplication by the forward filter matrix of the MMSE-GDFE. Moreover, instead of just adding the dither signal $\mathbf{u}$ at the receiver (as in [2]), we add the dither signal filtered by the upper triangular feedback filter matrix of the MMSE-GDFE. The output is then fed to a lattice decoder. The detailed description of the MMSE-Lattice decoder is available in [3].

III. The Main Result

Theorem 1 *There exists a sequence of nested LAST codes with block length* $T \geq M + N - 1$ *that achieves the optimal diversity-vs-multiplexing tradeoff curve* $d^\star(r)$ *for all* $r \in [0, \min\{M, N\}]$ *under the mod-*Λ *scheme.*

Moreover, one can establish the optimality of LAST coding/decoding without resorting to the nesting approach as shown in [3]. For a detailed discussion of the random coding arguments associated with this result and the insights that can be gleaned from it, the reader is referred to the journal version of this work [3].

References

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