

A BAYESIAN METHOD FOR ITERATIVE JOINT DETECTION AND DECODING IN THE PRESENCE OF PHASE NOISE

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ABSTRACT

In this paper we describe a new Bayesian algorithm for joint iterative decoding and parameter estimation. In particular, we focus on joint decoding and synchronization of low-density parity-check (LDPC) codes in the presence of phase noise. The performance of the proposed algorithm is analyzed by computer simulations showing that a time-varying channel phase, with a rate of change typical of the instabilities of the transmit and receive oscillators, does not entail significant degradation with respect to the case of a known phase.

1. INTRODUCTION

In the recent technical literature, the problem of LDPC decoding in the presence of an unknown channel phase has become quite popular. In [1, 2], a simple noncoherent channel model is considered that tries to capture the phase dynamics: the unknown carrier phase is considered constant over a block of N symbols and independent from block to block. In [3] an approximate quantized model for the unknown phase is considered and the receiver is designed based on this model. Finally, in [4], based on the approach in [5], different phase models are considered and approximate solutions are derived.

A non-Bayesian approach is adopted in [6, 7, 8]. The unknown parameters, modeled as deterministic, are estimated by using the *Expectation-Maximization* (EM) algorithm [6, 7] or an ad-hoc procedure [8], and this estimation algorithm is embedded into the iterative decoding process. For all these algorithms, when the channel is time-varying, the performance rapidly degrades since the receiver is not designed to exploit the statistical or a priori knowledge of the phase variations.

Bayesian methods for joint decoding and channel parameter estimation amount, roughly speaking, to construct a Factor Graph (FG) modeling the statistical dependency of the transmitted data, of the channel parameters to be estimated, and of the observed signal, and by applying the Sum-Product (SP) algorithm. The resulting algorithms are naturally iterative, and are well-suited to the decoding of codes such as LDPC and turbo codes, whose decoding algorithms are typically iterative even in the fully coherent setting (all channel parameters known).

Two possible approaches can be adopted for building a factor graph (FG) which takes into account the channel model along with the code constraints. In the first one, by means of a factorization of the joint a posteriori probability

of the transmitted symbols, a factor graph representing both the code constraints and the channel model but not explicitly the channel parameters can be built. The application of the SP algorithm to this factor graph leads to a scheme for joint detection and decoding. This approach is not further discussed here and an interested reader may refer to [1, 9].

In the second approach, suggested by [5], variable nodes representing the channel parameters are explicitly introduced in the FG. The marginalization with respect to the unknown channel parameters, that is performed with respect to the *a priori* probability distribution in the previous approach, is now performed directly by the SP algorithm. The problem with these methods is that, while the SP algorithm is well-suited to handle probability mass functions (i.e., discrete random variables), the channel parameters are typically continuous random variables, statistically described by some conditional probability density function (pdf). There are two classical solutions to this problem. One is based on the use of *canonical distributions*, i.e., on pdfs that are efficiently parameterized. Hence, the SP has just to forward the parameters of the distribution. The other method is based on the quantization of the parameter space. Obviously, this latter approach becomes “optimal” (in the sense that it approaches the performance of the exact SP algorithm) for a sufficiently large number of quantization levels, at the expenses of an increased computational complexity.

In this paper we derive a novel low-complexity Bayesian algorithm based on the canonical distribution paradigm.

2. SYSTEM MODEL

We consider a coded transmission system where codewords $\mathbf{c} = (c_0, \dots, c_{K-1}) \in \mathcal{C}$ are transmitted over a channel affected by additive white Gaussian noise (AWGN) and by a random time-varying carrier phase (phase noise). The code \mathcal{C} is defined over some complex signal set \mathcal{X} (e.g., PSK or QAM). In addition, to avoid phase ambiguity problems, pilot symbols may be also inserted in the transmitted symbol sequence. Assuming linear modulation with Nyquist pulses and slow enough phase time variations so as no intersymbol interference arises, the discrete-time basedband complex equivalent channel model at the receiver is given by

$$r_k = c_k e^{j\theta_k} + n_k, \quad k = 0, \dots, K-1 \quad (1)$$

where $\{n_k\}$ is a discrete-time proper complex WGN process with per-component variance equal to σ^2 . The phase noise process $\{\theta_k\}$ is modeled as a discrete-time Wiener process:

$$\theta_k = \theta_{k-1} + \Delta_k \quad (2)$$

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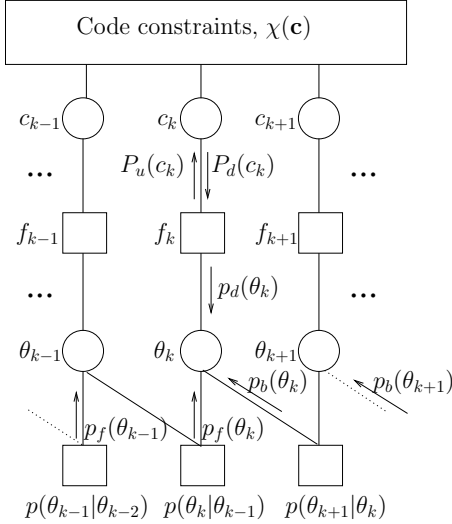


Figure 1: Factor graph corresponding to eqn. (5).

where T is the signaling interval and the i.i.d. Gaussian increments Δ_k have zero mean and standard deviation σ_Δ .¹ Hence

$$p(\theta_k|\theta_{k-1}, \theta_{k-2}, \dots, \theta_0) = p(\theta_k|\theta_{k-1}) \quad (3)$$

$$p(\theta_0) = \frac{1}{2\pi}, \theta_0 \in [0, 2\pi). \quad (4)$$

In the following, the vector of channel phase values will be denoted as $\underline{\theta} = (\theta_0, \theta_1, \dots, \theta_{K-1})$.

3. PROPOSED ALGORITHM

The joint distribution of coded symbols and channel parameters can be expressed as

$$\begin{aligned} P(\mathbf{c}|\underline{\theta}, \mathbf{r})p(\underline{\theta}|\mathbf{r}) &\sim p(\mathbf{r}|\mathbf{c}, \underline{\theta})\chi(\mathbf{c})p(\underline{\theta}) \\ &= \chi(\mathbf{c})p(\theta_0) \prod_{k=0}^{K-1} p(r_k|c_k, \theta_k) \prod_{k=1}^{K-1} p(\theta_k|\theta_{k-1}) \\ &= \chi(\mathbf{c})p(\theta_0) \prod_{k=0}^{K-1} f_k(c_k, \theta_k) \prod_{k=1}^{K-1} p(\theta_k|\theta_{k-1}) \end{aligned} \quad (5)$$

where \sim indicates that two quantities are monotonically related with respect to the variables of interest, $\chi(\mathbf{c})$ denotes the code constraint function ($\chi(\mathbf{c}) = 1$ for all codewords $\mathbf{c} \in \mathcal{C}$, and zero elsewhere), and we have defined

$$\begin{aligned} f_k(c_k, \theta_k) &= p(r_k|c_k, \theta_k) \sim \exp\left\{-\frac{1}{2\sigma^2}|r_k - c_k e^{j\theta_k}|^2\right\} \\ &\sim \exp\left\{\frac{1}{\sigma^2}\Re[r_k c_k^* e^{-j\theta_k}] - \frac{|c_k|^2}{2\sigma^2}\right\}. \end{aligned} \quad (6)$$

The corresponding factor graph is sketched in Fig. 1.

Omitting the explicit reference to the current iteration and assuming that the SP algorithm works in the natural domain, let us denote by $P_d(c_k)$ the message from variable node c_k to factor node f_k , and by $P_u(c_k)$ the message in the opposite

direction (see Fig. 1). The message $p_d(\theta_k)$ from factor node f_k to variable node θ_k can be expressed as

$$p_d(\theta_k) = \sum_{x \in \mathcal{X}} P_d(c_k = x) f_k(c_k = x, \theta_k). \quad (7)$$

We also assume that in the lower part of the factor graph, describing the channel, a forward-backward schedule is adopted. Hence, messages $p_f(\theta_k)$ from factor node f_k to variable node θ_k , and $p_b(\theta_k)$ from factor node f_{k+1} to variable node θ_k , may be recursively computed as follows:

$$p_f(\theta_k) = \int_0^{2\pi} p_d(\theta_{k-1}) p_f(\theta_{k-1}) p(\theta_k|\theta_{k-1}) d\theta_{k-1} \quad (8)$$

$$p_b(\theta_k) = \int_0^{2\pi} p_d(\theta_{k+1}) p_b(\theta_{k+1}) p(\theta_{k+1}|\theta_k) d\theta_{k+1} \quad (9)$$

with the following starting conditions:

$$p_f(\theta_0) = p(\theta_0) \quad (10)$$

$$p_b(\theta_{K-1}) = \frac{1}{2\pi}, \theta_{K-1} \in [0, 2\pi). \quad (11)$$

The probability $P_u(c_k)$ can be finally computed as

$$P_u(c_k) = \int_0^{2\pi} p_f(\theta_k) p_b(\theta_k) f_k(c_k, \theta_k) d\theta_k. \quad (12)$$

We now show a method for the computation of the probability $P_u(c_k)$ in the form of a series expansion.

The function $f_k(c_k, \theta_k)$ is periodic in θ_k . Hence, it can be expanded in Fourier series. We use the following known result [10, eqn. (9.6.34)]:

$$e^{x \cos \theta} = I_0(x) + 2 \sum_{\ell=1}^{\infty} I_\ell(x) \cos(\ell\theta) \quad (13)$$

where $I_\ell(x)$ is the modified Bessel function of the first kind of order ℓ . Defining, for a complex number z , $\phi(z) = \arg(z)$, after some straightforward manipulations we obtain

$$f_k(c_k, \theta_k) = e^{-\frac{|c_k|^2}{2\sigma^2}} \sum_{\ell=-\infty}^{\infty} I_\ell\left(\frac{|r_k||c_k|}{\sigma^2}\right) e^{-j\ell\phi(r_k c_k^*)} e^{j\ell\theta_k}. \quad (14)$$

Substituting (14) into eqn. (7), we may express

$$p_d(\theta_k) = \sum_{\ell=-\infty}^{\infty} A_k^{(\ell)} e^{j\ell\theta_k} \quad (15)$$

having defined

$$\begin{aligned} A_k^{(\ell)} &= \sum_{x \in \mathcal{X}} P_d(c_k = x) e^{-\frac{|x|^2}{2\sigma^2}} I_\ell\left(\frac{|r_k||x|}{\sigma^2}\right) e^{-j\ell\phi(r_k x^*)} \\ &= -j\ell\phi(r_k) \sum_{x \in \mathcal{X}} P_d(c_k = x) e^{-\frac{|x|^2}{2\sigma^2}} I_\ell\left(\frac{|r_k||x|}{\sigma^2}\right) e^{j\ell\phi(x)} \\ &= e^{-j\ell\phi(r_k)} \sum_{x \in \mathcal{X}} P_d(c_k = x) e^{-\frac{|x|^2}{2\sigma^2}} I_\ell\left(\frac{|r_k||x|}{\sigma^2}\right) \frac{x^\ell}{|x|^\ell}. \end{aligned} \quad (16)$$

Note that for M -PSK signals, the expression of coefficients $A_k^{(\ell)}$, neglecting irrelevant terms, simplifies to

$$A_k^{(\ell)} = e^{-j\ell\phi(r_k)} I_\ell\left(\frac{|r_k|}{\sigma^2}\right) \sum_{x \in \mathcal{X}} P_d(c_k = x) x^\ell. \quad (17)$$

¹Note that, since the channel phase is defined modulus 2π , the pdf of the random variable Δ_k can be approximated as Gaussian only if $\sigma_\Delta \ll 2\pi$.

In this case, at the first iteration, when the probabilities of symbols $P(c_k)$ are all equal to $1/M$ (excepting for pilot symbols), these coefficients are zero for $\ell \neq 0, \pm M, \pm 2M, \pm 3M, \dots$. In general, a reduced number N of coefficients must be taken into account due to the fact that, for a given x , functions $I_\ell(x)$ are monotonically decreasing for increasing values of ℓ . When this truncation is performed, it is suitable to apply a window to the truncated coefficients. By means of computer simulations, we found that the Kaiser window with an optimized parameter β [11] assures the better performance.

Pdfs $p_f(\theta_k)$ and $p_b(\theta_k)$ will be of the same form, i.e., they will be periodic and can be expanded in Fourier series:

$$p_f(\theta_k) = \sum_{\ell=-\infty}^{\infty} B_{f,k}^{(\ell)} e^{j\ell\theta_k} \quad (18)$$

$$p_b(\theta_k) = \sum_{\ell=-\infty}^{\infty} B_{b,k}^{(\ell)} e^{j\ell\theta_k}. \quad (19)$$

Substituting (15) and (18) into eqn. (8), we obtain

$$\begin{aligned} & \sum_{\ell=-\infty}^{\infty} B_{f,k}^{(\ell)} e^{j\ell\theta_k} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{k-1}^{(m)} B_{f,k-1}^{(n)} \int_0^{2\pi} e^{j(m+n)\theta_{k-1}} p(\theta_k|\theta_{k-1}) d\theta_{k-1} \\ &= \sum_{\ell=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{k-1}^{(m)} B_{f,k-1}^{(\ell-m)} \int_0^{2\pi} e^{j\ell\theta_{k-1}} p(\theta_k|\theta_{k-1}) d\theta_{k-1}. \end{aligned} \quad (20)$$

For practical values of σ_Δ , the pdf $p(\theta_k|\theta_{k-1})$ is strictly limited to an interval of duration less than 2π . Hence we may write

$$\int_0^{2\pi} e^{j\ell\theta_{k-1}} p(\theta_k|\theta_{k-1}) d\theta_{k-1} = \int_{-\infty}^{\infty} e^{j\ell\theta_{k-1}} p(\theta_k|\theta_{k-1}) d\theta_{k-1}. \quad (21)$$

By direct computation, it is easy to show that in the case of the phase model (2) it is

$$\int_{-\infty}^{\infty} e^{j\ell\theta_{k-1}} p(\theta_k|\theta_{k-1}) d\theta_{k-1} = D_\ell(\sigma_\Delta) e^{j\ell\theta_k} \quad (22)$$

where

$$D_\ell(\sigma_\Delta) = e^{-\frac{\sigma_\Delta^2 \ell^2}{2}}. \quad (23)$$

Hence

$$\sum_{\ell=-\infty}^{\infty} B_{f,k}^{(\ell)} e^{j\ell\theta_k} = \sum_{\ell=-\infty}^{\infty} \left[D_\ell(\sigma_\Delta) \sum_{m=-\infty}^{\infty} A_{k-1}^{(m)} B_{f,k-1}^{(\ell-m)} \right] e^{j\ell\theta_k} \quad (24)$$

obtaining a recursive equation for the computation of the coefficients $B_{f,k}^{(\ell)}$:

$$B_{f,k}^{(\ell)} = D_\ell(\sigma_\Delta) \sum_{m=-\infty}^{\infty} A_{k-1}^{(m)} B_{f,k-1}^{(\ell-m)} = D_\ell(\sigma_\Delta) [A_{k-1}^{(\ell)} \otimes B_{f,k-1}^{(\ell)}] \quad (25)$$

where \otimes denotes convolution between sequences. From condition (10), we derive the following starting condition:

$$B_{f,0}^{(\ell)} = \delta(\ell) \quad (26)$$

where $\delta(\ell)$ denotes the Kronecker delta. Similarly, to compute coefficients $\{B_{b,k}^{(\ell)}\}$, we have the following backward recursion:

$$B_{b,k}^{(\ell)} = D_\ell(\sigma_\Delta) [A_{k+1}^{(\ell)} \otimes B_{b,k+1}^{(\ell)}] \quad (27)$$

with starting condition

$$B_{b,K-1}^{(\ell)} = \delta(\ell). \quad (28)$$

Note that the computation of these coefficient can be simplified taking into account that $A_k^{(-\ell)} = A_k^{(\ell)*}$, $B_{f,k}^{(-\ell)} = B_{f,k}^{(\ell)*}$, and $B_{b,k}^{(-\ell)} = B_{b,k}^{(\ell)*}$. Finally, substituting (14), (18), and (19) into eqn. (12) and defining

$$E_k^{(\ell)} = e^{-\frac{|c_k|^2}{2\sigma^2}} \left\{ B_{f,k}^{(\ell)} \otimes B_{b,k}^{(\ell)} \otimes \left[I_\ell \left(\frac{|r_k||c_k|}{\sigma^2} \right) e^{-j\ell\phi(r_k c_k^*)} \right] \right\} \quad (29)$$

we have

$$P_u(c_k) = \sum_{\ell=-\infty}^{\infty} E_k^{(\ell)} \int_0^{2\pi} e^{j\ell\theta_k} d\theta_k = E_k^{(0)}. \quad (30)$$

4. NUMERICAL RESULTS

The performance of the proposed schemes is assessed by computer simulations in terms of bit error rate (BER) versus E_b/N_0 , E_b being the received signal energy per information bit and $N_0/2$ the two-sided noise power spectral density. The considered code is a (3,6)-regular LDPC code with codewords of length 4000 found in [12]. Binary PSK (BPSK) and quaternary PSK (QPSK) modulations are considered and a maximum of 200 iterations of the SP algorithm on the overall graph is allowed. A pilot symbol every 19 coded bits is added in order to make the iterative decoding algorithms bootstrap. This corresponds to a decrease in the effective transmission rate, resulting in an increase in the required signal-to-noise ratio of about 0.223 dB which has been introduced artificially in the curve labeled “known phase” for the sake of comparison. Hence, the gap between the “known phase” curve and the others is uniquely due to the need for phase estimation/compensation, and not to the rate decrease due to pilot symbols insertion.

In Fig. 2, the performance of the proposed algorithm is shown for $\sigma_\Delta = 6$ degrees and different values of the number N of considered Fourier coefficients. Values of $N > 17$ are not considered since they do not produce any performance improvement. Therefore, the value of $N = 17$ (i.e., $-8 \leq \ell \leq 8$ in all the equations of the previous Section) can be considered as optimal for $\sigma_\Delta = 6$ degrees. The gap of about 0.2 dB with respect to the curve labeled “known phase” is only due to the loss in channel capacity for a time-varying channel phase. In fact, the proposed algorithm performs as well as the algorithm based on the phase quantization [5] which is also shown for comparison assuming $L = 16$ quantization levels (no improvement has been observed for increasing values of L and this is in agreement with a result in [13] where the authors state that for M -PSK signals, $L = 8M$ values are sufficient to have no performance loss). This latter algorithm can be regarded as a “practically optimal” benchmark. In Fig. 2, the performance of the proposed algorithm for $N = 17$ and that of the quantized-based algorithm for $L = 16$ are also shown for $\sigma_\Delta = 12$ degrees.

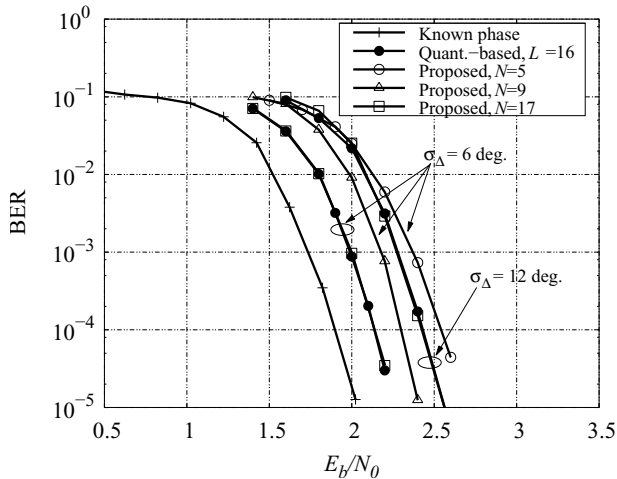


Figure 2: Performance of the proposed algorithm for BPSK and different values of the number N of considered Fourier coefficients.

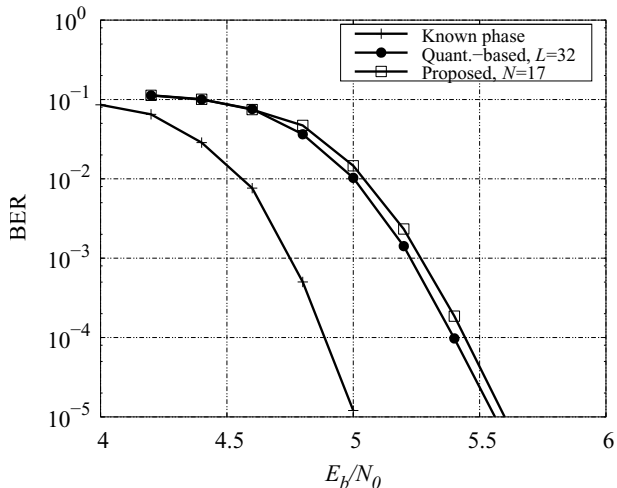


Figure 3: Performance of the proposed algorithm for QPSK.

In the binary case considered in the previous figure, the proposed algorithm has a practically optimal performance and a complexity similar to that of the algorithm based on phase quantization. However, for a modulation format characterized by a more dense constellation, if for the quantized-based algorithm the optimal number of quantization level, and thus the complexity, must be increased, it can be expected that the number N of considered Fourier coefficients in the proposed algorithm remains practically the same. This aspect is shown in Fig. 3 where a QPSK modulation is considered. The phase noise has $\sigma_{\Delta} = 6$ degrees. For the quantized-based algorithm $L = 8M = 32$ quantization levels are considered whereas for the proposed algorithm, the number of Fourier coefficients is still $N = 17$.

5. CONCLUSIONS

In this paper, the problem of joint detection and decoding of LDPC codes, transmitted over a channel affected by phase noise, has been considered. A factor graph, taking into account both the code constraints and the channel behavior, was built and by means of the sum-product algorithm, the marginal a posteriori probabilities of the transmitted code symbols were computed. To overcome the problem of an ex-

change of messages in the graph representing the probability density functions of continuous random variables, we used the method of canonical distributions. In this case, the above mentioned probability density functions were represented by means of a finite number of parameters which become the messages to exchange. The proposed algorithm exhibits a practically optimal performance and an affordable complexity.

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